The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

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To cite this article: M. L. Glasser (2019) A Note on Beukers’s and Related Double Integrals, The American Mathematical Monthly, 126:4, 361-363, DOI: 10.1080/00029890.2019.1565856

To link to this article: https://doi.org/10.1080/00029890.2019.1565856

Published online: 22 Apr 2019.

A Note on Beukers's and Related Double Integrals

M. L. Glasser

Abstract. An elementary transformation formula is derived, allowing double integrals of the type introduced by F. Beukers to be reduced, and new ones to be constructed.

1. INTRODUCTION. As an aid to studying the irrationality of certain mathematical constants, including Apery's constant $\zeta(3)$, in 1979 F. Beukers [1] introduced double integrals of the type

$$\zeta(2) = \int_0^1 \int_0^1 \frac{dx\,dy}{1 - xy}. \tag{1}$$

This is easily verified by expanding the denominator as a geometric series and integrating termwise. Since that time this class of representation has been studied intensively and extended to nearly all transcendental numbers important to number theory: Euler's constant, polylogarithms, etc. (see [3–5, 7, 8] where further references are given). For example [3],

$$\int_0^1 \int_0^1 \frac{\ln(1 + xy)}{1 - xy}\,dx\,dy = \frac{\pi^2}{4}\ln 2 - \zeta(3) \tag{2}$$

$$\int_0^1 \int_0^1 \frac{\ln(1 - xyz)}{1 - xy}\,dx\,dy = \frac{\pi^2}{6}\ln(1 - z) - \sum_{n=1}^{\infty} \frac{H_{n,2}}{n} z^n \tag{3}$$

where $H_{n,m} = 1 + 2^{-m} + 3^{-m} + \cdots + n^{-m}$ is a generalized harmonic number.

The aim of this note is to prove the following.

Theorem 1. *If $f : \mathcal{R} \to \mathcal{R}$ is integrable over the unit interval, then*

$$\int_0^1 \int_0^1 f(xy)\,dx\,dy = -\int_0^1 \ln(x) f(x)\,dx. \tag{4}$$

Equation (4) can serve to dispel any mystery one might feel surrounding formulas such as (1); since (1) becomes the familiar representation

$$\zeta(2) = -\int_0^1 \frac{\ln x}{1 - x}\,dx,$$

doi.org/10.1080/00029890.2019.1565856
MSC: Primary 11M30, Secondary 33B15

it also provides an easy route to constructing additional representations and might even provide a few new definite integrals.

We also have the following corollary.

Theorem 2. *Under the same assumption as in Theorem 1,*

$$\int_0^1 \int_0^1 \frac{f(xy)}{\ln(xy)}\, dx\, dy = -\int_0^1 f(x)\, dx.$$

2. CALCULATION. By symmetry, the left-hand side of (4) is

$$I = 2\int_R f(xy)\, dx\, dy,$$

where R is the right triangle (0, 0), (1, 0), (1, 1). Under the transformation

$$x = \frac{1}{2}[\sqrt{t^2 + 4u} + t], \quad y = \frac{1}{2}[\sqrt{t^2 + 4u} - t]$$

$$u = xy, \quad t = x - y$$

having Jacobian $J = -(t^2 + 4u)^{-1/2}$, in the (u, t)-plane R becomes the right triangle (0, 0), (0, 1), (1, 0) yielding

$$I = 2\int_0^1 du f(u) \int_0^{1-u} \frac{dt}{\sqrt{t^2 + 4u}} = 2\int_0^1 f(u) \sinh^{-1}\left(\frac{1-u}{2\sqrt{u}}\right) du.$$

Next, let $u = \operatorname{sech}^2 y$, so $(1-u)/2\sqrt{u} = (\cosh y - \operatorname{sech} y)/2$ and with $\cosh y = e^x$, $\sinh^{-1}[(1-u)/2\sqrt{u}] = x$. Therefore, with $s = e^{-2x}$, we have

$$I = 4\int_0^\infty x e^{-2x} f(e^{-2x})\, dx = -\int_0^1 \ln(s) f(s)\, ds.$$

This completes the proof of (4).

It was emphasized by an (anonymous) referee that the proof above is adapted to the continuous functions f in the examples and the referee outlined a more general argument based on partitioning the unit square into infinitesimal hyperbolic strips of the form $u = xy$. In this way the logarithm appearing in (4) emerges more naturally. This is left as an enlightening exercise.

3. DISCUSSION. Let us apply this to the Guillera–Sondow [3] formula (2). Equation (4) immediately gives

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}\, dx = \zeta(3) - \frac{\pi^2}{4}\ln 2.$$

Mathematica is able to reproduce this and it can be reproduced using formula (A.3.5) in [6], for example. However, in the case of (3), *Mathematica* and Lewin [6, (A.3.5)] give, for $0 < z < 1$,

$$\int_0^1 \frac{\ln(x)\ln(1-xz)}{1-x}\, dx, \tag{3}$$

a complicated expression containing di- and trilogarithms, meaning that the generating function for the generalized harmonic numbers $H_{n,2}/n$ can be expressed in closed form. In the case $z = 1/2$ the many terms simplify and one finds

$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n n} = \frac{5}{8}\zeta(3).$$

As an example of the corollary we have

$$\int_0^1 \int_0^1 \frac{e^{-xyz}}{\ln(x) + \ln(y)}\, dx\, dy = \frac{1 - e^z}{e^z}, \qquad Re[z] > 0,$$

which may be new.

Finally, many of the important double integrals, such as that representing Euler's constant, are of the form

$$J = \int_0^1 \int_0^1 (1 - x) f(xy)\, dx\, dy = \frac{1}{2} \int_0^1 \int_0^1 (2 - x - y) f(xy)\, dx\, dy.$$

However, by a similar calculation (see [8]) one finds

$$J = \int_0^1 (1 - x - \ln x) f(x)\, dx.$$

This gives an immediate proof of Hadjicostas's conjecture [2, 4]

$$\Gamma(n+2)\zeta(n+2) - \Gamma(n+1) = \int_0^1 \int_0^1 \frac{(1-x)(-\ln xy)^n}{1 - xy}\, dx\, dy.$$

ACKNOWLEDGMENT. Partial financial support from Spanish Grants MTM2014-57129-C2-I-P(MINECO) and VA057U16 (Junta de Castilla y León and FEDER) are gratefully acknowledged.

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