THE QUANTUM HARMONIC OSCILLATOR AND CATALAN'S CONSTANT

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(*Received March 1, 2021 — Revised May 18, 2021*)

In this note we provide a representation of Catalan's constant in terms of a series involving the values at the origin of the even eigenfunctions of the quantum harmonic oscillator.

Keywords: quantum harmonic oscillator, eigenfunctions, Hermite polynomials, Catalan's constant.

2000 Mathematics Subject Classification: Primary 11M36.

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1. Introduction

As is well known, Catalan's constant, denoted by G throughout our paper, as is customary nowadays, is defined by (see, e.g., [1, 2])

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 1 - \frac{1}{9} + \frac{1}{25} + \cdots \approx 0.91597. \tag{1}$$

Catalan's constant appears in the number theory, combinatorics, and different areas of mathematical analysis, but is not so frequent in applied fields, and in particular in the quantum realm, where only scarce and abstruse references are found: it appears when studying finite-size effects in two-dimensional lattice six vertex models working with relativistic string models [3]; it also comes out when evaluating form factors as auxiliary objects in order to compute correlation functions in homogeneous sine-Gordon models [4]; the so-called "quantum deformations" of Catalan's constant were studied [5], although the name "quantum" is purely ornamental; finally, G emerges when solving an integral that originates from the use of hyperbolic geometry in quantum field theory and also from the reduction of a multidimensional Feynman integral in [6].

As pointed out in [2], what is remarkable about G is that it is not yet known whether this number is irrational. By quoting those authors, "this remains a stubbornly unsolved problem". Furthermore, an intriguing fact about Catalan's constant is that quite a few definite integrals, arising in various contexts, can be expressed in terms of G, most of which are listed in [2].

In this work we are going to show how Catalan's constant appears in relation to a certain integral operator that can be written exclusively in terms of the eigenfunctions of the quantum harmonic oscillator.

2. Evaluation of the series

The following series

$$S = \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{2n} \tag{2}$$

may be of some interest in mathematical physics given that a certain quantity related to

$$H_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right), \tag{3}$$

the Hamiltonian of the quantum harmonic oscillator, can be written explicitly in terms of (2).

Here we are using the well-known eigenfunctions of the harmonic oscillator

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}}\; e^{-x^2/2}\; H_n(x) \quad \Longrightarrow \quad \psi_{2n}(0) = \frac{H_{2n}(0)}{\sqrt{2^{2n}(2n)!\sqrt{\pi}}}, \tag{4}$$

$H_n(x)$ being the n-th Hermite polynomial (see [1, 7, 8]). It is well known (see [1]) that

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}.$$

Then, taking the fourth power of (4) and using the previous result, we get

$$\psi_{2n}^4(0) = \frac{(2n)!^2}{\pi\, 2^{4n}\, (n)!^4} = \frac{[(2n-1)!!]^2}{\pi\, 2^{2n}\, (n)!^2}. \tag{5}$$

At this stage we wish to provide an example attesting the relevance of the aforementioned series (2) in relation to the harmonic oscillator. First, for any $E < \frac{1}{2}$ we consider the integral operator whose kernel is given by

$$[H_0 - E]^{-\frac{1}{4}}\,(x, y) = \sum_{n=0}^{\infty} \frac{\psi_n(x)\psi_n(y)}{(n + \frac{1}{2} - E)^{\frac{1}{4}}}.$$

After denoting the restriction of H_0 to $[\psi_0]^{\perp}$ by $H_0^{\geq 1}$ and setting $E = \frac{1}{2}$, we consider the integral operator whose kernel is given by

$$B_{1/2} = e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{4}} (x, y) = e^{-\frac{x^2}{2}} \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(y)}{(n)^{\frac{1}{4}}} = \pi^{1/4}\psi_0(x) \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(y)}{(n)^{\frac{1}{4}}}. \tag{6}$$

Of course, the products involving $B_{1/2}$ and its adjoint $B_{1/2}^*$ give rise to the positive operators

$$B_{1/2}^* B_{1/2} = \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{4}} e^{-x^2} \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{4}} \geq 0,$$

and

$$B_{1/2} B_{1/2}^* = e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \geq 0.$$

The latter operators are reminiscent of

$$[H_0 - E]^{-\frac{1}{2}}\, e^{-x^2}\, [H_0 - E]^{-\frac{1}{2}}$$

and

$$e^{-\frac{x^2}{2}}\, [H_0 - E]^{-1}\, e^{-\frac{x^2}{2}},$$

the Birman–Schwinger type operators used in our papers on the harmonic oscillator perturbed by an attractive Gaussian potential [9, 10] (see also the analogous papers on the negative Laplacian perturbed by the same attractive Gaussian potential in one dimension [11, 12]).

The operator defined by (6), being the limit of a sequence of finite rank operators in the norm topology, is compact. Furthermore, it is worth noting that the sequence

$$\left\|B_{1/2}\psi_n\right\|_2 = \left\|e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{4}} \psi_n\right\|_2$$

is not square summable since, by taking advantage of Wang's results [9, 10, 13, 14],

$$\left\| e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2} \right]^{-\frac{1}{4}} \psi_n \right\|_2^2 = \left(\psi_n, B_{1/2}^* B_{1/2} \psi_n \right) = \frac{(\psi_n, e^{-x^2} \psi_n)}{\sqrt{n}} = \sqrt{\pi} \frac{\psi_{2n}^2(0)}{\sqrt{2n}},$$

diverges given that $\psi_{2n}^2(0)$ behaves like $n^{-\frac{1}{2}}$ (see [9, 10, 14–26]). However, it is p-summable for any $p > 2$, so that

$$\sum_{n=1}^{\infty} \left\| e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2} \right]^{-\frac{1}{4}} \psi_n \right\|_2^4 = \pi \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{2n}. \tag{7}$$

It is worth pointing out that this series contains $\psi_{2n}^4(0)$ in the numerator of its sequence, differently from the series appearing in various models involving point perturbations of the quantum harmonic oscillator (see the articles cited earlier), which instead share $\psi_{2n}^2(0)$ in the numerator of their sequences. As a result, we cannot expect to express the series in terms of a ratio of Gamma functions, as was done in the above-mentioned papers. As will be shown next, the series (2) can be computed analytically. Although this fact is not entirely new (see [27] p. 472 as well as [28] investigating a series very closely related to ours by means of hypergeometric functions), we believe it may be worth obtaining the result by relying exclusively on the properties of the eigenfunctions of the harmonic oscillator combined with those of the complete elliptic integral of the first kind.

THEOREM 1. *The positive series* (2) *converges and*

$$S = \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{2n} = \frac{2}{\pi^2} \left[\pi \ln 2 - 2G \right], \tag{8}$$

where G is Catalan's constant.

Proof: As anticipated earlier, $\psi_{2n}(0)$ behaves like $n^{-1/4}$ as $n \longrightarrow +\infty$, so that the strictly positive sequence inside the sum behaves like n^{-2} as $n \longrightarrow +\infty$, which ensures the convergence of the series.

First of all, we notice that

$$S = \sum_{n=1}^{\infty} \frac{\psi_{2n}^4(0)}{2n} = \sum_{n=1}^{\infty} \psi_{2n}^4(0) \int_0^1 x^{2n-1}\, dx = \int_0^1 \left[\sum_{n=1}^{\infty} \psi_{2n}^4(0)\, x^{2n-1} \right] dx. \tag{9}$$

As follows in a rather straightforward manner from the definition of the normalised eigenfunctions of the harmonic oscillator, the value of $\psi_{2n}^4(0)$ is given by (4), so that the series inside the square brackets on the rhs of (9) becomes

$$f(x) = \sum_{n=1}^{\infty} \psi_{2n}^4(0)\, x^{2n-1} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{[(2n-1)!!]^2}{2^{2n}(n!)^2}\, x^{2n-1} = \frac{1}{\pi x} \sum_{n=1}^{\infty} \frac{[(2n-1)!!]^2}{2^{2n}(n!)^2}\, x^{2n}. \tag{10}$$

As can be checked by using mathematical induction, the latter series can be written in terms of the expansion of $K(x^2)$, the complete elliptic integral of the first kind defined by

$$K(x^2) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}. \tag{11}$$

Hence,

$$x\,f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(2n-1)!!]^2}{2^{2n}(n!)^2}\,x^{2n} = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{[(2n-1)!!]^2}{2^{2n}(n!)^2}\,x^{2n} - \frac{1}{\pi} = \frac{2K(x^2)-\pi}{\pi^2}. \tag{12}$$

Therefore,

$$f(x) = \frac{1}{\pi x}\sum_{n=1}^{\infty}\frac{[(2n-1)!!]^2}{2^{2n}(n!)^2}\,x^{2n} = \frac{2K(x^2)-\pi}{\pi^2 x}. \tag{13}$$

By integrating the rhs of (13) from 0 to 1, we get

$$S = \int_0^1 f(x)\,dx = \int_0^1 \frac{2K(x^2)-\pi}{\pi^2 x}\,dx = \int_0^1 \frac{K(y)-\frac{\pi}{2}}{\pi^2 y}\,dy. \tag{14}$$

The latter integral integral can be evaluated as follows, taking into account that $K(0) = \pi/2$,

$$\begin{aligned}\pi^2 S &= \int_0^1 \left(K(m)-\frac{\pi}{2}\right)\frac{dm}{m} = \int_0^1 \frac{dm}{m}\left(\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} - \int_0^1 \frac{dt}{\sqrt{1-t^2}}\right)\\ &= \int_0^1 \frac{dm}{m}\int_0^1 \frac{1-\sqrt{1-mt^2}}{\sqrt{(1-t^2)(1-mt^2)}}\,dt = \int_0^1 \frac{dt}{\sqrt{1-t^2}}\int_0^1 \frac{dm}{m}\,\frac{1-\sqrt{1-mt^2}}{\sqrt{1-mt^2}},\end{aligned} \tag{15}$$

where we have just interchanged the integrals in the variables t and m. The integral in m can be evaluated straightforwardly, and then

$$\begin{aligned}\pi^2 S &= \int_0^1 \frac{dt}{\sqrt{1-t^2}}\left[-2\ln\left(1+\sqrt{1-mt^2}\right)\right]_{m=0}^{1}\\ &= \int_0^1 \frac{dt}{\sqrt{1-t^2}}\left[2\ln 2 - 2\ln\left(1+\sqrt{1-t^2}\right)\right]\\ &= \pi\ln 2 - 2\int_0^1 \frac{\ln\left[1+\sqrt{1-t^2}\right]}{\sqrt{1-t^2}}\,dt = \pi\ln 2 - 2J.\end{aligned} \tag{16}$$

After setting $t = \sin 2\varphi$ in the last integral, we get

$$J = 2\int_0^{\pi/4} \ln\left[1+\cos 2\varphi\right] \; d\varphi = 2\int_0^{\pi/4} \ln\left[2\cos^2\varphi\right] \; d\varphi$$

$$= 2\int_0^{\pi/4} (\ln 2 + 2\ln\cos\varphi) \; d\varphi = \frac{\pi}{2}\ln 2 + 4\int_0^{\pi/4} \ln\cos\varphi \; d\varphi. \tag{17}$$

As is well known, one of the representations of Catalan's constant is the following integral

$$G = 2\int_0^{\pi/4} \ln(2\cos\varphi) \; d\varphi = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln\cos\varphi \; d\varphi. \tag{18}$$

Then, the remaining integral in (17) can be evaluated and we get

$$J = \frac{\pi}{2}\ln 2 + 2\left(G - \frac{\pi}{2}\ln 2\right) = 2G - \frac{\pi}{2}\ln 2, \tag{19}$$

implying from (16) that

$$\pi^2 S = 2\pi\ln 2 - 4G, \tag{20}$$

which completes the proof of our claim (8). □

As an immediate consequence of the previous theorem we get the following evaluation of the series (7),

$$\sum_{n=1}^{\infty} \left\| e^{-\frac{x^2}{2}} \left[H_0^{\geq 1} - \frac{1}{2}\right]^{-\frac{1}{4}} \psi_n \right\|_2^4 = \frac{2}{\pi}\left[\pi\ln 2 - 2G\right] \approx 0.220051. \tag{21}$$

We wish to remind the reader acquainted with the theory of operators belonging to the Schatten classes, that, as stated in Theorem A in [29], (21) does not guarantee that the operator belongs to the Schatten class of index 4 since the convergence of the series is required to hold not only for the eigenfunctions of the harmonic oscillator but for any orthonormal basis in our Hilbert space.

Acknowledgements

We are grateful to the anonymous referees for their remarks and for pointing out reference [28]. S. Fassari's contribution to this work has been made possible by the financial support granted by the Government of the Russian Federation through the ITMO University Fellowship and Professorship Programme. S. Fassari would like to thank Prof. Igor Yu. Popov and the entire staff at the Departament of Higher Mathematics, ITMO University, St. Petersburg for their warm hospitality throughout his stay. L. M. Nieto and C. San Millán acknowledge partial financial support to Junta de Castilla y León and FEDER (Project BU229P18).

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