Commun.Fac.Sci.Univ.Ank.Ser. A2-A3
Volume 00, Number 0, Pages 000–000
ISSN1303-6009

AVERAGES OF OBSERVABLES ON GAMOW STATES.

Manuel GADELLA[1] Carlos SAN MILLAN[1]

[1]Departamento de Física Teórica, Atómica y Optica and IMUVA, Universidad de Valladolid, Valladolid, SPAIN

Abstract. We propose a formulation of Gamow states, which is the part of unstable quantum states that decays exponentially, with two advantages in relation with the usual formulation of the same concept using Gamow vectors. The first advantage is that this formulation shows that Gamow states cannot be pure states, so that they may have a non-zero entropy. The second is the possibility of correctly defining averages of observables on Gamow states.

1. INTRODUCTION

Textbooks in Quantum Mechanics mostly deal with the so called bound states, which are described by eigenvectors (in explicit models, eigenfunctions) of a given Hamiltonian, H. Bound states are invariant under the time evolution governed by H. Thus, bound states are stable states, unless that are exposed to interactions with new external forces.

However, a substantial amount of quantum systems really existing in Nature are unstable. The range of unstable quantum systems include excited atoms, nuclei and particularly elementary particles. Needless to say that a Quantum Theory for unstable quantum states has been developed [1–5], although it may be considered as incomplete for various reasons, some of them will be the object of our study in the present paper.

Unstable quantum systems have historically received different names: quasi-stable states, meta-stable states, scattering resonances, etc, which have received different definitions, but that all have been proven to be the same. They have a common feature: they have an experimental exponential decay at *most* observable times. We have underlined the word *most*, since the ranges of time for which this decay is not exponential are very difficult to observe in practice. These deviations that have been foreseen by the general theory [5, 6] have been observed for very short [7] or very high times [8]. Nevertheless, the observed exponential decay for

Keywords. Quantum resonances, Gamow states, Algebraic formulation of states and observables.

Corresponding author; Manuel Gadella .

most times is not exact due to noise [5]. Thus, exponential decay may be considered as a very good approximation for the behaviour of quantum unstable systems for most purposes.

Generally speaking, quantum unstable states, henceforth *resonances*, are produced in resonance scattering [1] due to the presence of some external forces, which are usually given by a potential V. A particle, otherwise looked as free, evolves under the free dynamics given by H_0 up to it enters on the region where the interaction of V takes place. Then, the particle finally escapes from this interaction. If this particle stays in the interaction region a much higher time it would have stayed should this interaction not exist, we say that a resonance has been produced. Thus, for the creation of a scattering resonance, we need a Hamiltonian pair, $\{H_0, H\}$, where $H = H_0 + V$, V being the potential responsible for the creation of the resonance. Typically, point potentials [10] give often resonances.

Resonances are characterized in various forms from the physical point of view, not always equivalent [1]. Possibly the most popular among the physicists is given by poles of the analytic continuation of the scattering function (or matrix) $S(k)$ in representation of momenta. These poles are located on the lower half plane and appear in pairs symmetric to the negative imaginary axis. If instead of this representation, we go to energies, resonance poles are pairs of conjugate complex numbers located on the analytic continuation of $S(E)$ through the cut given by the spectrum of the total Hamiltonian H, and these poles lie on the second sheet of the Riemann surface associated to the transformation $k = \sqrt{2mE}$. Each pair of resonance poles have the form $E_R \pm i\Gamma/2$ with $E_R, \Gamma > 0$.

Is it possible to define a vector state for a unstable quantum state as is for a bound state? The vector state, ψ, for an unstable quantum state should have the following property: its probability amplitude in terms of time, t, which is given by

$$\alpha(t) := \langle\psi|e^{-itH}|\psi\rangle \,, \tag{1}$$

where $H = H_0 + V$, must be approximately exponential for almost all times, with the exception of very short or very long times, which may be unobservable in most cases.

Due to this condition, Nakanishi [9] in 1959 proposed to define the vector state for a resonance as an eigenvector of the Hamiltonian with complex eigenvalues. For instance, if a resonance pole is located at the point $z_R = E_R - i\Gamma_R/2$, this vector state, $|\psi\rangle$, should be characterized by the property:

$$H\,|\psi^D\rangle = z_R\,|\psi^D\rangle = (E_R - i\Gamma_R/2)\,|\psi^D\rangle \,, \tag{2}$$

since then if t means time, formally:

$$e^{-itH}\,|\psi^D\rangle = e^{-itE_R}\,e^{-t\Gamma/2}\,|\psi^D\rangle \,, \tag{3}$$

so that the time evolution for $t \longmapsto \infty$ is a decaying exponential.

Some important comments:

i.) Each resonance is given by a pair of complex conjugate poles, so that along $|\psi^D\rangle$ it must exist another vector, $|\psi^G\rangle$, so that

$$H\,|\psi^G\rangle = z_R^*\,|\psi^G\rangle = (E_R + i\Gamma_R/2)\,|\psi^G\rangle \Longrightarrow e^{-itH}\,|\psi^G\rangle = e^{-itE_R}\,e^{+t\Gamma/2}\,|\psi^G\rangle\,. \quad (4)$$

Consequently, the vector $|\psi^G\rangle$ grows exponentially as $t \longmapsto \infty$. The superindices D and G stand precisely for *decaying* and *growing*, respectively. Note that the time behaviour of $|\psi^D\rangle$ and $|\psi^G\rangle$ as $t \longmapsto -\infty$ is the opposite.

ii.) This opposite time behaviour of the vectors $|\psi^D\rangle$ and $|\psi^G\rangle$ suggests that they may be defined as the time reverse of each other. This is exactly what happens in [11]. Both vectors represent equally well the state of one unique and the same resonance, given by the poles $z_R = E_R - i\Gamma_R/2$ and $z_R^* = E_R + i\Gamma/2$. The real part E_R is called the *resonance energy* and the imaginary part is proportional to the inverse of the mean life.

iii.) The vectors $|\psi^D\rangle$ and $|\psi^G\rangle$ are called the *decaying Gamow vector* and the *growing Gamow vector*. Decaying and growing always with reference to the *future* of increasing times.

iv.) Realistic vector states for resonances show deviations of exponential law for very small and very large values of time. At the same time the time interval with exponential decay always shows certain amount of noise, so that this exponential is usually not exact. Thus, if $|\psi\rangle$ is the vector state for a resonance, it must be a sum of two contributions:

$$|\psi\rangle = |\psi^D\rangle + |\psi^{\rm BACK}\rangle = |\psi^G\rangle + |\psi^{\rm BACK^*}\rangle\,, \quad (5)$$

so that we have split $|\psi\rangle$ into the sum of the Gamow vector plus the vector state of a certain *background*, which is a vector state including all possible effects so that diverts $|\psi\rangle$ from the purely exponential behaviour. The effect of the background vector is much smaller than the effect of the Gamow vector for the effectively observable time interval.

v.) This is the most important point. Since H is a self adjoint operator, how is it possible that it shows complex eigenvalues? Not as an operator on Hilbert space! We need to extend the Hilbert space as well as H to a larger space on which H may have complex eigenvalues.

This extension comes after the rigging of the Hilbert space of states, $\mathcal{H}$, with another two spaces, so as to make a triplet of spaces,

$$\Phi \subset \mathcal{H} \subset \Phi^\times\,, \quad (6)$$

where:

i.) The Hilbert space $\mathcal{H}$ is infinite dimensional, usually of the type of space of square integrable functions such as $L^2(\mathbb{R})$. If the space were finite dimensional a construction like (6) would not be possible unless the other two spaces were equal to $\mathcal{H}$.

ii.) The space Φ is a dense subspace of $\mathcal{H}$. Dense means that for any $\psi \in \mathcal{H}$ and any $\varepsilon > 0$, there exists a $\phi \in \Phi$ such that $||\phi - \psi|| < \varepsilon$, where $||-||$ is the Hilbert norm ($||\psi||^2 = \langle\psi|\psi\rangle$, where $\langle -|-\rangle$ is the scalar product on the Hilbert space). In addition, it has a topology which is strictly finer than the Hilbert topology. Finer means that it has more open sets. In general, although not always, this topology is constructed with a countably infinite set of norms [12], one of which is the Hilbert space norm. This implies in particular that the canonical injection:

$$i : \Phi \longmapsto \mathcal{H}\,, \quad i(\phi) = \phi\,, \quad \forall \phi \in \Phi\,, \tag{7}$$

is continuous.

iii.) A continuous linear functional, or in short, a functional on Φ is a mapping $F : \Phi \longmapsto \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers with its usual topology, such that

a) F is anti-linear:

$$F(\alpha\,\psi + \beta\,\varphi) = \alpha^*\,F(\psi) + \beta^*\,F(\varphi)\,, \quad \forall\,\alpha, \beta \in \mathbb{C}\,, \quad \forall\,\psi, \varphi \in \Phi\,. \tag{8}$$

where the star denotes complex conjugation.

b) F is continuous with the topologies on Φ and $\mathbb{C}$. Since we have not given details on the topology on Φ, we cannot give details on some properties of F. Let us call $\Phi^\times$ to the set of functionals on Φ. It forms a vector space over the complex field. The sum of functionals and the multiplication of functionals by complex numbers is given by

$$(\alpha\,F + \beta\,G)(\varphi) := \alpha\,F(\varphi) + \beta\,G(\varphi)\,, \quad \forall\,\alpha, \beta \in \mathbb{C}\,, \quad \forall\,F, G \in \Phi^\times\,, \tag{9}$$

so that $\Phi^\times$ is a linear space over the complex field.

We can endow $\Phi^\times$ with a topology compatible with the topology on Φ [13]. Each vector $\psi \in \mathcal{H}$ determines a unique functional F_ψ by $F_\psi(\varphi) := \langle\varphi|\psi\rangle$ (we take the anti-linear part of the scalar product at the left). By an *abus de langage*, we may identify F_ψ with ψ. With this idea in mind, the canonical mapping $i : \mathcal{H} \longmapsto \Phi^\times$ is one to one (and also continuous).

iv.) We call the structure (6) with all the above defined properties, a *rigged Hilbert space* (RHS) or a *Gelfand triplet* [14–29].

Next, let H be a self adjoint Hamiltonian with domain with domain $\mathcal{D} \subset \mathcal{H}$, dense in the (separable) infinite dimensional Hilbert space $\mathcal{H}$. Assume that we have a RHS like (6), with $\Phi \subset \mathcal{D}$ and $H\Phi \subset \Phi$. This means that for each $\varphi \in \Phi$, $H\varphi \in \Phi$. Then, H may be extended to a unique operator into $\Phi^\times$, that we also call H for simplicity in the notation, using the following *duality formula*:

$$\langle F|H\varphi\rangle = \langle HF|\varphi\rangle\,, \qquad \forall\,\varphi \in \Phi\,. \tag{10}$$

This duality formula defines HF for each $F \in \Phi^\times$. The operator H defined on $\Phi^\times$ by (10) is linear. Its restriction to Φ coincides with the original operator, as can be easily seen from (10). Moreover, if we assume that H is continuous on Φ, its extension by (10) is continuous on $\Phi^\times$ with all topology on this space compatible with the topology on Φ [13].

In [30–32], we have obtain a pair of RHS where the Gamow vectors are well defined as functionals and verify properties (3-4). In addition, we may define RHS so that the time evolution of the decaying Gamow vector $|\psi^D\rangle$ given by (3) be properly given for $t > 0$ only [31,32] and the time evolution of $|\psi^G\rangle$ given for $t < 0$, only. This clarifies further the meaning of the indices *Decaying* and *Growing* given to these vectors.

Once we have clarified the notion of Gamow vectors, we ask ourselves whether these vectors behave as ordinary state vectors, so that they may give expectation values of observables. A first attempt was given by Berggren [33, 34]. A possible definition of the energy average on Gamow vectors was given in [35]. None of these results were convincing. Berggren averages were intended to be used with many observables, although the average for the energy on a Gamow state was not what our intuition would say it is (the real part of the resonance pole, or resonance energy) and was rather sophisticated. The solution given in [35] was valid just for H and could not be applied to other observables, even for H^2, the square of the Hamiltonian. In summary, expressions like $\langle\psi^D|O|\psi^D\rangle$ for a given observable O are not defined. Even $\langle\psi^D|\psi^D\rangle$ does not admit a natural definition.

In the more widespread formulation of quantum mechanics, pure states are represented by vector states. The most popular representation for quantum unstable states is given by the Gamow vectors, which may suggest that Gamow states are pure states. Is this correct? This argument has two contradictions. The former is of mathematical nature. Pure states are represented by square integrable functions or normalizable vectors on a Hilbert space, which is not the case for Gamow vectors. Second, Gamow vectors somehow represent dissipative states which cannot have zero entropy. Pure states have zero entropy. The entropy for unstable quantum states via Gamow vectors have been investigated [36, 37] and certainly, it is not zero.

The generalization of all of the above to systems with N resonances is obvious. Although many models exhibit an infinite number of resonances, one may always keep a finite number since those resonances with large values of Γ_R are practically unobservable, since Γ_R is related to the inverse of the mean life. Note that poles of a meromorphic function (analytic except poles) are always isolated points.

In the present review paper, we show that Gamow states are not pure states and suggest a receipt to obtain mean values of observables on Gamow states. To this end, we have propose a formalism based on the notion of a state as a functional over an algebra of operators including relevant observables. This construction is the objective of the next Section.

2. The algebra of observables and the states

Let us make the simplest possible assumptions for the Hamiltonian pair that produces a resonance phenomena, $\{H_0, H = H_0 + V\}$. For instance that both H_0 and H have a simple (non-degenerate) absolutely continuous spectrum coinciding to $[0,\infty)$. Since Hamiltonians are always semi-bounded, the choice of the spectrum as $[0,\infty)$ may be done without restriction to generality. We also assume some scattering properties like the existence of the Møller wave operators and asymptotic completeness [38, 39]. The Møller wave operators, Ω^- and Ω^+, relate the free incoming and outgoing free states with the incoming and outgoing perturbed states, respectively.

For the moment, let us focus our attention to the free Hamiltonian H_0. According to a Theorem by Gelfand and Maurin [40, 41], and under the above working hypothesis, there exists a RHS like (6) such that, i.) $H_0\Phi \subset \Phi$ and H_0 is continuous on Φ, so that H_0 may be continuously extended to the anti-dual $\Phi^\times$; ii.) For each $E \in [0,\infty)$ there exists a functional $|E\rangle \in \Phi^\times$ such that $H_0\,|E\rangle = E\,|E\rangle$; iii.) For all $\varphi, \psi \in \Phi$, one has the following spectral decompositions:

$$\langle\varphi|H_0\psi\rangle = \int_0^\infty E\,\langle\varphi|E\rangle\langle E|\psi\rangle\,dE\,, \tag{11}$$

which may be written, by omitting the arbitrary $\varphi, \psi \in \Phi$ as

$$H_0 = \int_0^\infty E\,|E\rangle\langle E|\,dE\,. \tag{12}$$

It is also valid that

$$H_0^n = \int_0^\infty E^n\,|E\rangle\langle E|\,dE\,, \qquad n = 0, 1, 2, \dots\,. \tag{13}$$

Note that for $n = 0$, we have a spectral decomposition of the identity. After the aforementioned hypothesis, we have that $H = \Omega^\pm\,H_0\,(\Omega^\pm)^\dagger$, where the dagger means the adjoint. Using a new couple of RHS, where we define the spaces $\Phi^\pm := \Omega^\pm\,\Phi$, we may consider the functionals $|E^\pm\rangle := \Omega^\pm\,|E\rangle$ in $(\Phi^\pm)^\times$. Then, $H|E^\pm\rangle = E\,|E^\pm\rangle$ and one has the following spectral decompositions for H^n, $n = 0, 1, 2, \dots$:

$$H^n = \int_0^\infty E^n\,|E^\pm\rangle\langle E^\pm|\,dE\,. \tag{14}$$

Again for $n = 0$, we have two distinct spectral decompositions for the identity, which are different from those given in (13). As a matter of fact, n could have been any real number. The above kets, $|E\rangle\,, |E^\pm\rangle$, satisfy the following product relations [42]:

$$\langle E|E'\rangle = \delta(E - E')\,, \qquad \langle E^\pm|{E'}^\pm\rangle = \delta(E - E')\,. \tag{15}$$

Now, we say that the operators $O^\pm$ are *compatible* with H if they satisfy the following spectral decompositions:

$$O^\pm = \int_0^\infty dE\, O(E)\, |E^\pm\rangle\langle E^\pm| + \int_0^\infty dE \int_0^\infty dE\text{"}\, O(E,E')|E^\pm\rangle\langle {E'}^\pm|$$

$$\int_0^\infty dE\, O(E)\, \Omega^\pm|E\rangle\langle E|(\Omega^\pm)^\dagger + \int_0^\infty dE \int_0^\infty dE\text{"}\, O(E,E')\Omega^\pm|E\rangle\langle E'|(\Omega^\pm)^\dagger\,, \quad (16)$$

where $O(E)$ and $O(E,E')$ belong to certain spaces of test functions. The functions $O(E,E')$ should admit analytic continuation to analytic functions on both variables separately. Test functions should form linear spaces, which implies that the operators of the form (16) built two different linear spaces, one for $+$ and the other for $-$. In addition, we assume that the spaces of test functions $O(E)$ and $O(E,E')$ form respective algebras, which include the particular cases of $O(E) = E^n$, $n = 0, 1, 2, \dots$ and $O(E,E') = 0$, which give respective spectral decompositions of powers of the Hamiltonian H. Then, the product of two operators is given by

$$O_1^\pm\, O_2^\pm = \int_0^\infty dE\, O_1(E)\, O_2(E)\, |E^\pm\rangle\langle E^\pm|$$

$$+ \int_0^\infty dE \int_0^\infty dE'\, O_1(E,E')\, O_2(E,E')\, |E^\pm\rangle\langle {E'}^\pm|\,. \quad (17)$$

This, we have two algebras of operators with identity that we represent as $\mathcal{A}^\pm$. These algebras are isomorphic. They have a topology induced by the topology of the algebras of test functions $O(E)$ and $O(E,E')$, although we will not insist on this particular point.

It is convenient to simplify the notation. To this end, let us introduce the following symbols:

$$|E^\pm) := |E^\pm\rangle\langle E^\pm|\,, \qquad |E{E'}^\pm) := |E^\pm\rangle\langle {E'}^\pm|\,. \quad (18)$$

With this convention, the operators in (16) are written as

$$O^\pm = \int_0^\infty O(E)\, |E^\pm)\, dE + \int_0^\infty dE \int_0^\infty dE'\, O(E,E')\, |E{E'}^\pm)\,. \quad (19)$$

In addition, we have some formal relations or “products” such that

$$(E^\pm|w^\pm) = \delta(E-w)\,, \quad (E{E'}^\pm|w{w'}^\pm) = \delta(E-w)\delta(E'-w')\,,$$

$$(E{E'}^\pm|w) = (w|E{E'}^\pm) = 0\,. \quad (20)$$

These relations permit to perform operations such as that in (17). With this notation, we may write the functions $O(E)$ and $O(E,E')$ as a sort of products such as

$$(E^\pm|O^\pm) = O(E)\,, \qquad (EE'^\pm|O^\pm) = O(E,E')\,. \tag{21}$$

The algebras $\mathcal{A}^\pm$ have respective identities

$$I^\pm := \int_0^\infty dE\,|E^\pm)\,. \tag{22}$$

In addition, these algebras have an involution. It is natural to define the adjoints of the operators $O^\pm$ as

$$(O^\pm)^\dagger = \int_0^\infty dE\,O^*(E)\,|E^\pm) + \int_0^\infty dE\int_0^\infty dE'\,O^*(E',E)\,|EE'^\pm)\,. \tag{23}$$

As always, the star denotes complex conjugation. Observe on the transposition of variables on the function under the double integral in (23). Clearly, (23) defines respective involutions on $\mathcal{A}^\pm$.

Operators of the form (16) (or analogously (19)) and (23) are linear mappings $\Phi^\pm \longmapsto (\Phi^\pm)^\times$, which are, in addition, continuous. The proof of this continuity requieres a detailed contruction of the topologies and, therefore, lies outside the scope of the present article. Observe that definition (17) allows for the definition of a product of operators and, therefore, for the structure of algebra on $\mathcal{A}^\pm$.

2.1. Functionals over the algebras.

The algebras $\mathcal{A}^\pm$ are endowed with a topology compatible with the structure of algebra, which depend on the topological structure of the algebras of test functions $O(E)$ and $O(E,E')$. Functionals, $\rho^\pm$, over these algebras are continuous linear mappings $\rho^\pm : \mathcal{A}^\pm \longmapsto \mathbb{C}$. In the present situation, these functionals should be written in the following form:

$$\rho^\pm := \int_0^\infty dE\,\rho(E)\,(E^\pm| + \int_0^\infty dE\int_0^\infty dE'\rho(E,E')\,(EE'^\pm|\,. \tag{24}$$

Here, $\rho(E)$ is a functional (or a generalized function in the sense of Gelfand [12]) over the space of functions $O(E)$ and $\rho(E,E')$ is a functional over the space of functions $O(E,E')$. We should not forget that these functions form spaces of test functions, which have their corresponding spaces of functionals defined as usual. The action of (24) on (16) after (21) is

$$(\rho^\pm|O^\pm) = \int_0^\infty dE\,\rho(E)\,O(E) + \int_0^\infty dE\int_0^\infty dE'\,\rho(E,E')\,O(E,E')\,. \tag{25}$$

The meaning of the integrals in (25) should be clear as the action of the generalized functions on the test functions. Thus, the first term of the right hand side of (25) represents the action of the functional $\rho(E)$ on the function $O(E)$. Similarly, the second. These are truly integrals if $\rho(E)$ and $\rho(E,E')$ are functions on a given space such that both integrals converge. The action of a functional, which does not admit a representation as a regular function may be exemplified taken for instance $\rho(E)=\delta(w-E)$, where this delta is the Dirac delta. Note that $(E^\pm|$ are functionals such that

$$\rho(E)=\delta(E-w)\,, \quad \text{and} \quad \rho(E,E')\equiv 0\,. \tag{26}$$

Analogously, the symbols $(EE'^\pm|$ denote the following functionals:

$$\rho(E)\equiv 0\,, \quad \text{and} \quad \rho(E,E')=\delta(E-w)\delta(E'-w')\,. \tag{27}$$

In both cases, w and w' are variables and E and E' are fixed positive real numbers.

Just a brief comment on the spaces of functions $O(E,E')$. For reasons to be understood next, these functions should be analytically continuable for each variable independently, preferable to the whole complex plane. The construction relies on some results of Mathematical Analysis that we do not want to mention here, since we do not want to involve the reader with mathematical details that, although important, distract from the objective of this presentation. They will be published in a forthcoming paper [43].

2.2. **On quantum states.** Possibly, the most general definition of state in non-relativistic Quantum Mechanics is given by the consideration of states and observables in its algebraic formulation [44, 45]. Let us define the notion of state in this context.

The point of departure is an algebra of operators, $\mathcal{A}$, with a topology, identity, I, and an involution $O \longmapsto O^\dagger$ for all $O\in\mathcal{A}$. An observable, O, in $\mathcal{A}$ is a self adjoint operator, i.e., $O=O^\dagger$. The algebra $\mathcal{A}$ should contain the relevant observables of a particular system.

Here, the notion of self adjointness is formal and differs from the same notion relative to operators on Hilbert spaces, which imples the identity between the domain of a *symmetric* [46], also called Hermitian, operator and the domain of its adjoint. By construction, an observable in this formalism is an operator of the type (16), which satisfies (compare to (19)), $O(E)=O^*(E)$ and $O(E,E')=O^*(E',E)$, where the asterisk stands for complex conjugation.

A state ρ on $\mathcal{A}$ is a linear mapping (with respect to the structure of linear space on the algebra) $\rho:\mathcal{A}\longmapsto\mathbb{C}$, such that:

i.) The mapping ρ is positive. This means that for each $O\in\mathcal{A}$, $\rho(O^\dagger\,O)\geq 0$.

ii.) The mapping ρ is normalized. This means that if I is the identity in $\mathcal{A}$, then, $\rho(I)=1$.

iii.) The mapping ρ is continuous with respect to the topology on $\mathcal{A}$ and the usual topology on $\mathbb{C}$.

Just restricting ourselves to states of the kind (24) on the algebras $\mathcal{A}^{\pm}$, there are three different types of states:

i.) *Pure states.* A state is pure if the exists a square integrable function $\psi(E) \in L^2(\mathbb{R}^+)$, $\mathbb{R}^+ \equiv [0,\infty)$, such that

$$\rho(E) = |\psi(E)|^2 \,, \qquad \rho(E,E') = \psi^*(E)\,\psi(E')\,. \tag{28}$$

Note that $\rho(E,E) = \rho(E)$.

ii.) *Mixtures.*- Just defined by the relation $\rho(E) = \rho(E,E)$. Note that pure states are a particular case of mixtures. For mixtures we do not need the existence of a square integrable function satisfying (28).

iii.) *Generalized states.*- All the others. These states have been introduced [47] in order to give a precise mathematical definition of the states with diagonal singular. This notion has been introduced by van Hove [48, 49] for systems far from the thermodynamic equilibrium. For the van Hove states, $\rho(E) \neq \rho(E,E)$, which still are regular functions, not generalized ones. Nevertheless, we may introduce in this group states for which $\rho(E)$ and $\rho(E,E)$ are generalized functions. Next, we see that Gamow states belong to this category.

2.3. **Gamow states.** Now, we define the notion of Gamow functional, which appeared for the first time in [50] and, then, in [42]. For the growing Gamow vector associated to the resonance pole $z_R = E_R + i\Gamma/2$, we begin with the generalized function $\delta_{z_R} \otimes \delta_{z_R^*}$ and for the decaying Gamow vector associated to the resonance pole $z_R^* = E_R - i\Gamma/2$, the generalized function $\delta_{z_R^*} \otimes \delta_{z_R}$, which act on the test function $O(E,E')$ as

$$(\delta_{z_R} \otimes \delta_{z_R^*} | O(E,E')) := O(z_R, z_R^*)\,, \quad (\delta_{z_R^*} \otimes \delta_{z_R} | O(E,E')) := O(z_R^*, z_R)\,, \tag{29}$$

respectively.

Recall that we have demanded that the functions $O(E,E')$ are analytically continuable independently in both variables. Once we have the generalized function (29), we define the following functional on the algebra $\mathcal{A}^-$:

$$\rho_G := \int_0^\infty dE\,\delta(E-E_R)\,(E^-| + \int_0^\infty dE \int_0^\infty dE'\,\delta_{z_R} \otimes \delta_{z_R^*}\,(EE'^-|\,, \tag{30}$$

and the following functional on the algebra $\mathcal{A}^+$:

$$\rho_D := \int_0^\infty dE\,\delta(E-E_R)\,(E^+| + \int_0^\infty dE \int_0^\infty dE'\,\delta_{z_R^*} \otimes \delta_{z_R}\,(EE'^+|\,. \tag{31}$$

Functionals (30) and (31) are, respectively, the *growing and the decaying Gamow functionals or Gamow states.* We need to show that ρ_G is a state on $\mathcal{A}^-$ and ρ_D is a state on $\mathcal{A}^+$. We sketch the proof for ρ_G, the proof for ρ_D being identical.

i.) *Positivity.* We have not given details on the construction of the functions $O(E,E')$, since this requires some mathematical subtleties. These subtleties are important in order to show the positivity of these functionals. In particular,

$$(\rho_G|(O^-)^\dagger\,O^-) = |O(E_R)|^2 + |O(z_R, z_R^*)|^2 \geq 0\,, \tag{32}$$

where the positivity of the last term in the right hand side of (32) come from the mentioned mathematical properties.

ii.) *Normalization.* This means that $(\rho_G|I^-) = 1$. The proof is obvious after (22).

iii.) *Continuity.* It comes from the topological properties of the algebras.

We denote the action of a state $\rho^\pm$ on an observable $O^\pm \in \mathcal{A}^\pm$ as $(\rho^\pm|O^\pm)$. Then, the action of ρ_G on $O^- \in \mathcal{A}^-$ is $(\rho_G|O^-)$ and the action of ρ_D on $(\rho_D|O^+)$.

From the definitions of ρ_G and ρ_D, we see that these lie on the kind of *generalized states*, although they are different from the above mentioned van Hove states.

Next, let us show that it is always possible to define averages of observables on Gamow states. The average of the observable $O^\pm$ on the state $\rho^\pm$ should be defined, as usual as the action of $\rho^\pm$ on $O^\pm$, as above. This definition is valid for all observables in $\mathcal{A}^\pm$ (respectively), so that is valid for a wide range of observables. In particular, the averages of the identities in $\mathcal{A}^\pm$ are both one. We also may obtain the averages of the powers of the Hamiltonian using the spectral decomposition (14). They are as may be directly checked from (14,30,31):

$$(\rho_G|H^n) = (\rho_D|H^n) = E_R^n\,. \tag{33}$$

This result coincides with that one given in [35] for $n = 1$, where no average for $n > 1$ can be defined. Thus, we go beyond other attempts in the same direction. This result in (33) seems more reasonable than other proposals [33, 34].

2.3.1. *On the time evolution.* First of all, let us go back to equations (30) and (31) defining the Gamow functionals. Both are the sum of two contributions, so that we may split ρ_G and ρ_D as the sum of these terms as:

$$\rho_D = \rho_{DR} + \rho_{DS}\,, \qquad \rho_G = \rho_{GR} + \rho_{GS}\,. \tag{34}$$

In both cases, the first term in this decomposition is called the regular part and the second one the singular part. This terminology is purely conventional.

In order to study the time evolution of Gamow states, we need first to define the time evolution on the algebras $\mathcal{A}^\pm$. This is nothing else than the Heisenberg time evolution for observables. This is given by

$$e^{itH}\,O^{\pm}\,e^{-itH}=\int_0^\infty dE\,O(E)\,e^{itH}\;|E^{\pm}\rangle\langle E^{\pm}|\,e^{-itH}$$

$$+\int_0^\infty dE\int_0^\infty dE'\,O(E,E')\,e^{itH}\,|E^{\pm}\rangle\langle {E'}^{\pm}|\,e^{-itH}$$

$$=\int_0^\infty dE\,O(E)\,|E^{\pm}\rangle\langle E^{\pm}|+\int_0^\infty dE\int_0^\infty dE'\,O(E,E')\,e^{it(E-E')}\,|E^{\pm}\rangle\langle {E'}^{\pm}|\,. \quad (35)$$

The Schrödinger evolution of the states is related to the Heisenberg evolution of observables through a well known duality formula, given by the first identity in (36).

$$(e^{-itH}\,\rho_G\,e^{itH}|O^-)=(\rho_G|e^{itH}\,O^-\,e^{-itH})=\int_0^\infty dE\,\delta(E-E_R)\,O(E)$$

$$+\int_0^\infty dE\int_0^\infty dE'\,\delta_{z_R}\otimes\delta_{z_R^*}\,O(E,E')\,e^{it(E-E')}=O(E_R)+O(z_R,z_R^*)\,e^{it(z_R-z_R^*)}$$

$$=O(E_R)+O(z_R,z_R^*)\,e^{t\Gamma}=(\rho_{GR}|O^-)+e^{t\Gamma}\,(\rho_{GS}|O^-)\,. \quad (36)$$

Thus, by omitting the arbitrary $O^-\in\mathcal{A}^-$, we obtain the following time decay for the Gamow state:

$$\rho_G(t)=e^{-itH}\,\rho_G\,e^{itH}=\rho_{GR}+e^{t\Gamma}\,\rho_{GS}\,. \quad (37)$$

Analogously,

$$\rho_D(t)=\rho_{DR}+e^{-t\Gamma}\,\rho_{DS}\,. \quad (38)$$

Observe that, while (36) grows for increasingly positive values of time and decreases for increasingly negative values of time, the opposite is true for (38). A choice of $O(E,E')$ using a kind of complex analytic functions called Hardy functions on a half plane [31, 32] implies the validity of (37) for $t<0$ *only* and the validity of (38) for $t>0$ *only*. Thus, the evolution group splits into two distinct semigroups, one for values of t negatives (the growing part) and the other for values of t positives (the decaying part). This point of view postulates the existence of an origin of times for decaying processes [51].

3. CONCLUDING REMARKS

Unstable quantum states, usually called quantum scattering resonances, can be split into two parts: one which have exponential decay at all times, starting from an origin $t=0$. The other shows unescapable big deviations of the exponential law for very small, next to $t=0$, and very large values of time. These are hardly observable, so that exponential decay is assumed to be a good approximation for practically all observable values of time. Then, it looks practical to use resonance states which show an exponential decay for all times. Vector states for these exponential decay states are called Gamow vectors or Gamow states. Different resonances have distinct Gamow states.

The use of Gamow vectors as a representation of Gamow states has been popular since the years 1950'. However, this representation has two problems. One is the difficulty to define averages of observables on Gamow states. The other is the presumption that Gamow states should have non-zero entropy, which looks incompatible with the representation of Gamow state as vectors, as is the case for pure quantum states. It is certainly true that Gamow vectors are not normalizable, but we still need to show that notwithstanding they are not pure states.

We have used an algebraic formulation of observables and states that solves both problems. In one side, averages on Gamow states are well defined for a wide range of observables. In addition it is shown that Gamow states are not pure (neither mixtures, although they are quantum states).

Author Contribution Statements

All authors have equally contributed to this paper.

Declaration of Competing Interests

The Authors declare no conflict of interests.

Acknowledgement Partial financial support is acknowledged to Ministry of Science of Spain (project PID2020-113406GB-I00) and to Junta de Castilla y León (project BU229P18)

References

[1] Bohm, A. Quantum Mechanics: Foundations and Applications, 3$^{\rm rd}$ Ed. Springer Verlag, New York and Berlin 2001.

[2] Newton, R.G. Scattering Theory of Waves and Particles, 2$^{\rm rd}$ Ed. Springer Verlag, Berlin and Heidelberg, 1982.

[3] Nussenzveig, H.M. Causality and Dispersion Relations, Academic, New York and London, 1972.

[4] Kukulin, V.I.; Krasnopolski, V.M.; Horáček, Theory of Resonances, Principles and Applications, Academia, Prag, 1989.

[5] Fonda, L.; Ghirardi, G.C.; Rimini, A . Decay theory of unstable quantum systems. *Rep Progr. Phys.*, 41 (1978), 587-631.

[6] Khalfin, L.A., Contribution to the decay theory of a quasi stationary state, *Sov. Phys., JETP-USSR* 6 (6) (1958), 1053-1063.

[7] Fischer, M.C.; Gutiérrez-Medina, B.; Raizen, M.G., Observation of the quantum Zeno and anti-Zeno effects in an unstable system, *Phys. Rev. Lett.*, 87 (4) (2001), 040402.

[8] Rothe, C.; Hintschich, S.I.; Monkman, A.P., Violation of the Exponential-Decay Law at Long Times, *Phys. Rev. Lett.*, 96 (16) (2006), 163601.

[9] Nakanishi, N., A note on the physical state of unstable particles, *Progr. Theor. Phys.*, 21 (1) (1959), 216-217.

[10] Albeverio, S., Kurasov, P., *Singular Perturbations of Differential Operators*, Lecture Note Series, vol. 271, Cambridge, UK, London Mathematical society.

[11] Losada, M., Fortín, S., Gadella, M. Holik, F., Dynamics of algebras in quantum unstable systems, *Int. J. Mod. Phys. A*, 33 (18-19) 2018, 1850109.
[12] Gelfand, I.M., Shilov, G.E., Generalized Functions, Vol. II, Academic Press, New York, 1964.
[13] Horváth, J., Topological Vector Spaces and Distributions, Addison-Wesley, London, 1966.
[14] Bohm, A., Rigged Hilbert space and the mathematical description of physical systems, Boulder Lecture Notes in Theoretical Physics, Vol. 9A, Gordon and Breach Science Publishers, New York, 1967, 255-317.
[15] Bohm, A., The Rigged Hilbert Space and Quantum Mechanics, Lecture Notes in Physics **78**, Springer, New York, 1978.
[16] Roberts, J.E., Rigged Hilbert spaces in quantum mechanics, *Commun. Math. Phys.*, 2 (1966), 98-119.
[17] Antoine, J.P. Dirac formalism and symmetry problems in quantum mechanics. I. General Dirac formalism. *J. Math. Phys.*, 10 (1969), 53-69.
[18] Melsheimer, O. Rigged Hilbert space formalism as an extended mathematical formalism for quantum systems. I. General theory. *J. Math. Phys.*, 15 (1973), 902-916.
[19] Bellomonte, G.; Trapani, C. Rigged Hilbert spaces and contractive families of Hilbert spaces. *Monatshefte Math.* 164 (2011), 271-285.
[20] Bellomonte, G.; di Bella, S.; Trapani, C. Operators in rigged Hilbert spaces: some spectral properties. *J. Math. Anal. Appl.*, 411 (2014), 931-946.
[21] Chiba, H. A spectral theory of linear operators on rigged Hilbert spaces under analyticity conditions. *Adv. Math.*, 273 (2015), 324-379.
[22] Chiba, H. A spectral theory of linear operators on rigged Hilbert spaces under analyticity conditions II: Applications to Schrödinger operators. *Kyushu J. Math.*, 72 (2018), 375-405.
[23] Gadella, M.; Gómez, F. A unified mathematical formalism for the Dirac formulation of quantum mechanics. *Found. Phys.*, 32 (2002), 815-869.
[24] Gadella, M.; Gómez, F. On the mathematical basis of the Dirac formulation of Quantum Mechanics. *Int. J. Theor. Phys.*, 42 (2003), 2225-2254.
[25] Gadella, M.; Gómez-Cubillo, F. Eigenfunction Expansions and Transformation Theory. *Acta Appl. Math.*, 109 (2010), 721-742.
[26] Celeghini, E., Gadella, M., del Olmo, M.A., Applications of rigged Hilbert spaces in quantum mechanics and signal processing, *J. Math. Phys.*, 57 (2016), 072105.
[27] Celeghini, E., Gadella, M., del Olmo, M.A., Spherical harmonics and rigged Hilbert spaces, *J. Math. Phys.*, 59 (5) (2018), 053502.
[28] Celeghini, E., Gadella, M., del Olmo, M.A., Zernike functions, rigged Hilbert spaces and potential applications, *J. Math. Phys.*, 60 (2019), 083508.
[29] Celeghini, E., Gadella, M., del Olmo, M.A., Groups, Jacobi Functions and rigged Hilbert spaces, *J. Math. Phys.*, 61 (2020), 033508
[30] Bohm, A., Resonance poles and Gamow vectors in the rigged Hilbert space formulation of quantum mechanics, *J. Math. Phys.*, 22 (12) (1980), 2813-2823.
[31] Bohm, A., Gadella, M., Dirac Kets, Gamow Vectors and Gelfand Triplets, Springer Lecture Notes in Physics, 348. Springer Verlag, Berlin 1989.
[32] Civitarese, O., Gadella, M., Physical and Mathematical Aspects of Gamow States, *Phys. Rep.*, 396 (2004), 41-113.
[33] Berggren, T., On the use of resonant states in eigenfunction expansions of scattering and reaction amplitudes, *Nucl. Phys. A*, 109 (2) (1968), 265-287.
[34] Berggren, T., Expectation value of an operator in a resonant state, *Phys. Lett. B*, 373 (1-3) (1996), 1-4.
[35] Civitarese, O., Gadella, M., Id Betan, R., On the mean value of the energy for resonance states, *Nucl. Phys. A*, 660 (1999), 255-266.
[36] Civitarese, O., Gadella, M., On the concept of entropy for quantum decaying systems, *Found. Phys.*, 43 (2013), 1275-1294.

[37] Civitarese, O., Gadella, M., The definition of entropy for quantum unstable systems: A view-point based on the properties of Gamow states, *Entropy*, 20 (4) (2018), 231.

[38] Amrein, W.O., Jauch, J.M., Sinha, K.B., Scattering Theory in Quantum Mechanics. Physical Principles and Mathematical Methods, 917 Bejamin, Reading, Massachusetts, USA, 1977.

[39] Reed, M., Simon, B., Scattering Theory, Academic, New York, 1979.

[40] Gelfand, I.M., Vilenkin, Generalized Functions. Applications of Harmonic Analysis, Academic, New York, 1970.

[41] Maurin, K.it Generalized eigenfunction expansions and unitary representations of topological groups, Polish Scientific Publishers, Warszawa, 1968.

[42] Castagnino, M., Gadella, M., Id Betán, R., Laura, R., Gamow functionals on operator algebras, *J. Phys. A: Math. Gen.* 34 (2001), 10067-10083

[43] Gadella, M., et al., To appear in Entropy.

[44] Segal, I.E., Postulates for General Quantum Mechanics, *Annal. Math.*, 48 (1947), 930-948.

[45] Brateli, O., Robinson, B., *Operator Algebras and Quantum Statistical Mechanics*, Vol I and II, Springer, New York, 1979.

[46] Reed, M., Simon, B., *Functional Analysis*, Academic Press, New York, 1972.

[47] Antoniou, I.E., Laura, R., Suchanecki, Z, Tasaki, S., Intrinsic irreversibility of quantum systems with diagonal singularity, *Phys. A*, 241 (1997), 737-772.

[48] van Hove, L., The approach to equilibrium in quantum statistics, *Physica*, 23 (1957), 441-480.

[49] van Hove, L., The ergodic behaviour of quantum many-body systems, *Physica*, 25 (1959), 268-276.

[50] Castagnino, M., Gadella, M., Id Betán, R., Laura, R., The Gamow functional, *Phys. Lett.*, 282 (2001), 2450-250.

[51] Bohm, A., Bryant, P.W., Uncu, H., Wickramasekara, S., Schleich, W.P., The beginning of time observed in quantum jumps, *Fort. Phys.*, 65 (6-8) (2017), 1700015.