A Note on Integrals Containing the Univariate Lommel Function

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ABSTRACT

This short note investigates a number of index integrals of products of the Lommel functions $s_{\mu,\nu}(a)$ and uncovers an integral relationship. between this function and the Chebyshev polynomials $T_{2n}(x)$.

Key Words:

Classification:

Introduction

Index integrals, such as the well-known Kontorovich-Lebedev transform [1]

$$\int_0^\infty f(x)K_{ix}(y)dx \tag{1}$$

have been investigated and tabulated for over two centuries. A survey of their evaluation and applications can be found in the book *Index Transforms* [2] by S. Yakubovich. index integrals of the Lommel functions

$$s_{\mu,\nu}(z)=\frac{z^{\mu+1}}{(\mu+1)^2-\nu^2}\,{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{z^2}{4}\right) \tag{2}$$

seem not to be found in the extensive table by Prudnikov et al.[3] and elsewhere [4] and therefore even a few that can be obtained by unsophisticated means would seem to be worthy of notice, and that is the aim here.

In the next section we present a number of these integrals and sketch their derivation and follow this by a few corollary results.

Results and Derivations

Theorem 1.

$$\int_0^\infty x\sin(\pi x)s_{-1,x}(a)s_{0,x}(b)dx=\frac{\pi^2}{4}[\mathbf{H}_0(a-b)-\mathbf{H}_0(a+b)] \tag{a}$$

$$\int_0^\infty x\sin(\pi x)s_{-1,x}(a)s_{0,x}(a)dx=-\frac{\pi^2}{4}\mathbf{H}_0(2a). \tag{a'}$$

$$\int_0^\infty \cos^2(\pi x/2)s_{0,x}(a)s_{0,x}(b)dx=\frac{\pi^2}{8}[J_0(|a-b|)-J_0(a+b)] \tag{b}$$

$$\int_0^\infty \cos^2(\pi x/2)s_{0,x}^2(a)dx=\frac{\pi^2}{8}(1-J_0(2a)). \tag{b'}$$

$$\int_0^\infty x^2\sin^2(\pi x/2)s_{-1,x}(a)s_{-1.x}(b)=\frac{\pi^2}{8}[J_0(|a-b|)+J_0(a+b)] \tag{c}$$

$$\int_0^\infty x^2\sin^2(\pi x/2)s_{-1,x}^2(a)dx=\frac{\pi^2}{8}(1+J_0(2a)). \tag{c'}$$

Proof: These are all consequences of the Fourier transforms[5,(1.7.(49),(50))] whose inversion gives

$$\sin(a\cos(x))\theta\left(\frac{\pi}{2}-x\right)=\frac{2}{\pi}\int_0^\infty \cos(xy)\cos\left(\frac{\pi y}{2}\right)s_{0,y}(a)dy \tag{3a}$$

$$\cos(b\cos x))\theta\left(\frac{\pi}{2}-x\right)=-\frac{2}{\pi}\int_0^\infty \cos(xy)y\sin\left(\frac{\pi y}{2}\right)s_{-1,y}(b)dy \tag{3b}$$

where θ denotes the unit step function: 1 for positive argument, 0 for negative. By Parseval's identity and the double angle identity for the sine we have

$$\int_0^{\pi/2} \sin(a\cos x)\cos(b\cos x)dx = -\frac{1}{\pi}\int_0^{\infty} x\sin(\pi x)s_{-1,x}(a)s_{0,x}(b)dx. \tag{4}$$

Since

$$-\pi\int_0^{\pi/2} \sin(a\cos x)\cos(b\cos x)dx = \frac{\pi^2}{4}[\mathbf{H}_0(a+b)+\mathbf{H}_0(a-b)] \tag{5}$$

one has (a) and (a'). The derivations of the pairs b and c proceed similarly.

Next, the Fourier inversions of (3a,b) with $x = cos^{-1}u$ read

$$\int_0^1 \sin(au)\frac{\cos(y\cos^{-1}u)}{\sqrt{1-u^2}}du = \cos\left(\frac{\pi y}{2}\right)s_{0,y}(a) \tag{6a}$$

$$\int_0^1 \cos(bu)\frac{\cos(y\cos^{-1}u)}{\sqrt{1-u^2}}du = -y\sin\left(\frac{\pi y}{2}\right)s_{-1,y}(b). \tag{6b}$$

so, again by Fourier inversion,

$$\frac{\cos(y\cos^{-1}u)}{\sqrt{1-u^2}}\theta(1-u) = \frac{2}{\pi}\cos\left(\frac{\pi y}{2}\right)\int_0^{infty} \sin(ut)s_{0,y}(t) \tag{7a}$$

$$\frac{\cos(y\cos^{-1}u)}{\sqrt{1-u^2}}\theta(1-u) = -\frac{2}{\pi}y\sin\left(\frac{\pi y}{2}\right)\int_0^{\infty} \cos(ut)s_{-1,y}(t)dt. \tag{7b}$$

Therefore, with $y = 2n, 2n+1$, respectively, one has the explicit connection with Tchebyshev polynomials

Theorem 2

$$\frac{T_{2n}(u)}{\sqrt{1-u^2}}\theta(1-u) = (-1)^n\frac{2}{\pi}\int_0^{\infty} \sin(ut)s_{0,2n}(t)dt \tag{8a}$$

$$\frac{T_{2n+1}(u)}{\sqrt{1-u^2}}\theta(1-u) = (-1)^{n+1}(2n+1)\int_0^{\infty} \cos(ut)s_{-1,2n+1}(t)dt \tag{6b}$$

$$s_{0,2n}(t) = (-1)^n\int_0^1 \sin(ut)\frac{T_{2n}(u)}{\sqrt{1-u^2}}du \tag{9a}$$

$$s_{-1,2n+1}(t) = \frac{(-1)^{n+1}}{2n+1}\int_0^1 \cos(ut)\frac{T_{2n+1}(u)}{\sqrt{1-u^2}}du. \tag{9b}$$

Discussion

The results of Theorem 1 can be extended to other integer values of the first index by means of documented[6] recursion relations for the Lommel function. For example

$$s_{0,x}(a) = \frac{a - s_{2,x}(a)}{1 - x^2}, \quad s_{-1,x}(a) = x^{-2} s_{1,x}(a) \tag{10}$$

$$s_{m,x}(a) = \frac{a}{2x}[(m + x - 1)s_{m-1,x-1}(a) - (m - x = 1)s_{m-1,x+1}(a)] \tag{111}$$

$$\frac{d}{da} s_{m,x}(a) = \frac{1}{2}[(m + x - 1)s_{m-1,x-1}(a) + (m - x - 1)s_{m-1,x+1}(a)]. \tag{12}$$

Since the Tchebyshev functions are related to other families of orthogonal polynomials[7] it is possible to connect these with the Lommel function as well. For example,

$$U_{2n}(x) = \frac{(-1)^n}{\sqrt{1 - x^2}} T_{2n+1}(\sqrt{1 - x^2}) \tag{13}$$

gives for the Tchebyshev polynomial of the second kind

$$U_{2n}(x) = -(2n + 1) \int_0^\infty \cos(t\sqrt{1 - x^2}) s_{-1,2n+1}(t) dt \tag{14}$$

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