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A Note on the Riemann $\xi-$Function.

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Abstract. This note investigates a number of integrals of and integral equations satisfied by Riemann's $\xi-$function and its integer powers. A somewhat less restrictive derivation of one of Riemann's identities is provided. The work centers on the critical strip and it is argued that the line $s = 3/2 + it$, e.g., contains a kind of holographic image of the critical region.

1. Introduction

The notation used throughout this note is:

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma(1+s/2)\zeta(s) \qquad (i)$$

$$\rho = \sigma + i\tau, \quad 0 < \sigma < 1 \qquad (ii)$$

$$\Xi(\tau) = \xi(1/2 + i\tau) \qquad (iii)$$

$$E_z(a) = \int_1^\infty \frac{dt}{t^z} e^{-at}, \qquad (iv)$$

$$\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} = \frac{1}{2}[\theta_3(0, e^{-\pi x}) - 1], \quad x > 0 \qquad (v)$$

$$J(\rho) = \int_0^1 dt[t^{\rho-2} + t^{(1-\rho)-2}]\psi(1/t^2) \qquad (vi).$$

γ denotes the contour consisting of the two parallel lines $[c - i\infty, c + i\infty]$, $[1 - c + i\infty, 1 - c - i\infty]$, where, unless indicated otherwise, $1 < c < 2$, which span the *critical strip* $0 < \sigma < 1$, $\rho_n = 1/2 + i\tau_n$ is the n-th zero of $\zeta(s)$ on the critical line in the upper half plane.

The function $\xi(s)$, $s \in \mathcal{C}$, introduced by Riemann[1], satisfies the simple functional equation $\xi(1-s) = \xi(s)$, is analytic, decays exponentially as $|s| \to \infty$ and possesses the

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same zeros in the critical strip as $\zeta(s)$. By Cauchy's theorem, one has, for $0 < Re[s] < 1$ and any positive integer k

$$\xi^k(s) = \int_\gamma \frac{dt}{2\pi i}\frac{\xi^k(t)}{t-s}, \tag{1}$$

which, in view of the functional equation, can be written

$$\xi^k(s) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\xi^k(t)\left(\frac{1}{t-s} + \frac{1}{t-1+s}\right) \tag{2}$$

and expresses the values of ξ^k inside the critical strip entirely in terms of its values in a region where $\zeta(s)$ is completely known from its defining series, say. Here, we concentrate on the case $k = 1$. In the following section the representation (2) will be exploited to obtain several known and some, perhaps, unfamiliar identities. In particular, a number of new integrals containing Riemann's function are evaluated, which should prove useful in further investigations.

Calculation

We begin by recalling the tabulated inverse Mellin transform[2]

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-t}\Gamma(t)\zeta(2t) = \sum_{n=1}^{\infty} e^{-n^2 x} \tag{3}$$

from which, by differentiation, one finds the useful inverse Mellin transform

$$F(x) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-t}\xi(t) = 4\pi^2 x^4 \sum_{n=1}^{\infty} n^4 e^{-\pi n^2 x^2} - 6\pi x^2 \sum_{n=1}^{\infty} n^2 e^{-\pi . n^2 x^2}, \tag{4}$$

Eq.(4) has been presented, in different form, by Patkowski[3] for example.

Parenthetically, we note that if f is integrable and odd, then $f(1-2t)\xi(t)$ is odd under $t \to (1-t)$ so that

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} f(1-2t)\xi(t) = \int_\gamma \frac{dt}{2\pi i} f(1-2t)\xi(t) = 0. \tag{5}$$

Thus, by making use of Romik's formulas[4] for the values of the Theta function $\theta_3(0, q)$ and its derivatives we have, from (4)

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\frac{\xi(t)}{t} = \frac{1}{2} - \frac{\Gamma(5/4)}{\sqrt{2}\pi^{3/4}} \qquad (6).$$

and from (5)

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\xi(t)\left[\frac{(1-2t)}{4\tau_n^2 + (1-2t)^2}\right] = 0 \tag{7}$$

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\xi(t)(2t-1)^{2n+1} = 0, \quad n = 0, 1, 2, \cdots .. \tag{8a}$$

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} t\xi(t) = \frac{1}{2}\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i}\xi(t) = \tag{8b}$$

$$\pi^2 \sum_{n=-\infty}^{\infty} n^4 e^{-\pi n^2} - \frac{3\pi}{2} \sum_{n=-\infty}^{\infty} n^2 e^{-\pi n^2} \tag{8c}$$

$$= \frac{\Gamma(5/4)}{128\sqrt{2}\pi^{19/4}}[\Gamma^8(1/4) - 96\pi^4]. \tag{8d}$$

$$\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \frac{\xi(t)}{t(1-t)} = \frac{1}{2}\left(1 - \frac{\pi^{1/4}}{\Gamma(3/4)}\right) = \int_{1-c-i\infty}^{1-c+i\infty} \frac{dt}{2\pi i} \frac{\xi(t)}{t(1-t)} \tag{9}$$

None of these appears to have been recorded previously.

Next, by rewriting (2), we have

Theorem 1

Within the critical strip Riemann's function $\xi(\rho)$ obeys the integral equation

$$\xi(\rho) = 1 - \frac{\pi^{1/4}}{2\Gamma(3/4)} - \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \frac{\xi(t)}{t} \left[\frac{2\rho(1-\rho) - t}{\rho(1-\rho) - t(1-t)}\right], \quad 1 < c < 2. \tag{10}$$

or

$$\xi(\rho) = \frac{1}{2} + \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \frac{\xi(t)}{t} \left[\frac{1}{1-t} - \frac{2\rho(1-\rho) - t}{\rho(1-\rho) - t(1-t)}\right]. \tag{11}$$

From this, one finds

Corollary 1

$$\xi(\rho) = 2\pi^2 \sum_{n=1}^{\infty} \int_1^{\infty} dt \left(t^{\rho/2} + t^{(1-\rho)/2}\right)\left(n^4 t - \frac{3}{2\pi} n^2\right) e^{-n^2\pi t}. \tag{12}$$

$$= \frac{\pi^{1/4}}{2\Gamma(3/4)} - \pi \sum_{n=1}^{\infty} n^2 \left[\rho E_{(1-\rho)/2}(\pi n^2) + (1-\rho) E_{-\rho/2}(\pi n^2)\right], \tag{13}$$

Eq. (10) is equivalent to the very important Eq(3.10) in Milgram's paper[5] and (13), apart from having summed a series, is LeClair's key formula (15) in [6]

To explore further consequences of (2), note that the Mellin transform

$$\phi(x) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-t} \frac{\xi(t)}{t-s} \tag{14}$$

satisfies the linear differential equation

$$\phi'(x) + \frac{s}{x}\phi(x) = -\frac{1}{x}F(x), \quad \phi(\infty) = 0 \tag{15}$$

where F is defined in (4), so after a bit of easy analysis

$$\phi(x) = 2\pi^2 \sum_{n=1}^{\infty} E_{\frac{x}{2}+1}(\pi n^2) - 3\pi \sum_{n=1}^{\infty} E_{\frac{x}{2}}(\pi n^2). \tag{16}$$

By applying (16) to (2) one has (note that τ here is not restricted to be real)

Theorem 2

In the critical strip, Riemann's function $\Xi(\tau)$ satisfies the integral equation

$$\Xi^k(\tau) = \frac{1}{\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{t\,\Xi^k(t)}{t^2-\tau^2} dt, \quad -3/2 < c < -1/2. \tag{17}$$
$$k = 1, 2, 3, \cdots$$

Corollary 2

For τ real and $x > 0$,

$$\Xi(\tau) = 4\pi^2 \sum_{n=1}^{\infty} \int_1^{\infty} dt\, t^{1/4} \cos(\tau \ln \sqrt{t}) \left(n^4 t - \frac{3}{2\pi} n^2 \right) e^{-n^2 \pi t}. \tag{17a}$$

$$= 4 \int_1^{\infty} dt\, t^{1/4} \cos[\frac{1}{2}\tau \ln t][t\psi''(t) + \frac{3}{2}\psi'(t)] \tag{17b}$$

$$= \frac{1}{2} - (\tau^2 + 1/4) \sum_{n=1}^{\infty} Re\, E_{\frac{3}{4}+i\frac{\tau}{2}}(\pi n^2) \tag{17c}$$

So

$$\Xi(\tau) = 1/2 - (\tau^2 + 1/4) \int_1^{\infty} \frac{dt}{t^{3/4}} \cos(\frac{\tau}{2} \ln t) \psi(t) \tag{18}$$

Now, (18), which appears in Riemann's paper[8], can be rewritten

$$\xi(\rho) = \frac{1}{2} - (\alpha + i\beta) \int_0^1 dt[t^{\rho-2} + t^{(1-\rho)-2}]\psi(1/t^2). \tag{19}$$

where $\alpha = \sigma(1-\sigma) + \tau^2$ and $\beta = (1-2\sigma)\tau$. For $\sigma = 1/2$, (18) gives

Corollary 3

ρ is a zero of $\zeta(s)$ on the critical line, $\sigma = 1/2$, if and only if,

$$Re \int_0^1 t^{\rho-2} \psi(1/t^2) dt = \frac{1}{4|\rho|^2}. \tag{20}$$

From the key result (17), for simplicity we choose $c = -1$,

Corollary 4

For ρ in the critical strip

$$\xi^k(\rho) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, \xi^k(3/2+it)\frac{1+it}{(1+it)^2-(\rho-\frac{1}{2})^2} \qquad (21)$$

$$k = 1, 2, 3, \cdots$$

Before leaving this section, we note that (17) is the source of a great number of fascinating integral identities found by multiplying both sides by a suitable function $g(\tau)$ and integrating over τ. In the next section a few of these are presented, omitting details.

Additional integrals

$$\int_0^{\infty} \cos(xt)\Xi(t)dt = \frac{1}{2}e^{-x}\int_{-\infty}^{\infty} e^{-ixt}\xi(3/2+it)dt, \quad x > 0 \qquad (a1),$$

so, from [2], for $x > 0$

$$\frac{d}{d\sigma}\int_{-\infty}^{\infty} \xi(\sigma+it)dt = 0, \quad 1/2 < \sigma <\leqslant \frac{3}{2}. \qquad (a2)$$

Similarly,

$$\int_0^{\infty} t^{s-1}\Xi(t)dt = \frac{1}{2\sin(s\pi/2)}\int_{-\infty}^{\infty} (1+it)^{s-1}\xi(3/2+it)dt, \quad 0 < s < 1. \qquad (a3)$$

while for $x = i\alpha$ (a3) reduces to the Hardy-Littlewood integral[1,(10.2.1)].

$$\int_0^{\infty} J_0(at)\Xi(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} \xi(3/2+it)I_0[a(1+it)]dt, \quad a > 0. \qquad (a4)$$

$$\int_0^{T} \Xi(t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, \xi(3/2+it)\tan^{-1}\left(\frac{T}{1+it}\right). \qquad (a5)$$

For $p > 0$,

$$\int_0^{\infty} \frac{\Xi(t)}{p+t}dt = \frac{1}{\pi}\int_{-\infty}^{\infty} dt\xi(3/2+it)\frac{1+it}{(1+it)^2+p^2}\left[\frac{p\pi}{2(1+it)} - \log[\frac{p}{1+it}]\right]. \qquad (a6)$$

$$\int_0^{\infty} \frac{\Xi(t)}{p^2+t^2}dt = \frac{1}{2p}\int_{-\infty}^{\infty} dt\frac{\xi(3/2+it)}{1+p+it}. \qquad (a7)$$

$$\int_{-\infty}^{\infty} \xi(z+it)dt = \sqrt{\frac{\sqrt{\pi}}{32}}\Gamma(1/4)\left[\frac{\Gamma^8(1/4)}{32\pi^4} - 3\right] = \int_{-\infty}^{\infty} \Xi(t)dt \qquad (a8)$$

What these examples have in common is that all information on the critical line is equivalent to information on the line $\sigma = 3/2$.

Discussion

The thrust of the preceding sections, and the key formulas have all been confirmed numerically with Mathematica, is that all the features of ξ in the critical strip are encoded on the lines $\sigma = c$, $1 < c < 2$ in a "holographic "manner made explicit by (21). This has the form

$$\xi(\rho) = \int \xi(3/2 + it)R(\sigma, \tau; t)dt \tag{22}$$

where R is a *rational* function. As functions go, $\xi(3/2 + it)$ is uncomplicated: it has no zeros or poles, it is infinitely differentiable and it decays exponentially. Its real and imaginary parts are even and odd, respectively, and related to each other by the Cauchy-Riemann equations. Both of the latter functions are, except near $t = 0$, almost featureless. Furthermore, the zeta function component is nearly equal to unity over most of the integration range. Thus any interesting feature in the critical strip must be ascribed largely to features of R which ought be easily analyzed.

Equation (20) (in essence due to Riemann) has been confirmed for the first 1000 critical zeros, which are available on Mathematica, in its exact form, but even if $\psi(x)$ is truncated to one exponential it is satisfied to many decimal place accuracy for large magnitude zeros. In this case, the integral is $E_{3/4+i\tau_n/2}(\pi)$ and by asymptotic expansion should be capable of producing a formula for τ_n similar to LeClair's[6] and Milgram's[7], but with less complexity. That is, to derive an expression for ρ_n, one expresses (20) in the form

$$Re\left[e^{-(\rho+1/2)\ln\sqrt{\pi}}\Gamma\left[\frac{1}{2}(\rho+\frac{1}{2})\right]+g(\rho)\right]=0 \tag{23a}$$

$$g(\rho) = \frac{\rho + 1/2}{|\rho + 1/2|^2}\ {}_1F_1(\rho + 1/2; \rho + 3/2; -\pi) - \frac{1}{4|\rho|^2} \tag{23b}$$

Now, along the critical line the real part of the function $g(\rho)$ in (23b) is non-oscillatory, monotonically decreasing and smaller than the accuracy we are trying to achieve, although much larger than the first term in (23a).which is oscillatory. However, in the spirit of [6] one ignores g and thus approximates (23a) as

$$Re\left[e^{-i\tau\ln(\sqrt{\pi})}\Gamma\left(\frac{1}{2}+i\frac{\tau}{2}\right)\right]=0 \tag{23c}$$

Following [6] one now applies Stirling's formula in (22) and solves for τ_n to obtain an analogue of LeClair's formula[6. (20)].

This note concludes with speculations on the Riemann hypothesis(RH) which claims that $\xi(\rho)$ does not vanish for $0 < \sigma < 1/2$. However, as remarked above, this is a question of the simultaneous vanishing of the real and imaginary parts of the integral (22) for which the zeta function itself seems to play a small role. For $\sigma = 1/2$, the imaginary part goes away and it is known that the real part has countably many roots τ_n. Otherwise, the situation appears to depend mainly on the nature of the rational function R, which depends on σ, more than on ξ, which does not.

Since the critical strip is known to be free of non-critical zeros to astronomical values of $|\rho|$, the resolution of this matter might be settled by extracting a low order asymptotic estimate of the Mellin transform $J(\rho)$ and analyzing the resulting algebraic equation. Since truncating ψ to a single term appears to yield accurate results for large n, it may be reasonable to conjecture:

The Riemann hypothesis is true if for large t the equation

$$\frac{(1-s)s+t^2}{\left((1-s)s+t^2\right)^2-(1-2s)^2t^2}-\Re\left(E_{-\frac{s}{2}-\frac{it}{2}+1}(\pi)+E_{\frac{1}{2}(s+it+1)}(\pi)\right)=0 \tag{26}$$

has no real solution t for $0 < s < 1/2$. However, such simple expedients tend to be illusory since no matter how small it is, a positive number is not zero, as exemplified by Lehmer's phenomenon[7].

One can, I believe, do better. From (21), n by noting that

$$f(\rho,t)=\frac{1+it}{(1+it)^2-(\rho-1/2)^2}=\int_0^\infty du\, e^{-(1+it)u}\cosh[(\rho-1/2)u], \tag{27}$$

one sees that for $\rho=\sigma+i\tau$ in the critical strip

$$\xi(\rho)=\frac{1}{\pi}\int_{-\infty}^{\infty}dt\,\xi(3/2+it)f(\rho,t)$$

$$=\frac{1}{\pi}\int_0^\infty du\, e^{-u}\cosh[(\rho-1/2)u]\int_{-\infty}^{\infty}dt\, e^{-iut}\xi(3/2+it), \quad u>0. \tag{28}$$

but,

$$\int_{-\infty}^{\infty}dt\, e^{-iut}\xi(3/2+it)=8\pi e^{-3u/2}[e^{-2u}\psi''(e^{-2u})+\frac{3}{2}\psi'(e^{-2u})]. \tag{29}$$

Therefore

$$\xi(\rho)=8\int_0^\infty du\, e^{-5u/2}[e^{-2u}\psi''(e^{-2u})+\frac{3}{2}\psi'(e^{-2u})]\cosh[(\rho-1/2)u]. \tag{30a}$$

and

$$\Xi(\tau)=8\int_0^\infty du\, e^{-5u/2}[e^{-2u}\psi''(e^{-2u})+\frac{3}{2}\psi'(e^{-2u})]\cos(\tau u). \tag{30b}$$

The above shows that

Theorem 3.
$\rho = \sigma + i\tau$ can be a zero of $\zeta(\rho)$ in the critical region only if

$$\int_0^\infty du\, Q(u) \cosh[(\sigma - 1/2)u] \cos(\tau u) \tag{31a}$$

and

$$\int_0^\infty du\, Q(u) \sinh[(\sigma - 1/2)u] \sin(\tau u) \tag{31b}$$

are simultaneously zero, where

$$Q(u) = e^{-5u/2} \left[e^{-2u} \psi''(e^{-2u}) + \frac{3}{2} \psi'(e^{-2u}) \right]. \tag{32}$$

A glance at the graph of $Q(u)$ shows that it is essentially zero for $u > 1$ where its value drops from about 2 at $u = 0$ to 5.51×10^{-7} so this range is of major concern. Furthermore, for $1/2 < \sigma < 1$ the magnitude of the oscillation of the integrand is larger for (3a1) than for (31b) hence the vanishing of (31a) and (31b) together for any value of τ is highly unlikely. For $\sigma = 1/2$, however, (31b) is eliminated and the critical zeros are the solutions to

$$\int_0^\infty Q(u) \cos(\tau u) du = 0.$$

Of course, Titchmarsh could have noted this by Fourier inversion of his integral, but did not. Therefore, in a sense (30) is merely a generalization of his oft-cited formula [2].

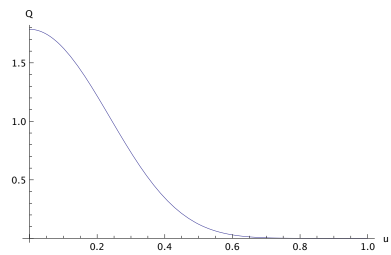

FIGURE 1. Plot of Q(u)

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