Casimir energy through transfer operators for weak curved backgrounds

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The Casimir energy between a pair of two-dimensional plates represented by Dirac delta potentials and embedded in the topological background of a sine-Gordon kink is studied in [L. Santamaría-Sanz, letter (2023)] through an extension of the $TGTG$-formula, firstly discovered by O. Kenneth and I. Klich, to weak curved backgrounds. More details of the calculations are provided here, not only regarding the spectrum of the corresponding associated non-relativistic quantum mechanical problem but also concerning the Green's function and the transfer operators of the corresponding Quantum Field Theory. These details allow a better understanding of the issue. In fact, a more general potential consisting of a Dirac delta as well as its first derivative would be used to represent each plate. Moreover, the relation between the phase shift and the density of states (the well-known Dashen-Hasslacher-Neveu formula) is also exploited to characterize the quantum vacuum energy.

I. INTRODUCTION

The vacuum state of an arbitrary either classical or Quantum Field Theory (QFT) is the state of minimum energy of the Hamiltonian. This state is not empty but instead contains electromagnetic waves and infinite pairs of virtual short-lived particles and antiparticles, or vacuum bubbles. These pairs annihilate each other very quickly in accordance with the Heisenberg energy-time uncertainty principle. Nevertheless, when some objects are introduced into the space, sometimes nearby particles collide with them and get reflected in such a way that they do not combine again with their antipaticles. Quantum forces may thus appear in the system as a result of the presence of these frontiers. The physical properties of the vacuum state and the vacuum energy show a strong dependence on the type of boundaries. One of the most important boundary phenomenon is the Casimir effect [1], in which the presence of two parallel, uncharged and conducting plates restrics the modes of the fluctuations of the electromagnetic field between them. Vacuum polarization occurs and a finite force between both plates appears. This effect, predicted in 1948 by H.B.G Casimir, was experimentally measured for the first time in 1958 by M.J. Sparnaay [2]. Since then, many studies have been performed for bodies with different geometries and materials and important applications have emerged in the fields of condensed-matter physics, nanoscience and cosmology [3].

A crucial point in the theory is that the vacuum energy is ultraviolet divergent. It needs to be regularized and renormalized in order to obtain the finite contribution related to the interaction betwen objects. There are several approaches to achieve this goal: computing the 00-component of the energy-momentum tensor in terms of Green's function and the scattering data and then subtracting the first terms in its Born expansion [4], calculating the transfer operator to apply the $TGTG$-formula[1] [5], or using zeta functions, complex integrals and heat traces [6–8], to name just a few of methods. The $TGTG$-formula allows to compute the quantum vacuum interaction energy between two disjoint objects represented by a smooth classical background in a flat spacetime. For instance, it has been used to compute the vacuum interaction energy between two sine-Gordon kinks [9] and two plates mimicked by $\delta\delta'$ potentials [10]. Since in both cases the potentials representing the two objects do not overlap (i.e. they are potentials with disjoint compact supports), the $TGTG$-formula provides exact results. The major advantage of using the $TGTG$-formalism is that only the scattering problem for the background potential together with one single object is necessary to compute the vacuum interaction energy between the pair of bodies. This could be a crucial factor whenever the scattering problem for the complete potential with two objects in a classical background is hard to be solved. Furthermore, using the $TGTG$-formula significantly reduces the complexity of the analytical and numerical computation.

There are relevant differences between studying QFT for flat metrics and doing it for those that characterize curved manifolds. Varying the Hilbert-Einstein action of a scalar quantum field ϕ in a curved background with boundaries with respect to ϕ yields the field equations [11–13]:

$$g^{\mu\nu} D_\mu D_\nu \phi + (m^2 + \xi R)\phi = 0, \tag{1}$$

being $g^{\mu\nu}$ the metric tensor, D_μ the covariant derivative obtained from the connection, R the Ricci scalar curvature and ξ the coupling to the gravitational field. For scalar fields, D_μ reduces to the usual partial derivative ∂_μ. Concerning QFT, the main difference of the scalar field equations in a generic curved spacetime with respect to those present in the flat Minkowski one are the terms

[1] The name of the formula is due to the fact that it involves the use of Green's functions and Lippmann-Schwinger T-operators related to the objects in the order specified by the formula name.

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proportional to the scalar curvature R in (1). They are indispensable when renormalising the theory with counterterms. Moreover, the fundamental problems regarding the abscence of the concept of particle, the abscence of a reference vacuum state and the unitarily inequivalent representations of the algebra of the observables in curved spacetimes are arisen for instance in [14, 15]. From these references, it is clear that only for globally hyperbolic curved spacetimes endowed with a global temporal Killing vector, the solutions of the field equations and the temporal coordinate are globally defined and one could perform the usual canonical quantization. When there exists a goblal Killing vector, the spacetime is a fiber bundle with a set of spatial slices or Cauchy surfaces which evolve in time. For each fixed value of the temporal coordinate, one could solve the spectra of the Laplacian-Beltrami operator in the spatial slice as done for the Minkowski metric. However, in another more general case, if the curved spacetime is such that the fiber bundle do not allow an interpretation in terms of particle spectra independent of the observer, talking about scattering is ambiguous. Consequently, computing the quantum vacuum energy either from the 00-component of the energy-momentum tensor and from the transfer operators defined in terms of the scattering data will not offer a universal outcome independent of the observer. In fact, there are not many results about the $TGTG$-formula in curved backgrounds and sometimes it does not even exist [16]. In these last cases, the correct way to compute the Casimir energy is considering the wave function of the fundamental state of the field configuration and using spectral functions associated to the Laplacian operator. In this way, the trace of the determinant of the Laplacian operator is interpreted as the energy and by using zeta regularization, it will be possible to find a universal result.

There is a special case to be taken into account. When the frequencies of the particles created by the gravitational background are much smaller than the Planck frequency, one could use the perturbation theory for this curved spacetime as a semiclassical approach to quantum gravity [17, 18]. In doing so, this weak gravitational backgrounds are treated classically and the matter fields are the ones which will be quantized. The key point is that this gravity would be strong enough to produce some effects to the quantum matter, but not so strong as to require an own quantization. This is exactly the case which is going to be considered in this work. The main objective will be the study of the quantum vacuum interaction energy between a pair of two-dimensional homogeneous plates placed[2] at $z = a, b$ and embedded in a weak curved background potential centered at the origin of the direction orthogonal to the plates. The general procedure for calculating the transfer operators and Green's functions will be given. However, it will also be illustrated by one concrete example where the plates are modeled by the Dirac punctual potential

$$V_{\delta\delta'}(z) = v_0\delta(z-a) + w_0\delta'(z-a) + v_1\delta(z-b) + w_1\delta'(z-b),$$

being $v_0, v_1, w_0, w_1, a, b \in \mathbb{R}$ and $a < b$. Above, δ' denotes the first derivative of the delta function. Punctual potentials or contact interactions have attracted much attention so far. Dirac delta potentials are widely used as toy models for realistic materials like quantum wires [19], and to analyze physical phenomena such as Bose-Einstein condensation in periodic backgrounds [20] or light propagation in 1D relativistic dielectric superlattices [21]. Despite being a rather simple idealization of the real system, the δ function has been proved to correctly represent surface interactions in many models related to the Casimir effect. For instance, Dirac δ functions have been set on the plates acting as the electrostatic potential [22], to represent two finite-width mirrors [23], or to describe the permittivity and magnetic permeability in an electromagnetic context, by associating them to the plasma frequencies in Barton's model on spherical shells [24, 25]. On the other hand, the first derivative of the delta potential has been used to study monoatomically thin polarizable plates formed by lattices of dipoles [26] and resonances in 1D oscillators [27]. There is some controversy in the definition of the δ' potential since different regularizations produce different scattering data (see [28] and references therein). Here, I will use the one presented in [29], in which the authors define it by introducing a Dirac delta potential at the same point to regularize the whole potential. As they explained, the major advantage of this choice is that it enables defining this singular potential in terms of matching conditions at the origin which do not depend on the choice of a regularization method.

Once the plates have been described, it is interesting to point out that the classical background of the specific example will be the Pöschl-Teller (PT) potential

$$V_{PT}(z) = -2\,\mathrm{sech}^2\, z.$$

It models the propagation of mesons moving in a sine-Gordon kink background [30–32]. Kinks in $1+1$ dimensions can be embedded into a $3+1$ dimensional theory as solutions which are independent of all but one spatial direction. These types of solutions with finite energy per unit area are known as domain walls [31]. Previous work concerning the 00-component of the energy momentum tensor $T_{\mu\nu}$ in a system with a kink in the real line, and the scattering problem of two delta potentials simetrically placed around a kink can be found in [32, 33]. Nevertheless, here I tackle a more general situation, as already introduced. It is important to highlight that I deal with a QFT in which the ultraviolet divergences present in the vacuum energy are not eliminated just by taking

[2] The position four-vector will be expressed from now on as $x^\mu = (t, \vec{x}_\parallel, z) \in \mathbb{R}^{1,3}$. Notice that $\vec{x}_\parallel \in \mathbb{R}^2$. Likewise, the four-momentum will be $K^\mu = (E, \vec{k}_\parallel, k)$. Here, z and k are the position and the momentum coordinates in the direction orthogonal to the surfaces of the plates.

the normal ordering of the operators. In fact, even if there were no plates and only the PT kink theory was considered, mass corrections of the order of $\hbar$ would appear (the well-known Dashen-Hasslacher-Neveu (DHN) correction [34]). Another interesting property of the PT background is that it is transparent in the sense that the fields could be asymptotically interpreted as particles. Consequently, it is possible to define incoming and outgoing waves and to derive a S-matrix in a similar way to the usual for flat spacetimes [35–37].

It is important to note that the thermodynamics at non-zero temperature are not going to be considered. This is due to the fact that for weak gravitational backgrounds, once the thermal fluctuations be the dominant ones, the effects of the curved background will not be noticeable. One will find the same results as in a flat background and more specifically, the results presented in [37]. Furthermore, it is worth mentioning that only effective theories [38, 39] are going to be considered. When computing the Casimir force between plates or objects, the microscopic details of the material that the object is built of should also be taken into account. The QED Lagrangian should also be introduced to add the contribution of the atoms of the objects. However, this is not what happens. In fact, if one were to do it, the theory would become as complicated that it probably could not be solved. Consequently, one traces over the microscopic degrees of freedom concerning the fermions in the plates to work with an effective theory which analytically describes the system properly enough. This approach has also been implicitly followed in this work.

The discussion given in the following sections constitutes a technical companion to Ref. [40], providing a detailed derivation and several extensions of the results mentioned there. The work is organised as follows: in Sec. II, a brief discussion regarding the curved background to be considered is presented. Sections III and IV involve the computation of the spectrum of scattering and bound states of the associated Schrödinger operator as well as the derivation of the Green's functions, respectively. In Sec. V and VI, the $TGTG$-formula and the DHN one [41] will be used to analyze the quantum vacuum interaction energy and to study the Casimir pressure between plates, giving a more general overview than the one shown in [40]. Finally, Sec. VII summarizes the main conclusions. The natural system of units $\hbar = c = 1$ will be used.

II. DOMAIN WALL BACKGROUND

The next question that arises is whether it is possible to determine a metric for a curved spacetime, in such a way that the equation describing the dynamics of the quantum vacuum fluctuations around a kink solution in a flat spacetime[3], i.e.

$$\partial_t^2 \phi - \nabla^2 \phi - \left(-m^2 + 2\,\mathrm{sech}^2\, z\right) \phi = 0,$$

be the equation of motion for a scalar field coupled to the gravitational background of a domain wall[4]. It is possible to find a solution for the metric from the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi \mathtt{G}\, T_{\mu\nu}$$

(being $R_{\mu\nu}$ the Ricci curvature tensor, $\mathtt{G}$ the universal gravitational constant and Λ the cosmological one), but it is undoubtedly complicated. One of the several difficulties is that very little is known about the distribution of momentum and energy in such a curved spacetime. Is it sufficient for the domain wall to be the only gravitational source of mass? If yes, and considering z as the spatial coordinate in which the one-dimensional domain wall extends along, can $T_{\mu\nu}$ be written as

$$T_{\mu\nu} = \begin{pmatrix} \rho(z) & 0 \\ 0 & f_\Pi \end{pmatrix},$$

with $\rho(z)$ the energy density? If so, what is the flux of momentum f_Π? As it can be seen, being able to derive the components of the metric from the Einstein's equations without knowing in advance the exact form of the $T_{\mu\nu}$ tensor may not be guaranteed. However, taking into account the symmetry of the system, perhaps it might be possible to apply the same reasoning given in [43]. In this work, the authors derive the metric components just by solving two differential equations that arise when imposing the spherical symmetry characteristic of their example on the Einstein's equations, written in terms of the sectional curvatures[5] [44, 45], and the Bianchi's identities. Due to the spherical symmetry all sectional curvatures, as well as the proper energy density and the proper pressures across the transverse spatial planes, depend only on the radial coordinate. The rest of the components of $T_{\mu\nu}$ are zero. All this significantly reduces the problem of finding the sectional curvatures and then, the metric components. Back to the case of the metric for the domain wall, there is also a certain degree of symmetry in the configuration of the system. Notice that

[3] The equation of motion incorporates a mass term m^2 in order for the Hamiltonian of the QFT to be a non-negative self-adjoint operator and for the theory to be well-defined and unitary.

[4] Domain walls can be thought as membrane-like two-dimensional structures embedded in three-dimensional spaces. In the early stages of the Universe, the spontaneous breaking of discrete symmetries produced this kind of topological defects [42].

[5] Sectional curvatures allows to compute the second derivative of the separation between any two nearby geodesic curves, with tangent vectors at a given point contained in the corresponding two-plane indicated by the subindeces. They are geometrical quantities independent of the coordinate system.

the domain wall extends along a spatial coordinate but the other two are totally symmetric. Consequently, the sectional curvatures $\{K_{tx}, K_{ty}, K_{tz}, K_{yz}, K_{zx}, K_{xy}\}$ to be found only depend on the spatial coordinate z. Moreover, the properties $K_{tx}(z) = K_{ty}(z)$ and $K_{yz}(z) = K_{zx}(z)$ are fulfilled. The Einstein's equations

$$\begin{aligned} 2K_{tx}(z) + K_{tz}(z) &= 4\pi\mathtt{G}\,(\rho(z) + \mathrm{tr}\, f_\Pi), \\ K_{xy}(z) + 2K_{yz}(z) &= 8\pi\mathtt{G}\,\rho(z), \\ K_{xy}(z) + K_{tz}(z) &= 4\pi\mathtt{G}\,(\rho(z) + \mathrm{tr}\, f_\Pi - 2\sigma^{zz}), \end{aligned}$$

should be solved together with the Bianchi's identities but due to the symmetry, the problem can be tackled. In the equations above, σ^{zz} is the flux of the z component of the momentum transferred per unit time across the unit area along the xy parallel two-dimensional plane.

Nevertheless, the aim of this paper is not to derive the metric of the spacetime but to obtain a generalization of the $TGTG$-formula. The advantage of dealing with weak curved spacetimes is that it is not necessary to solve the problem of the metric in order to understand what will appear in the rest of paper. Since the weak gravitational field is going to be treated classically without an own quantization, and furthermore the PT potential is transparent so that the notion of incoming and outgoing particle applies here, studying the propagation of mesons moving in a domain wall background while interacting with two Dirac delta plates reduces to analyzing a quantum scalar field in the presence of an external classical background potential, in a similar way to what is done for flat spacetimes. In this way, the starting point would be the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi + 2\,\mathrm{sech}^2(z)\,\phi^2 - V_{\delta\delta'}(z)\,\phi^2\right), \quad (2)$$

being $v_0, v_1, w_0, w_1, a < b \in \mathbb{R}$, and the problem of solving the metric can be left for future investigation.

III. SCATTERING DATA AND SPECTRUM

For computing the interaction energy between plates due to the quantum vacuum fluctuations, it is necessary to characterize the normal modes of a scalar field whose dynamics is described by the action involving the Lagrangian density (2). Consider for simplicity a real massless scalar field ϕ confined between two parallel $(D-1)$-dimensional plates separated by a distance $b-a$ in the axis orthogonal to the plates, i.e. the z axis. For isotropic and homogeneous plates, there exists a translational symmetry along the surface of the plates and the theory of free fields without boundaries is recovered for the parallel direction coordinates $\vec{x_\parallel} \in \mathbb{R}^{D-1}$. Splitting the spatial coordinate as $x = (\vec{x_\parallel}, z)$ with $\vec{x_\parallel} \in \mathbb{R}^{D-1}$ and taking into account the Fourier decomposition of the field

$$\phi(t,x) = \int d\omega\, e^{-i\omega t}\phi_\omega(x),$$

the equation for the modes of the fluctuations field is given by the non-relativistic Schrödinger separable eigenvalue problem

$$(-\Delta + V_{PT}(z) + V_{\delta\delta'}(z))\phi_\omega(x) = \omega^2\phi_\omega(x).$$

In the equation above, $\Delta = \Delta_\parallel + \partial_z^2$ and the frequencies follow the dispersion relation $\omega^2 = \vec{k}_\parallel^{\,2} + k^2$. Therefore, the problem is separable and only the direction orthogonal to the plates needs to be studied, as the spectrum is trivial in the other directions. From now on, I will consider $D = 3$, just for simplicity.

Notice that the non-relativistic Schrödinger operator related to the background in the dimension orthogonal to the plates,

$$\hat{K}_{PT} = -\partial_z^2 + V_{PT}(z),$$

is not essentially self-adjoint in the Sobolev space of functions $W_2^2(\mathbb{R} - \{a,b\}, \mathbb{C})$. It is necessary to add some matching conditions, concerning the continuity of the wave function and the discontinuity of its derivative at the boundary points $\{a,b\}$, in order to define the self-adjoint extensions of $\hat{K}_{PT}$ in the aforementioned domain. These boundary conditions will be determined by the specific potential that represents the plates. For instance, if they are mimicked by the Dirac delta potential $V_{\delta\delta'}$, the domain of the self-adjoint extension of $\hat{K}_{PT}$ is given by the suitable matching conditions [29] collected in (3), which come from the original work of Kurasov in one-dimensional systems [46].

$$\mathcal{D}_{\hat{K}_{PT}} = \left\{ \phi \in W_2^2(\mathbb{R} - \{a,b\}, \mathbb{C}) \;\middle|\; \begin{pmatrix} \phi(a^+) \\ \phi'(a^+) \\ \phi(b^+) \\ \phi'(b^+) \end{pmatrix} = \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ \beta_0 & \alpha_0^{-1} & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & \beta_1 & \alpha_1^{-1} \end{pmatrix} \begin{pmatrix} \phi(a^-) \\ \phi'(a^-) \\ \phi(b^-) \\ \phi'(b^-) \end{pmatrix} \right\},$$

$$\text{being} \qquad \alpha_i = \frac{1 + w_i/2}{1 - w_i/2}, \quad \beta_i = \frac{v_i}{1 - (w_i/2)^2}, \qquad i = 0,1. \quad (3)$$

The system of two plates in the background chosen has an open geometry so the positive energy spectrum will be continuous. Scattering states correspond to solutions of the Schrödinger equation

$$(\hat{K}_{PT} + V_{\delta\delta'}(z))\phi_{\omega,\vec{k}_{\parallel}}(z) = k^2\phi_{\omega,\vec{k}_{\parallel}}(z),$$

with $k \in \mathbb{R}$ (such that $k^2 > 0$) and keeping in mind that the delta potential must be understood as the boundary conditions at $z = a, b$ aforementioned. Given a linear momentum k there are two independent scattering solutions to be found. However, these solutions are constructed differently depending on the problem geometry. Theoretically speaking, there are several ways to place two objects embedded in a background potential. In order for the Casimir energy between plates to be a non-negligible magnitude, the two objects have to be very close to each other. Consequently, two different cases could be examined. For the problem of two plates placed symmetrically around the origin and a background potential centered at $z = 0$ with compact support ϵ much smaller than the distance between plates, the solutions for "*diestro*" and "*zurdo*" scattering solutions (incoming particles from the left or from the right, respectively) are of the form:

$$\psi_k^R(z) = \begin{cases} e^{ikz} + r_R\, e^{-ikz}, & \text{if} \quad z < -\frac{\epsilon}{2}, \\ F_R\, f_k(z) + P_R\, f_{-k}(z), & \text{if} \quad -\frac{\epsilon}{2} < z < \frac{\epsilon}{2}, \\ t_R\, e^{ikz}, & \text{if} \quad z > \frac{\epsilon}{2}, \end{cases}$$

$$\psi_k^L(z) = \begin{cases} t_L\, e^{-ikz}, & \text{if} \quad z < -\frac{\epsilon}{2}, \\ F_L\, f_k(z) + P_L\ f_{-k}(z), & \text{if} \quad -\frac{\epsilon}{2} < z < \frac{\epsilon}{2}, \\ r_L\, e^{ikz} + e^{-ikz}, & \text{if} \quad z > \frac{\epsilon}{2}. \end{cases}$$

Notice that $f_{\pm k}(z)$ would be the eigenfunctions of the operator $-\partial_z^2 + V_{background}(z)$ and $\{t, r, F, P\}_{L,R}(k)$ the scattering data for each object. To obtain them, some boundary conditions have to be imposed at the position where the plates are located, i.e. $z = a \ll -\epsilon/2$ and $z = b \gg \epsilon/2$.

There is another possible configuration. If the distance between plates is smaller than the support of the background potential in such a way that the plates are placed within this support, the scattering solutions are of the form:

$$\psi_k^R(z) = \begin{cases} f_k(z) + r_R\, f_{-k}(z), & \text{if} \quad z < a, \\ B_R\, f_k(z) + C_R\, f_{-k}(z), & \text{if} \quad a < z < b, \\ t_R\, f_k(z), & \text{if} \quad z > b, \end{cases} \qquad (4)$$

$$\psi_k^L(z) = \begin{cases} t_L\, f_{-k}(z), & \text{if} \quad z < a, \\ B_L\, f_k(z) + C_L\ f_{-k}(z), & \text{if} \quad a < z < b, \\ r_L\, f_k(z) + f_{-k}(z), & \text{if} \quad z > b. \end{cases} \qquad (5)$$

This second case is the one which will be considered in the example of two Dirac delta plates in a Pöschl-Teller kink background. Thus, $f_k(z) = e^{ikz}(\tanh(z) - ik)$ are the free waves of the Pöschl-Teller potential (i.e. plane waves times first order Jacobi polynomials). It is relevant to highlight that the transmission amplitudes $t_R(k), t_L(k)$ are identical to each other due to the time-reversal invariance of the Schrödinger operator. Consequently, they will be substituted by $t(k)$ from now on. Replacing (4) and (5) in the matching conditions (3) and solving the resulting two systems of equations with unknowns $\{r_R, r_L, t, B_R, B_L, C_R, C_L\}(k)$, the scattering data are obtained (they are collected in eq. (A1) in Appendix A).

The denominator of all the scattering parameters, $\Upsilon(k)$, is the spectral function. The set of zeroes of $\Upsilon(k)$ can be the poles of the scattering matrix $S(k)$. Notice that the $S(k)$-matrix admits an analytic continuation to the entire complex momentum plane. The zeroes of the spectral function on the positive imaginary axis in the complex momentum k-plane gives the bound states of the spectrum of the non-relativistic Schrödinger operator. Making $k \to i\kappa$ in $\Upsilon(k) = 0$, one can study the bound states as the intersections between an exponential and a rational function via the transcendent equation $-\Upsilon_1(\kappa)/\Upsilon_2(\kappa) = e^{-2\kappa(b-a)}$, where

$$\begin{aligned} \Upsilon_1(\kappa) &= 4\,\Sigma(v_0, w_0, a, \kappa)\,\Sigma(v_1, w_1, b, \kappa), \\ \Upsilon_2(\kappa) &= 16(\kappa - \tanh a)(\kappa + \tanh b)\,\Lambda(v_0, w_0, a, \kappa) \\ &\quad \times \Lambda(-v_1, w_1, -b, \kappa), \\ \Lambda(v_i, w_i, x, \kappa) &= -2w_i \operatorname{sech}^2 x - (v_i - 2w_i\,\kappa)(\kappa - \tanh x), \\ \Sigma(v_i, w_i, x, \kappa) &= [2v_i + \kappa(4 + w_i^2)](\kappa^2 - 1) \\ &\quad + 2\operatorname{sech}^2 x\,(v_i + 2w_i \tanh x). \end{aligned} \qquad (6)$$

Once the momenta of the bound states are determined, their energies are given by $E = (i\kappa)^2 < 0$. The lowest energy state will be characterized by E_{min}. For some type of potentials (such as the one described in [40]), it is possible to give an analytic formula to bound the energy of the states with negative energy of the spectrum. In other configurations, E_{min} has to be obtained numerically.

Finding the $TGTG$-formula implies focusing only in the scattering problem for one single plate. If one of the delta plates is removed (for instance $v_1 = w_1 = 0$), the spectral function of the reduced system is:

$$\begin{aligned} &(4 + w_0^2)\,\kappa^3 + 2v_0\kappa^2 - (4 + w_0^2)\,\kappa - 2v_0 \tanh^2 a \\ &+ 4w_0 \tanh a \operatorname{sech}^2 a = 0. \end{aligned} \qquad (7)$$

By studying the asymptotic behavior of (7) as well as its maxima and minima for different values of the parameters, it can be seen that there may be several cases: only one bound state or two bound states. There is no zero mode because the state with wave vector $k = 0$ does not constitute a pole of the $S(k)$-matrix. The scattering data for the reduced system can be obtained from equation (A1) and they are given by:

$$\begin{aligned} t^\ell &= \frac{-\alpha_0 W}{\alpha_0 f_{-k}(a)\,(\alpha_0 f_k'(a) - \beta_0 f_k(a)) - f_k(a) f_{-k}'(a)}, \\ r_R^\ell &= \frac{-f_k(a)\left[\left(\alpha_0^2 - 1\right) f_k'(a) - \alpha_0\beta_0 f_k(a)\right]}{\alpha_0 f_{-k}(a)\,(\alpha_0 f_k'(a) - \beta_0 f_k(a)) - f_k(a) f_{-k}'(a)}, \\ r_L^\ell &= \frac{-f_{-k}(a)\left[\left(\alpha_0^2 - 1\right) f_{-k}'(a) - \alpha_0\beta_0 f_{-k}(a)\right]}{\alpha_0 f_{-k}(a)\,(\alpha_0 f_k'(a) - \beta_0 f_k(a)) - f_k(a) f_{-k}'(a)}, \end{aligned} \qquad (8)$$

where W is the Wronskian

$$W \equiv W[f_k(a), f_{-k}(a)] = -2ik(k^2+1).$$

Notice that $B_R^\ell = t^\ell, B_L^\ell = r_L^\ell, C_R^\ell = 0, C_L^\ell = 1$. Similarly, setting $v_0 = w_0 = 0$ one obtains the reduced scattering data when the plate on the right is the only one present in the system. In this case, $\{t^r, r_R^r, r_L^r\}$ are given by (8) but replacing $\alpha_0 \to \alpha_1, \beta_0 \to \beta_1, a \to b$. Furthermore, $C_L^r = t^r, B_L^r = 0, B_R^r = 1, C_R^r = r_R^r$. The superscript ℓ, r in the scattering data indicates which plate is being considered: ℓ for the plate placed on the left of the system and r on the right. The subscript R, L refers to "*diestro*" and "*zurdo*" scattering.

A rather important fact is that due to the Pöschl-Teller background potential, the translational invariance of the system is broken. $V_{PT}(z)$ breaks the isotropy of the space and consequently if $f_k(z)$ is a eigenfunction of the non-relativistic Schrödinger operator $\hat{K}_{PT}$, then $f_k(z+a)$ with $a \in \mathbb{R} - \{0\}$ will no longer be another. This means that the scattering data explicitly depend on the position of the plates in a non-trivial way.

For computing the vacuum interaction energy in the corresponding QFT for the general case in which $v_0, v_1, w_0, w_1 \in \mathbb{R} - \{0\}$, the value of the energy for the lowest energy bound state of the quantum mechanical problem explained in this section is essential. Since the bound state with the lowest energy is characterized by E_{min}, the mass of the fluctuations in the theory will be balance with this value E_{min} for making fluctuation absorption impossible. The unitarity of the QFT sets this lower bound for the mass of the quantum vacuum fluctuations, so that the total energy of the lowest energy state of the spectrum will be zero. Thus the spectrum of the associated Quantum Field Theory will consist of a set of discrete states with energies within the gap $[0, |E_{min}|]$ and a continuum of scattering states with energies above the threshold $E = |E_{min}| = m^2$. The number of discrete states will be determined by the value of the coefficients $\{v_0, v_1, w_0, w_1\}$, with a maximum of three being possible[6]. It is important to highlight that the value of E_{min} must be computed for the whole system of two objects plus the background potential. This is not contradictory to the fact that if the Casimir energy is calculated with the $TGTG$-formula, only the transmission and reflection coefficients of the reduced system of an object in the background potential are needed. It will be discussed in detail in Sec. V.

[6] Although each $\delta\delta'$ potential can hold at most two bound states, since the two plates are very close together, the whole system of two plates in the PT background acts as a well not deep and wide enough to accommodate four bound states, but three.

IV. GREEN'S FUNCTION

Once the spectral problem has been solved, the usual second quantization procedure could be applied to promote the non-relativistic quantum mechanical theory to a QFT in which to study the quantum vacuum interaction energy between objects. The $TGTG$-formula is based on two main elements: the Green's function and the transfer operators. The characteristic Green's function can be obtained by solving the differential equation

$$\left[\partial_\mu \partial^\mu - 2\,\mathrm{sech}^2 z + V_{\delta\delta'}(z) + m^2\right] G(x^\mu, y^\mu) = \delta(x^\mu - y^\mu)$$

for the complete Green's function

$$G(x, x') = \int \frac{d^2 k_\parallel}{(2\pi)^2} e^{i\vec{k}_\parallel(\vec{x}_\parallel - \vec{x'}_\parallel)} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G_k(z, z'),$$

or, equivalently, by solving

$$\left[-\partial_{z_1}^2 - k^2 - 2\,\mathrm{sech}^2 z_1 + V_{\delta\delta'}(z_1)\right] G_k(z_1, z_2) = \delta(z_1 - z_2)$$

for the reduced one. Solving this differential equation requires assuming the continuity of $G_k(z_1, z_2)$ and the discontinuity of its first derivative at the points $\{a, b\}$, as well as imposing an exponentially decaying behaviour of the solutions at infinity. Another way to compute the reduced Green's function in the spatial dimension orthogonal to the surfaces of the plates is by using [10]

$$G_k(z, z') = \frac{\mathrm{u}(z - z')\psi_k^R(z)\psi_k^L(z') + \mathrm{u}(z' - z)\psi_k^R(z')\psi_k^L(z)}{W[\psi_k^R, \psi_k^L]}$$

for the two linear independent scattering solutions given in (4)-(5) for the complete system of two plates in the PT background. Note that $\mathrm{u}(z - z')$ is the unit or Heaviside step function. Both aforementioned methods yield the same solution for the correlator.

Moreover, the Wronskian $W[\psi_k^R, \psi_k^L]$ has to be the same for the three zones in which the two delta plates divide the space. This imposes the following relation between the scattering coefficients: $t = B_R C_L - C_R B_L$. This relation is useful to simplify the solutions of the Green's function in the different zones which the plates divide the space into, and to rewrite them as

$$G_k(z_1, z_2) = G_k^{PT}(z_1, z_2) + \Delta G_k(z_1, z_2),$$

being $\Delta G_k(z_1, z_2)$ given by the equation (A2) in Appendix A.

Notice that the Green's function for the kink potential centered at the origin without any delta interactions (i.e. $v_0 = w_0 = v_1 = w_1 = 0$) takes the form

$$G_k^{PT}(z_1, z_2) = \frac{1}{W} f_{-k}(z_<) f_k(z_>) =$$
$$\frac{e^{ik|z_1 - z_2|}}{W}\left(k^2 + ik|\tanh z_1 - \tanh z_2| + \tanh z_1 \tanh z_2\right),$$

where $z_<$ and $z_>$ are the lesser or the greater of z_1 and z_2. It plays the same role as $G_k^0(z_1, z_2) = e^{ik|z_1 - z_2|}/(-2ik)$

in free plain backgrounds. This is due to the fact that the Pöschl-Teller potential is transparent (there is not additional reflection with respect to the free case). Furthermore, since the Pöschl-Teller potential breaks the isotropy of the space, the Green's function is such that $G_k^{PT}(z_1, z_2) \neq G(z_1 - z_2)$. In fact, $G^{PT}(z, z)$ is not a constant as happens in the free flat case, but depends on the spatial orthogonal coordinate in a non-trivial way and thus, spatial translations are no longer symmetries of the system.

V. CASIMIR ENERGY AND *TGTG* FORMULA

The quantum vacuum energy per unit area of the plates,

$$\frac{\tilde{E}_0}{A} = \frac{1}{2}\sum\!\!\!\!\!\!\int_k \int_{\mathbb{R}^2} \frac{d\vec{k}_\parallel}{(2\pi)^2}\sqrt{m^2 + k^2 + \vec{k}_\parallel^2},$$

involves the summation of the frequencies of the fields modes ω such that $\omega^2 = \vec{k}_\parallel^2 + k^2 + m^2$ is in the spectrum σ of the operator $-\partial^2_{\vec{x}_\parallel} + \hat{K}_{PT} + V_{\delta\delta'}(z)$. The result is divergent due to the contribution of the energy density of the free theory in the bulk and the self-energy of the infinite area plates. As a consequence, it is necessary to introduce a regulator. For instance, one could introduce an exponentially decaying function and perform the integration over the parallel modes to obtain:

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}\frac{d\vec{k}_\parallel}{(2\pi)^2}\sqrt{\vec{k}_\parallel^2 + k^2 + m^2}\, e^{-\epsilon(\vec{k}_\parallel^2 + k^2 + m^2)} =$$
$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\,\chi(k,\epsilon)\, e^{-\epsilon(k^2+m^2)},$$

with

$$\chi(k,\epsilon) = \frac{\sqrt{\pi}}{4\,\epsilon^{3/2}} + \frac{\sqrt{\pi}(m^2+k^2)}{4\sqrt{\epsilon}} - \frac{1}{3}\left(k^2+m^2\right)^{3/2} + o(\epsilon).$$

Notice that the terms proportional to $\epsilon^{-3/2}$ and $\epsilon^{-1/2}$ must be removed before taking the limit $\epsilon \to 0$ to eliminate the contribution of the parallel modes to the dominant and subdominant divergences[7], respectively. In this way

$$\frac{E_0}{A} = -\frac{1}{2}\sum\!\!\!\!\!\!\int_k \frac{(m^2+k^2)^{3/2}}{6\pi}. \tag{9}$$

In the equation above, the sum over modes of the spectrum in the orthogonal direction splits into the summation over a finite number of states with positive energy in the gap $[0, |E_{min}|]$ (coming form the bound states of the associated quantum mechanical problem) and the integral over the continuous states with energies greater than $|E_{min}|$.

Now, it is necessary to remove in (9) the contribution to the divergences coming from the modes in the orthogonal direction, where the one-dimensional kink lives. These contributions are different from the ones of the modes in the parallel directions, because in the orthogonal direction the space is not longer a free one. The method to be used now is to put the system into a very large box of length L with periodic boundary conditions (p.b.c.) at its edges

$$\psi\left(-\frac{L}{2}\right) = \psi\left(\frac{L}{2}\right), \qquad \psi'\left(-\frac{L}{2}\right) = \psi'\left(\frac{L}{2}\right).$$

By so doing, all the spectrum of the Schrödinger operator $\hat{K} = -\partial_z^2 - 2\,\mathrm{sech}^2(z) + V_{\delta\delta'}(z)$ becomes discrete.

On the one hand, the contribution of the discrete set of N sates in the gap to the vacuum interaction energy is

$$-\frac{1}{2}\sum_{j=1}^{N}\frac{\left(\sqrt{(i\kappa_j)^2 + m^2}\right)^3}{6\pi}.$$

The frequencies of these bound states for each configuration $(v_0, v_1, w_0, w_1, a, b)$ will be determined numerically by solving $\Upsilon(i\kappa) = 0$, $\kappa > 0$ from the scattering problem. If there were half-bound states in the spectrum (i.e. states with energies that lie in the threshold $E = |E_{min}|$), they would have to be accounted for with a weight of $1/2$. But this will not be the case covered in the example of the two Dirac plates in a PT background.

On the other hand, concerning the states with energy $E > m^2$, it is necessary to compute

$$-\frac{1}{2}\sum_{k_n\in\sigma^+(\hat{K})}\frac{\left(\sqrt{k_n^2 + m^2}\right)^3}{6\pi}, \tag{10}$$

being $\sigma^+(\hat{K}) = \{k_n \in \sigma(\hat{K})\,|\, k_n^2 + m^2 > |E_{min}|\}$. The differential equation $\hat{K}\psi(z) = k_n^2\psi(z)$ with periodic boundary conditions at $\pm L/2$ must be solved. Notice that now $\psi(z) = A\psi_k^R(z) + B\psi_k^L(z)$ is a linear combination of the scattering solutions (4) and (5). The resulting system of equations admits a solution whenever the following spectral equation holds:

$$h_{p.b.c.}(k, L) \equiv 2tW - 2(t^2 - r_R r_L) f_k\left(\frac{L}{2}\right) f_k'\left(\frac{L}{2}\right)$$
$$+(r_R + r_L)\left[f_{-k}\left(\frac{L}{2}\right) f_k'\left(\frac{L}{2}\right) + f_k\left(\frac{L}{2}\right) f_{-k}'\left(\frac{L}{2}\right)\right]$$
$$+2 f_{-k}\left(\frac{L}{2}\right) f_{-k}'\left(\frac{L}{2}\right) = 0.$$

[7] Dominant or subdominant divergence refers to the degree of ultraviolet divergence of the terms. In [47] it is shown that for the system of a scalar field confined between two plates in a flat spacetime, the dominant divergence is a term proportional to a regularization parameter with units of energy raised to the power $(D+1)/2$ with D the spatial dimension of the theory, and the subdominant as the parameter raised to $D/2$. Both terms are divergent in the ultraviolet regime. Whilst the former is associated to the density energy of the free theory in the bulk, the latter is due to the self-energy of infinite area plates.

The scattering data are given in equation (A1) of Appendix A. The discrete set of zeroes $\{k_n\}$ of the secular function $h_{p.b.c.}(k, L)$ on the real axis will be the frequencies of the modes over which one has to perform the summation in (10). This sum can be computed through a complex integral over a contour enclosing all the zeroes of $h_{p.b.c.}(k, L)$. By using the residue theorem in complex analysis and also by taking into account the sates in the gap, the total quantum vacuum interaction energy reads:

$$\frac{E_0}{A} = -\frac{1}{2}\left[\oint_\Gamma \frac{dk}{2\pi i}\frac{(m^2+k^2)^{3/2}}{6\pi}\partial_k \log h_{p.b.c.}(k, L)\right] - \frac{1}{2}\sum_{j=1}^{N}\frac{\left(\sqrt{(i\kappa_j)^2+m^2}\right)^3}{6\pi},$$

being Γ the contour represented in FIG. 1. It can be proved that the integration over the circumference arc of the contour is zero in the limit $R \to \infty$. Hence, the integration over the whole contour Γ reduces to the integration over the straight lines $\xi_\pm = \pm i\xi + m$ with $\xi \in [0, R]$.

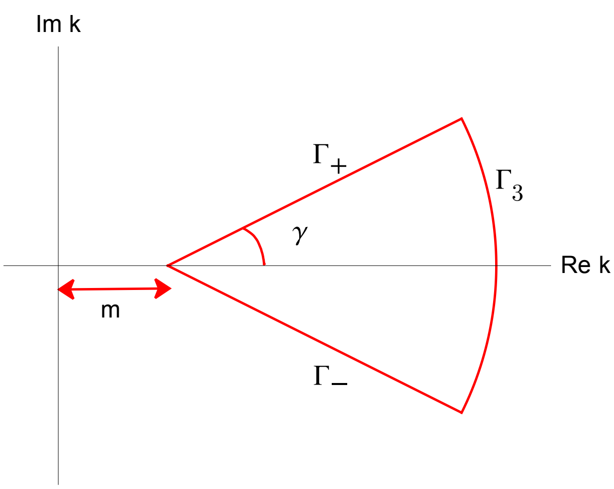

FIG. 1. Complex contour that encloses all the zeroes of the spectral function when $R \to \infty$. In this contour, one the one hand $\Gamma_\pm = \{m + \xi e^{\pm i\gamma}\,|\,\xi \in [0, R]\}$ and on the other hand $\Gamma_3 = \{m + Re^{i\nu}\,|\,\nu \in [-\gamma, \gamma]\}$. The angle $\gamma = \pi/2$ is going to be chosen. When integrating, the contour will be run counterclockwise.

Moreover, the dominant and subdominant divergent terms associated to the orthogonal modes and caused by the confinement of the system in a very large box must be subtracted as explained in [36, 37, 48], i.e. by computing:

$$\int_0^R \frac{d\xi}{12\pi^2 i}\left\{(m^2+\xi_+^2)^{\frac{3}{2}}\left[L - L_0 - \partial_\xi \log\frac{h_{p.b.c.}(\xi_+, L)}{h_{p.b.c.}(\xi_+, L_0)}\right] - (m^2+\xi_-^2)^{\frac{3}{2}}\left[L - L_0 - \partial_\xi \log\frac{h_{p.b.c.}(\xi_-, L)}{h_{p.b.c.}(\xi_-, L_0)}\right]\right\}$$

in the limits $L_0, R \to \infty$. The result of the integration does not depend on the box size and consequently, one could study the limit $L \to \infty$.

At this point, reversing the change of variable $\xi \to -ik$ due to the Wick rotation on the momentum yields the DHN formula [41]

$$E_0 = -\frac{A}{12\pi^2}\int_m^\infty dk\,\left(\sqrt{k^2+m^2}\right)^3\frac{d\delta(k)}{dk} - \frac{A}{2}\sum_{j=1}^{N}\frac{\left(\sqrt{-(\kappa_j)^2+m^2}\right)^3}{6\pi}, \tag{11}$$

with $m^2 = |E_{min}|$ and being $\delta(k)$ the phase shift related to the scattering problem in the direction orthogonal to the plate:

$$\delta(k) = \frac{1}{2i}\log_{-\pi}\left[t^2(k) - r_R(k)r_L(k)\right].$$

So far, I have considered the scattering problem for waves interacting with a system inside a large box, without specifying the type of system I was working with. In addition, the divergences related to putting the system in a box were eliminated. However, notice that the system is composed by two infinitely large plates and a background potential. Consequently, the subdominant divergences associated to the plates are still present and a renormalization mode-by-mode is necessary. This step is achieved by subtracting from the phase shift of the whole system with two plates, the corresponding phase shifts associated to a reduced problem with only one delta plate:

$$\tilde{\delta}(k) = \delta_{v_0w_0,v_1w_1}(k) - \delta_{v_0w_0}(k) - \delta_{v_1w_1}(k). \tag{12}$$

It is worth highlighting that this last equation constitutes a subtraction mode by mode of the spectrum to complete the renormalization. This method is different from the frequently used one of setting a cutoff in the integral over modes to remove the high energetic part of the spectrum that does not feel the background. The phase shift (12) has to be used in the DHN formula (11) to obtain a finite result.

Nevertheless, instead of using the aforementioned approach with the derivative of the phase shifts acting as the density of states, the *TGTG*-formula will be used. The results will be the same using either of these two procedures but with the *TGTG* representation we do not have to work with the scattering of the complete problem but with that of the reduced problem of a single object in the background. Hence, the numerical computational effort is much lower.

In the seminal paper [5], O. Kenneth and I. Klich give the following formula for the quantum vacuum interaction energy between two compact bodies 1, 2 in one spatial dimensional flat spacetime:

$$E_0 = -i\int_0^\infty \frac{d\omega}{2\pi}\,\mathrm{tr}\log(1 - T_1G_{12}T_2G_{21}). \tag{13}$$

Notice that the case the authors considered does not present bound states with negative energy in the spectrum of the corresponding Schrödinger operator in quantum mechanics. On the contrary, these type of states must be included in the case I am considering. Furthermore, in the appendices B and C of [5] the authors prove that for any pair of disjoint finite bodies separated by a finite distance and any Green's function that is finite away from the diagonal, the $TGTG$-operator is trace-class. The modulus of its eigenvalues is less than one and $\log(1-TGTG)$ is well defined. A similar reasoning can be followed here for the system of a pair of two-dimensional plates, that are assumed not to touch, in the curved background of a kink. Thus, the only step left to be taken is to calculate the T-operators for each one of the plates.

From the well-known Lippmann-Schwinger equation

$$\Delta G_k(z_1,z_2) = -\int dz_3\, dz_4\, G_k^{PT}(z_1,z_3)T_k(z_3,z_4)G_k^{PT}(z_4,z_2),$$

and $\tilde{K}_{z_1} G_k^{PT}(z_1-z_2)=\delta(z_1-z_2)$, it is easy to see that

$$-\tilde{K}_{z_2}\,\tilde{K}_{z_1}\,\Delta G_k(z_1,z_2) = \int dz_3\, dz_4\, \delta(z_1-z_3)T_k(z_3,z_4)\delta(z_4-z_2),$$

where $\tilde{K}_z = \hat{K}_{PT}(z)-k^2$. Notice that in the above formula, $\Delta G_k(z_1,z_2)$ corresponds to the Green's function of only one plate in the Pösch-Teller potential, i.e. the one given in (A2) with the coefficients of one of the plates equal to zero. Due to the absolute values contained in G_k^{PT}, to obtain the transfer matrix $T_k(z_1,z_2)$ corresponding to one plate, the only non-trivial contribution comes from the case in which one point is on the left and the other one on the right of the plate. Hence, since in that case the Green's function is given by $\Delta G_k(z_1,z_2)=(t-1)\,G_k^{PT}(z_1,z_2)$, one needs to compute:

$$T_k(z_1,z_2) = -(t-1)\tilde{K}_{z_2}\,\tilde{K}_{z_1}\,G_k^{PT}(z_1,z_2).$$

In order to obtain the transfer matrix associated to the plate on the right[8], one assumes the plate sitting at the origin ($b=0$) for simplicity and hence, one of the coordinates z_1, z_2 will be greater than zero and the other less than zero. Taking into account that

$$\begin{aligned}
&e^{ik|z_1-z_2|} = e^{ik(|z_1|+|z_2|)},\\
&|\tanh z_1 - \tanh z_2| = \tanh|z_1| + \tanh|z_2|,\\
&\tanh z_1 \tanh z_2 = -\tanh|z_1|\tanh|z_2|,
\end{aligned}$$

(both in the cases $z_1<0, z_2>0$ and $z_1>0, z_2<0$), it is possible to rewrite the free Green's function in the background of the kink as

$$G_k^{PT}(z_1,z_2) = -\frac{1}{W} f_k(|z_1|) f_k(|z_2|).$$

Because the Green's differential equation

$$\left(-\partial_{|z|}^2 - k^2 - 2\,\mathrm{sech}^2\, z\right) f_k(|z|) = 0$$

holds, and using the formulas for the derivatives of functions depending on absolute values:

$$\begin{aligned}
\frac{df_k(|z|)}{dz} &= f_k'(|z|)\,\mathrm{sign}\, z,\\
\frac{d^2 f_k(|z|)}{dz^2} &= f_k''(|z|) + f_k'(|z|)\,2\,\delta(z),
\end{aligned}$$

the transfer matrix for the right plate can be written as

$$\begin{aligned}
T_k(z_1,z_2) &= \frac{t-1}{W} 4\,\delta(z_1)\,\delta(z_2)\, f_k'(|z_1|) f_k'(|z_2|)\\
&= -4\,\delta(z_1)\,\delta(z_2)\,\Delta G(z_1,z_2)\,\frac{f_k'(|z_1|)}{f_k(|z_1|)}\,\frac{f_k'(|z_2|)}{f_k(|z_2|)}\\
&= \frac{|W|^2}{k^4}\delta(z_1)\delta(z_2)\Delta G_k(z_1,z_2).
\end{aligned}$$

The Green's function for one delta plate is defined as the following piece-wise function:

$$\Delta G_k(z_1,z_2) = \begin{cases} \dfrac{r_L}{W} f_k(z_1) f_k(z_2), & \text{if } z_1, z_2 > b,\\ \dfrac{r_R}{W} f_{-k}(z_1) f_{-k}(z_2), & \text{if } z_1, z_2 < b,\\ \dfrac{t-1}{W} f_k(z_>) f_{-k}(z_<), & \text{otherwise,} \end{cases} \quad (14)$$

being the scattering data given in (A1) but for the case $v_0 = w_0 = 0$. Consequently,

$$T_k(z_1,z_2) = -\frac{W^*}{k^2}\delta(z_1)\delta(z_2)\begin{cases} r_L(b=0),\\ r_R(b=0),\\ 1-t(b=0).\end{cases}$$

Asterisk means complex conjugate. The Lippmann-Schwinger operator is related to the scattering matrix by $S = 1 - i\delta(\omega-\omega')T$. This implies normalizing T so that the factor k^{-2} cancels out. When the delta potential which mimics the plate is evaluated at another point different from the origin, the just-computed result for $T_k(z_1,z_2)$ is valid once after performing the translation $z_1 \mapsto z_1 - b$ and $z_2 \mapsto z_2 - b$. Notice that in the definition of T at a point different to $z=0$, translation must be understood as replacing the scattering coefficients at $z=0$ contained in its definition by the ones at $z=b$. Due to the PT potential, the isotropy of the spacetime is broken and $r_{R,L}(b) \neq r_{R,L}(0)e^{ikb}$, so the translation $z_1 \mapsto z_1 - b$ and $z_2 \mapsto z_2 - b$ aforementioned must not be interpreted in this usual sense. The T operator for the right-hand side plate is thus given by

$$T(z_1,z_2) = -W^*\delta(z_1-b)\delta(z_2-b)\begin{cases} r_L(b), & z_1,z_2 \to b^+,\\ r_R(b), & z_1,z_2\to b^-,\\ 1-t(b), & \text{otherwise,}\end{cases} \quad (15)$$

and analogously for the plate located at $z=a$.

The Green's function or correlator represents the probability transition amplitude for a particle to propagate from one point to another while moving freely in the

[8] For computing the transfer matrix of the left-hand side plate, one sets $v_1 = w_1 = 0$ in the transmission coefficient (A1) involved in the Green's function (A2) and considers the case in which the left plate is centered at $a=0$.

background spacetime ($G^{PT}(z_1, z_2)$ term), or while interacting with different potentials ($\Delta G(z_1, z_2)$ contribution). Besides that, the T-matrix is the probability amplitude for a particle to interact with the potential but without propagation. Hence, in the system of two plates mimicked by punctual delta potentials in the background of a kink, the definition of the T-operator must depend on ΔG evaluated at the point at which the delta potential is centered, as it is the case in (15). It could not depend on an arbitrary point of the spacetime for the causality not to be violated. Notice that in more general cases, when the potential representing each object is not supported at a point but a compact interval, then T is local. Although in this case T would depend on the points that constitute the support of the potential, it does not violate causality because it does not depend on arbitrary points. When the particle interacts with a potential of compact support, $\Delta G(z_1, z_2)$ includes the probability of interaction with the potential (T-contribution) as well as the propagation of the particle within the support of the potential.

By definition $G^{PT}(z_1, z_2) = \langle \mathcal{T}\phi(z_1)\phi(z_2)\rangle$, expression in which the time-ordering operator product has been considered. All eigenstates of $\hat{K}_{PT}$ with fixed energy k^2 can be described in terms of the orthonormal basis of left and right Pöschl-Teller free waves. By labeling $R = f_k(z)$ and $L = f_{-k}(z)$, the Green's function or propagator can be written in this basis as

$$G^{PT}(a, b) = \frac{1}{W} \left| L(a) \right\rangle \left\langle L(b) \right| ,$$
$$G^{PT}(b, a) = \frac{1}{W} \left| R(b) \right\rangle \left\langle R(a) \right| ,$$

being $a < b$ and the trace of the $TGTG$-operator behaves as:

$$\begin{aligned} \mathrm{tr}\, T^{\ell} G^{PT} T^{r} G^{PT} &= \langle R(a)| T^{\ell} |L(a)\rangle \langle L(b)| T^{r} |R(b)\rangle \\ &= r_L^{\ell}(v_0, w_0, a, k)\, r_R^{r}(v_1, w_1, b, k). \end{aligned} \quad (16)$$

It has been taken into account that $T\,|R\rangle = |L\rangle$ and vice versa. So, it is clear that the $TGTG$-formula will involve the reflection coefficients, which depend explicitly on the position of each plate. The above formula is the only product of the T-matrix components that allows coincidences of the z_1, z_2 points in $[a, b]$ and contributes to the quantum vacuum interaction energy between plates. Since the modulus of the eigenvalues of the $TGTG$ operator is less than one, it is possible to use

$$\begin{aligned} &\mathrm{tr}\log(1 - TGTG) = \log\, \det(1 - TGTG) \\ &\approx \log(1 - \mathrm{tr}\ TGTG) \end{aligned} \quad (17)$$

as a good approximation up to first order to simplify (13). The demonstration is collected in Appendix B. In summary, replacing (16) and (17) in (13) and generalizing it to three dimensions leads the final expression (18), with $m^2 = |E_{min}|$.

There are some details that are worth highlighting. Firstly, if $f_{\pm}(z)$ were replaced by the usual plain waves, Kenneth and Klich's original $TGTG$-formula would be restored. The reason is that in flat isotropic spacetimes the scattering coefficients for plates placed at another point different from the origin are equal to the ones at $z = 0$ times an exponential factor that accounts for the translation that has taken place:

$$r_L^{\ell}(a) = r_L^{\ell}(0)\, e^{-2iak} = r_L^{\ell}(0)\, W\, G_{-k}^{(0)}(-a, a),$$
$$r_R^{r}(b) = r_R^{r}(0)\, e^{2ibk} = r_R^{r}(0)\, W\, G_{k}^{(0)}(-b, b).$$

On the contrary, the main difference when working with weak and transparent curved spacetimes that breaks the isotropy of the space is that this rule no longer applies. Thus, the scattering coefficients for plates placed at another point different form the origin are equal to the ones at $z = 0$ times the quotient between the transmitted probability amplitude at the generic point and the one at $z = 0$, and multiplied by another function related to the configuration of the space:

$$r_L^{\ell}(a) = r_L^{\ell}(0)\, \frac{G_{-k}^{PT}(-a, a)\, t(a)\, h^{+}(\alpha_0, \beta_0, -a)}{G_{-k}^{PT}(0, 0)\, t(0)\, h^{+}(\alpha_0, \beta_0, 0)},$$
$$r_R^{r}(b) = r_R^{r}(0)\, \frac{G_{k}^{PT}(-b, b)\, t(b)\, h^{-}(\alpha_1, \beta_1, b)}{G_{k}^{PT}(0, 0)\, t(0)\, h^{-}(\alpha_1, \beta_1, 0)},$$

being $h^{\pm}(\alpha_i, \beta_i, x) = \alpha_i\beta_i \pm (\alpha_i^2 - 1) f'(k, x)/f(k, x)$. This fact is crucial to generalize the $TGTG$-formula in the case studied.

$$\begin{aligned} \frac{E_0}{A} &= -\frac{1}{2}\sum_{j=1}^{N} \frac{\left(\sqrt{-\kappa_j^2 + m^2}\right)^3}{6\pi} + \frac{1}{8\pi^2}\int_m^{\infty} d\xi\, \xi\, \sqrt{\xi^2 - m^2}\, \log(1 - \mathrm{Tr}\, TGTG_{\xi}), \\ &= -\frac{1}{2}\sum_{j=1}^{N} \frac{\left(\sqrt{-\kappa_j^2 + m^2}\right)^3}{6\pi} + \frac{1}{8\pi^2}\int_m^{\infty} d\xi\, \xi\, \sqrt{\xi^2 - m^2}\, \log\!\big[1 - r_L^{\ell}(v_0, w_0, a, i\xi)\, r_R^{r}(v_1, w_1, b, i\xi)\big]. \end{aligned} \quad (18)$$

Secondly, by defining the potential $V_i(z)$ to describe each of the two plates as $V_i(z) = v_i\, \delta(z - z_i) + w_i \delta'(z - z_i)$ (with

$i = 0, 1$ and $z_0 = a, z_1 = b)$, the Schrödinger operators $\hat{K}_i = -\partial_z^2 - V_i(z)$ are defined over a Hilbert space that, in general, is not isomorphic to that of $\hat{K}_{PT}$. Hence, G^{PT} and $T^{r,\ell}$ do not act in the same spaces and $T^{\ell} G^{PT} T^r G^{PT}$ is ill-defined. To avoid this problem, a Wick rotation of the momentum k must be performed in order for all the operators to act in the same Hilbert space. The integral (18) is thus convergent and can be evaluated numerically with *Mathematica*. In the next section, the results of the Casimir energy for some configurations of the plates in the PT background potential are going to be discussed.

VI. CASIMIR PRESSURE

Once the quantum vacuum interaction energy is determined, one can study the Casimir force between plates as

$$F = -\frac{\partial E_0}{\partial d}.$$

being d the distance between plates. Nevertheless, as already explained, the translational invariance is broken due to the PT background, which means that the scattering data for the plates explicitly depend on the position in a non-trivial way. Hence, when computing the Casimir force, a non-trivial contribution coming from the derivatives of the scattering amplitudes of one of the plates with respect to the position will appear. There is an ambiguity yet not clarified in this calculation. One can either introduce the dependence on the distance between plates in three different ways:

1. Putting the left-hand side plate at $z_1 = a$ and the right-hand side one at $z_2 = a + d$. In this case, only the scattering data of the plate on the right will depend on the distance d, and only the derivative of $r_R^r(v_1, w_1, a + d, k)$ with respect to d will appear.

2. Considering the right-hand side plate placed at $z_2 = b$ and the left-hand side one at $z_1 = b - d$. Analogously to the previous case, the derivative of $r_L^{\ell}(v_0, w_0, b - d, k)$ with respect to the distance is the only possible contribution.

3. When one of the plates is to the left of the origin and the other one to the right, one could describe the location of the plates as the left one being at $z_1 = -d + b$ and the other one at $z_2 = d + a$, with $a < 0$ and $b, d > 0$. This case is different because the derivatives of the reflection coefficients of both plates $r_R^r(v_1, d + a, k)$ and $r_L^{\ell}(v_0, -d + b, k)$ will be taken into account.

It is work in progress to check that these three situations give rise to the same force. However, it seems reasonable to think that if the Casimir energy between plates has a change in the sign for some values of the parameters $\{v_0, v_1, w_0, w_1, a, b\}$, the Casimir force will present it too. Consequently, studying numerical results for the quantum vacuum interaction energy is enough to discuss whether this flip of sign appears as a consequence of the introduction of the δ' potential, as was the case in other configurations in flat spacetimes [10, 49].

Figures 2 and 3 show the quantum vacuum interaction energy per unit area of the plates for different configurations of the system of two plates in the PT background.

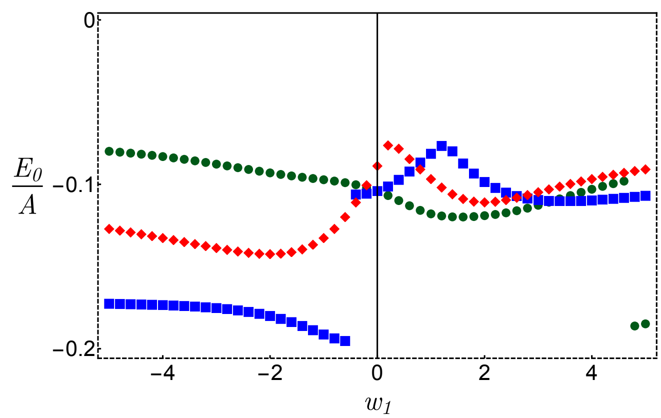

FIG. 2. Casimir energy per unit area between plates situated at the points $a = -0.2, b = 0.8$, as a function of the coefficient w_1. Different configurations are shown: a) Pure δ' plates (i.e. $v_0 = v_1 = 0$) with $w_0 = 3$ (rhombi), b) Identical plates characterized by $v_0 = v_1 = 1$ and $w_0 = w_1$ (circles), c) Opposite plates described by $v_0 = v_1 = 1$ and $w_0 = -w_1$ (squares).

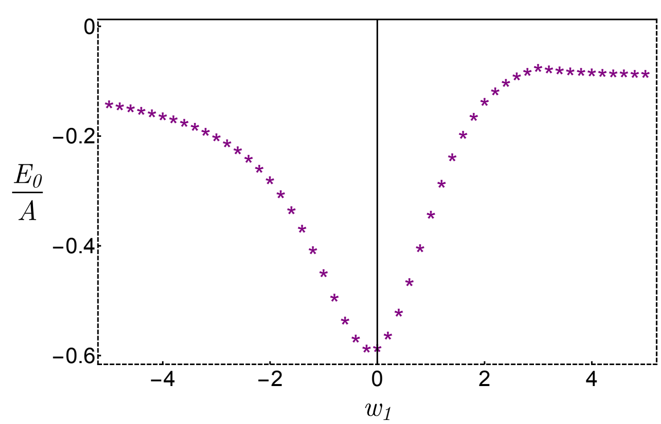

FIG. 3. Casimir energy per unit area between plates situated at the points $a = -0.2, b = 0.8$, as a function of the coefficient w_1. A generic $\delta\delta'$ potential with $v_0 = 1, v_1 = -4, w_0 = 2.5$ has been considered.

In [40], it was shown that for pure delta plates (i.e. $w_0 = w_1 = 0$) the energy is always negative independently on the value of the delta coefficients. Furthermore it could be checked numerically that the quantum vacuum interaction energy is definite negative regardless of the relative position of the plates with respect to the kink center as well. However, due to the changes of the spectrum of bound states as a function of $\{v_0, v_1\}$ and the relative position between the plates and the kink, there is a peculiar fact characteristic of the spectra shown in the figure in [40]. The sudden discontinuities present in the plot are related to the loss of one bound state with very low $k = i\kappa$ (nearly zero) in the spectrum of the system. At this point it is necessary to realize that the whole system of the two plates together with the PT potential acts as a well with a fixed depth and width. Consequently, at the configuration in the space of parameters at which the jump appears, the resulting well is not deep enough to hold more bound states with large negative energy. This loss of a bound state translates into a jump in the energy for the pure delta plates problem. Here, as can be seen in FIG. 2 and 3, the energy is also negative independently of the value of the coefficients of the $\delta\delta'$ potentials. This implies that the Casimir force between plates will always be attractive in this system too. In general terms, it can be also seen that the larger the magnitude of the delta coefficients $\{v_0, v_1\}$, the larger the one of the quantum vacuum interaction energy providing $w > -2$. The jump discontinuities in the energy appear for the $\delta\delta'$ plates case whenever $v_i > 0$. It can be checked that for the cases of identical and opposite plates, if $v_i < 0$, the results would be qualitatively similar to those shown in FIG. 3. Consequently, the introduction of the δ' potential modifies the spectrum in such a way that the well represented by the system can better accommodate bound states if $v_i < 0$. In these cases, for different configurations of the coefficients of the $\delta\delta'$ functions very close to each other, no bound states with momenta close to zero are lost.

Another important conclusion can be drawn. For flat spacetimes, it has been proved that the introduction of the δ' potential causes the sign of the force to change in different areas of the parameter space. This behaviour has been observed for instance for a scalar field and two concentric spheres defined by such a singular $\delta\delta'$ potential on their surfaces [49] or in [50] for $\delta\delta'$ plates. However, curiously, when considering this last configuration in a curved spacetime, the change of sign in the energy disappears. Namely, if one considers a curved spacetime that is confining (in the sense that the background acts as a well) and the plates are within the support of that well, then even in the case where the plates act as very repulsive barriers, there are still negative energy states in the spectrum of the associated Schrödinger operator and the Casimir energy between plates will be attractive. Consequently, although the background potential under consideration constitutes an example of weak curved background, the results are quite different from the flat case. When the Pöschl-Teller potential is not confined at all between the plates and they are far from the kink centre, the system of two $\delta\delta'$-plates in flat spacetime is recovered. This is the reason that the numerical representation for this situation, which has already been studied in the literature, is not included here. However, it is available in [50].

Finally, is worth pointing that even in the case where there is only one plate in the system, the other plate feels the interaction because there is still a non-zero quantum vacuum interaction energy in the system. This can be checked by looking at the non-zero values of the energy appearing in the vertical axis for the pure δ' plates (in this axis, $w_1 = 0$ and there is no right plate in the system) in Figure 2.

VII. CONCLUSIONS

In this work, a quantum scalar field between two parallel two-dimensional plates in a curved background at zero temperature is presented. The main result is the generalization of the $TGTG$-formula for weak and transparent gravitational backgrounds in which the frequencies of the particles created by the gravitational background are much smaller than the Planck frequency, and the fields could be asymptotically interpreted as particles. The quantum vacuum interaction energy has been calculated using this formula, which only depends on the reflection coefficients associated to the scattering problem. They involve a dependence on the analogous of the plane waves in flat spacetimes but for the specific curved background potential chosen. The Casimir energy thus depends on the parameters describing the potentials and on the distance from the plates to the center of the kink. For obtaining the energy, it has been necessary to compute the Green's functions from the scattering data. The transfer matrix has been determined with complete generality too, in terms of the Green's function. Although the $TGTG$-formula has the advantage that it depends only on the scattering data of one of the plates and it is not necessary to solve the scattering problem of the whole system, the well-known DHN formula has also been derived.

As an example, two plates mimicked by Dirac δ-potentials in a curved background of a topological Pöschl-Teller kink is studied. The quantum vacuum fluctuations around the kink solution could be interpreted as mesons propagating in the spacetime of a domain wall. One of the relevant characteristics of the configuration of the open system of two plates in a Pöschl-Teller background is that the translational symmetry is broken and the space is anisotropic. This translates into the fact that the scattering coefficients, as well as the Green's function, will depend on the position of the plates in a non-trivial way. The wave functions of the continuous spectrum of states with positive energy have been characterized by means of the scattering data. The bound states have also been studied, setting a threshold for the minimum negative energy in the system. The unitarity of the QFT requires this lower bound be fixed as the mass of the quantum vacuum fluctuations so that the total energy of the lowest energy state of the spectrum will be zero, making fluctuations absorption impossible. It is worth highlighting that the quantum vacuum energy for this (3+1)-dimensional problem is only negative, independently on the value of the coefficients of the $\delta\delta'$ potentials and its location in relation to the kink center. This implies that the Casimir force between plates will be attractive in this system. Furthermore, even in the case where there is only one plate in the system, the other plate feels the Casimir interaction because there is still a non-zero quantum vacuum interaction energy in the system.

Once the Casimir energy between plates in the curved background of a sine-Gordon kink has been computed by using the $TGTG$-formula, it would be enlightening to obtain the same result but from the integration of the 00-component of the energy-momentum tensor. In this way one could also study the spatial distribution of the energy density. It is left for further investigations. It is worth noticing that the virtue of the method presented here for obtaining a $TGTG$-formula valid in weak curved spacetimes is that it can be easily generalized to other type of configurations, either for another background and for other potentials that could properly mimic the plates.

VIII. ACKNOWLEDGMENTS

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Appendix A: Scattering data and Green's function for two $\delta\delta'$ plates in a PT background

The scattering data for the non-relativistic mechanical problem of scalar fields propagating in the curved background of a topological PT kink while interacting with two Dirac $\delta\delta'$ plates are given in (A1). The notations $\mathcal{A}_i(z) = -\beta_i\, f_k(z) + \alpha_i\, f'_k(z)$ together with $W = -2ik(k^2+1)$ have been used to simplify the expressions.

Notice that the denominator of all the scattering parameters, $\Upsilon(k)$, is the Jost function. Its zeroes on the positive imaginary axis of the complex momentum plane characterize the wave vector of the bound states in the quantum mechanical problem.

$$
\begin{aligned}
t(k) &= \frac{1}{\Upsilon(k)}\alpha_0\alpha_1 W^2,\\
r_R(k) &= \frac{1}{\Upsilon(k)}[-f_k(b)f_k(a)\left(f'_{-k}(b)\left(f'_k(a)-\alpha_0\mathcal{A}_0(a)\right)+\alpha_0^2 f'_{-k}(a)\left(-\alpha_1\mathcal{A}_1(b)+f'_k(b)\right)\right)\\
&\quad+\alpha_1\mathcal{A}_1(b)f_k(a)f_{-k}(b)\left(f'_k(a)-\alpha_0\mathcal{A}_0(a)\right)-f_k(b)f_{-k}(a)\left(f'_k(a)+\alpha_0\beta_0 f_k(a)\right)\left(\alpha_1\mathcal{A}_1(b)-f'_k(b)\right)],\\
r_L(k) &= -\frac{1}{\Upsilon(k)}[f_{-k}(b)f_k(a)f'_k(a)(-\alpha_1\mathcal{A}_1^*(b)+f'_{-k}(b))+\alpha_0\beta_0 f_{-k}^2(a)(-f_{-k}(b)f'_k(b)+\alpha_1 f_k(b)\mathcal{A}_1^*(b))\\
&\quad+f_{-k}(a)[f_{-k}(b)\alpha_0\mathcal{A}_0(a)(\alpha_1\mathcal{A}_1^*(b)-f'_{-k}(b))+(-1+\alpha_0^2)f'_{-k}(a)(-\alpha_1 f_k(b)\mathcal{A}_1^*(b)+f'_k(b)f_{-k}(b))]],\\
B_R(k) &= \frac{1}{\Upsilon(k)}[\alpha_0 W\left(f_k(b)f'_{-k}(b)-\alpha_1\mathcal{A}_1(b)f_{-k}(b)\right)],\\
B_L(k) &= -\frac{1}{\Upsilon(k)}[\alpha_1 W f_{-k}(a)\left(f'_{-k}(a)-\alpha_0\mathcal{A}_0^*(a)\right)],\\
C_R(k) &= -\frac{1}{\Upsilon(k)}[\alpha_0 W f_k(b)\left(f'_k(b)-\alpha_1\mathcal{A}_1(b)\right)],\\
C_L(k) &= -\frac{1}{\Upsilon(k)}[\alpha_1 W\left(\alpha_0\mathcal{A}_0(a)f_{-k}(a)-f_k(a)f'_{-k}(a)\right)],\\
\Upsilon(k) &= -\left(\alpha_0\mathcal{A}_0^*(a)-f'_{-k}(a)\right)\left[f_k(b)\left(f_{-k}(a)\left(\alpha_1\mathcal{A}_1(b)-f'_k(b)\right)+f_k(a)f'_{-k}(b)\right)-\alpha_1\mathcal{A}_1(b)f_k(a)f_{-k}(b)\right]\\
&\quad+\alpha_0^2\, W\left(f_k(b)f'_{-k}(b)-\alpha_1\mathcal{A}_1(b)f_{-k}(b)\right).
\end{aligned}
\tag{A1}
$$

The Green's function of the associated QFT can be written as $G_k(z_1,z_2) = G_k^{PT}(z_1,z_2) + \Delta G_k(z_1,z_2)$ with $\Delta G_k(z_1,z_2)$ given by (A2). The scattering data involved are collected in (A1). The points $\{a,b\}$ at which the plates are located are completely general and can be replaced by any other pair of points, independently of their position with respect to the origin, around which the PT kink is centered.

$$\Delta G_k(z_1,z_2)=\begin{cases}\frac{r_L}{W}f_k(z_1)f_k(z_2), & \text{if } z_1,z_2>b,\\ \frac{r_R}{W}f_{-k}(z_1)f_{-k}(z_2), & \text{if } z_1,z_2<a,\\ \frac{B_RB_L}{tW}f_k(z_1)f_k(z_2)+\frac{C_RC_L}{tW}f_{-k}(z_1)f_{-k}(z_2) & \text{if } a<z_1<b \text{ and } a<z_2<b,\\ +\frac{C_RB_L}{tW}(f_{-k}(z_>)f_k(z_<)+f_k(z_>)f_{-k}(z_<)), & \\ (t-1)\,G_k^{PT}(z_1,z_2), & \text{if } z_2>b \text{ and } z_1<a\,(\text{or } z_1\leftrightarrow z_2),\\ (C_L-1)\,G_k^{PT}(z_1,z_2)+\frac{B_L}{W}f_k(z_1)f_k(z_2), & \text{if } z_2>b \text{ and } a<z_1<b\,(\text{or } z_1\leftrightarrow z_2),\\ (B_R-1)\,G_k^{PT}(z_1,z_2)+\frac{C_R}{W}f_{-k}(z_1)f_{-k}(z_2), & \text{if } z_2<a \text{ and } a<z_1<b\,(\text{or } z_1\leftrightarrow z_2).\end{cases}\tag{A2}$$

Appendix B: Proof of equation (17)

In Appendices B and C of [5] the authors prove that if the Green's function $G(x,y)$ is smooth for $x\neq y$, then for any two disjoint objects A,B separated by a finite distance, G^{AB} is a trace-class operator. Moreover, T^A and T^B are bounded and $T^AG^{AB}T^BG^{BA}$ is a trace-class operator. They also prove that the modulus of the eigenvalues of $T^AG^{AB}T^BG^{BA}$ is less than one. The same reasoning holds here for the weak and transparent curved background considered.

The aim of this appendix is to demonstrate that

$$\begin{aligned}&\mathrm{tr}\log(1-TGTG)=\log\,\det(1-TGTG)\\&\approx\log(1-\mathrm{tr}\;TGTG),\end{aligned}\tag{B1}$$

is satisfied whenever there are two separate objects in a weak curved background as the one discussed in this paper. The above first equality can be proven by taking into account that any Hermitian matrix P representing an Hermitian operator can be transformed into a diagonal matrix, so that $P_D=QPQ^{-1}$. In this way:

$$\begin{aligned}&e^{\mathrm{tr}\log\,P}=e^{\mathrm{tr}\log\,(Q^{-1}P_DQ)}=e^{\mathrm{tr}\,[Q^{-1}(\log P_D)Q]}\\&=e^{\mathrm{tr}\log\,P_D}=e^{\sum_i\log\lambda_i}=\prod_i\lambda_i=\det P_D\\&=\det\,(QPQ^{-1})=\det P.\end{aligned}$$

The cyclic property of the trace has been used above. The eigenvalues of the diagonal matrix P_D have been denoted by λ_i. On the other hand, denoting by α_i the eigenvalues of $M=TGTG$, it is easy to prove the second claim in (B1) because

$$\begin{aligned}&\log\det\,(1-M)=\log\det\,[Q_1^{-1}(1-M_D)Q_1]\\&=\log\prod_i(1-\alpha_i)=\log[1-\sum_i\alpha_i+o(\alpha_i\alpha_j)]\\&\approx\log[1-\mathrm{tr}\;M_D]=\log[1-\mathrm{tr}\big(Q_1MQ_1^{-1}\big)]\\&=\log[1-\mathrm{tr}\;M].\end{aligned}$$

It constitutes a good approximation up to first order since the norm of M is less than one.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51** (1948).
[2] M. J. Sparnaay, Nature **180** (1957), 10.1038/180334b0.
[3] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, *Advances in the Casimir effect*, International Series of Monographs on Physics, Vol. 145 (Oxford University Press, Oxford, 2009).
[4] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, and H. Weigel, Nucl. Phys. B **645**, 49 (2002).
[5] O. Kenneth and I. Klich, Phys.Rev. B **78**, 014103 (2008).
[6] K. Kirsten, *Spectral Functions in Mathematics and Physics* (Chapman & Hall/CRC, New York, 2001).
[7] D. Vassilevich, Phys.Rept. **388**, 279 (2003).
[8] E. Elizalde, *Ten Physical Applications of Spectral Zeta Functions*, Lectures in Physics (Springer-Verlag Berlin, Heidelberg, 2012).
[9] M. Bordag and J. M. Muñoz-Castañeda, J. Phys. A: Math. Theor. **45**, 374012 (2012).
[10] J. M. Muñoz-Castañeda and J. M. Guilarte, Phys.Rev. D **91**, 025028 (2015).
[11] S. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime*, London Mathematical Society Student Texts (Cambridge University Press, Cambridge, 1989).
[12] N. Birrel and P. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 1982).
[13] R. M. Wald, *Quantum field theory in curved space-time and black hole thermodynamics* (University of Chicago Press, Chigaco, 1995).
[14] C. J. Fewster, *Lectures on quantum field theory in curved spacetime*, Tech. Rep. Lecture Note 39 (Max Planck Institute for Mathematics in the Natural Sciences, Leipzig, 2008).
[15] I. Khavkine and V. Moretti, "Algebraic QFT in Curved Spacetime and Quasifree Hadamard States: An Introduction," in *Advances in Algebraic Quantum Field The-*

ory, edited by R. Brunetti, C. Dappiaggi, K. Fredenhagen, and J. Yngvason (Springer International Publishing, Cham, 2015) pp. 191–251.

[16] J. M. Muñoz-Castañeda and M. Bordag, Phys. Rev. D **89**, 065034 (2014).

[17] C. Bär and K. Fredenhagen, *Quantum Field Theory on Curved Spacetimes: Concepts and Mathematical Foundations*, Lectures Notes in Physics No. 786 (Springer-Verlag Berlin, Heidelberg, 2009).

[18] B. Kay, in *Encyclopedia of Mathematical Physics*, edited by J.-P. Françoise, G. L. Naber, and T. S. Tsun (Academic Press, Oxford, 2006) pp. 202–212.

[19] J. M. Cerveró and A. Rodríguez, Eur. Phys. J. B **30**, 239 (2002).

[20] M. Bordag, J. Phys. A: Math. Theor. **53**, 015003 (2020).

[21] I. Alvarado-Rodríguez, P. Halevi, and J. J. Sánchez-Mondragón, Phys. Rev. E **59**, 3624 (1999).

[22] M. Bordag, D. Hennig, and D. Robaschik, J. Phys. A **25**, 4483 (1992).

[23] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. D **80**, 085004 (2009).

[24] G. Barton, J. Phys. A **37**, 1011 (2004).

[25] P. Parashar, K. A. Milton, K. V. Shajesh, and M. Schaden, Phys. Rev. D **86**, 085021 (2012).

[26] M. Bordag, Phys. Rev. D **89**, 125015 (2014).

[27] J. J. Álvarez, M. Gadella, L. Lara, and F. H. Maldonado-Villamizar, Phys. Lett. A **377**, 2510 (2013).

[28] F. M. Toyama and Y. Nogami, J. Phys. A: Math. Theor. **40**, F685 (2007).

[29] M. Gadella, J. Negro, and L. M. Nieto, Phys. Lett. A **373**, 1310 (2009).

[30] T. Vachaspati, *Kinks and Domain Walls: An Introduction to Classical and Quantum Solitons* (Cambridge University Press, Cambridge, 2006).

[31] N. Manton and P. Sutcliffe, *Topological Solitons*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2004).

[32] J. M. Guilarte and J. M. Muñoz-Castañeda, Int. J. Theor. Phys. **50**, 2227 (2011).

[33] I. Cavero-Peláez and J. Guilarte, in *Quantum Field Theory Under the Influence of External Conditions (QFEXT09)*, edited by K. A. Milton and M. Bordag (World Scientific, Singapore, 2010) pp. 457–464.

[34] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).

[35] M. Asorey, A. Ibort, and G. Marmo, Int. J. Mod. Phys. A **20**, 1001 (2005).

[36] M. Bordag, J. M. Muñoz-Castañeda, and L. Santamaría-Sanz, Mod. Phys. Lett. A **35**, 2040018 (2020).

[37] J. Muñoz-Castañeda, L. Santamaría-Sanz, M. Donaire, and M. Tello-Fraile, Eur. Phys. J. C **80**, 793 (2020).

[38] S. Weinberg, *The Quantum Theory of Fields. Vol. 2. Modern Applications* (Cambridge University Press, Cambridge, 1995).

[39] L. S. Brown, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1992).

[40] L. Santamaría-Sanz, "Casimir effect and $TGTG$-formula for curved backgrounds". Letter to be published (2023).

[41] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4130 (1974).

[42] W. H. Press, B. S. Ryden, and D. N. Spergel, Astrophys. J. **347**, 590 (1989).

[43] M. Santander, L. M. Nieto, and N. A. Cordero, Am. J. Phys. **65**, 1200 (1997).

[44] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity* (Academic Press, London, 1983).

[45] R. P. Feynman, F. Morinigo, and W. Wagner, *Lectures on Gravitation* (Addison Wesley, Reading, Mass., 1995).

[46] P. Kurasov, J. Math. Anal. Appl. **201**, 297 (1996).

[47] M. Asorey and J. M. Muñoz-Castañeda, Nucl. Phys. B **874**, 852 (2013).

[48] M. Bordag, J. Muñoz-Castañeda, and L. Santamaría-Sanz, Front. Phys. **7** (2019), 10.3389/fphy.2019.00038.

[49] I. Cavero-Peláez, J. M. Muñoz-Castañeda, and C. Romaniega, Phys. Rev. D **103**, 045005 (2021).

[50] J. M. Muñoz-Castañeda, J. M. Guilarte, and A. M. Mosquera, Phys. Rev. D **87**, 105020 (2013).