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Dirac-like Hamiltonians associated to Schrödinger factorizations

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Abstract In this work, we have extended the factorization method of scalar shape-invariant Schrödinger Hamiltonians to a class of Dirac-like matrix Hamiltonians. The intertwining operators of the Schrödinger equations have been implemented in the Dirac-like shape invariant equations. We have considered also another kind of anti-intertwining operators changing the sign of energy. The Dirac-like Hamiltonians can be obtained from reduction of higher dimensional spin systems. Two examples have been worked out, one obtained from the sphere $\mathcal{S}^2$ and a second one, having a non-Hermitian character, from the hyperbolic space $\mathcal{H}^2$.

1 Introduction

In this work, our starting point will be the one-dimensional Schrödinger equations that can be algebraically solved by the well known method of factorization [1]. This method has a long history of applications to quantum mechanics since the early times when it was already used by Dirac and Schrödinger [2,3]. The basis of the method can be described as the building of a sequence (or hierarchy) of shape invariant Schrödinger Hamiltonians connected by intertwining operators [4–6]. The knowledge of such intertwining operators and their annihilated functions solves the eigenvalue problem for all the eigenvalue equations in the Hamiltonian hierarchy. Our objective here is to widen the factorization method from the Schrödinger second order scalar equations to a kind of first order matrix Dirac-like equations. We will associate to each of these hierarchies of Schrödinger equations another hierarchy made up of matrix equations similar to Dirac Hamiltonians in one variable [7]. Note that L.M. Nieto et al extended the Darboux transformations (intertwining operators) for Schrödinger differential operators to the one dimensional Dirac equation in an exhaustive study [8]. However, we remark that in the present work our approach will be different and we will show other complementary applications of the matrix intertwining method. This type of Dirac-like factorization hierarchies will have also a kind of intertwining operators which

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correspond to those of the scalar case in a natural way. Let us remark that some particular cases of such Dirac-like Hamiltonian hierarchies have been found previously when solving the Dirac equation for some external magnetic fields [9–20]. In this work, we plan to undertake a systematic study of such a kind of matrix Hamiltonians.

Among the purposes of this work, we want to understand the origin of the "Dirac factorizations" from symmetry considerations. In fact, we will show that the class of Dirac-like Hamiltonians related to factorizable Schrödinger equations can be explained by means of a reduction of higher dimensional problems in flat and curved spaces. These initial systems must have symmetries, including spin, which allow the reducing process. Our program consist in the following steps. We will begin with a free Hamiltonian on a flat or curved space. If this Hamiltonian is scalar, it will lead, by reduction, to some of the Schrödinger factorized Hamiltonians. If that Hamiltonian is a kind of spin-orbit coupling then, after a reduction process, it will lead us to a particular kind of the Dirac-like Hamiltonians proposed for each factorization type. The intertwining operators of the reduced Dirac Hamiltonians will be recovered from the symmetries of the initial system, but we remark that there will appear new intertwining operators. We have restricted in this paper to some specific two-dimensional spaces, the sphere $\mathcal{S}^2$ and the hyperboloid $\mathcal{H}^2$, for the sake of simplicity and to keep it within a reasonable length, but this program can in principle be applied to all the factorization cases [1].

In a similar way as the Klein-Gordon equation is obtained by "squaring" the Dirac equation, we will have a similar property where the scalar second order Schrödinger factorized Hamiltonian is obtained by taking the square (in an appropriate way) of the associated Dirac-like Hamiltonian.

The organization of this work is as follows. In the second section, the factorization method is introduced briefly in order to set the notation of increasing and decreasing hierarchies used along the paper. The third section is devoted to Dirac-like Hamiltonians related with increasing hierarchies. Next, in the fourth section, an example on $\mathcal{S}^2$ is given and the connection with symmetry group reduction is examined. In the fifth section the same procedure is applied to find Dirac-like Hamiltonians for decreasing hierarchies which in this case are non-Hermitian [25]. Here, the space $\mathcal{H}^2$ will be used in order to get an example of this type of Dirac-like Hamiltonians. Finally, this work will end with some conclusions and remarks.

2 Hierarchies of 1D Schrödinger Hamiltonians

In the following sections we will specify how is the correspondence of 1D Schrödinger and Dirac Hamiltonian hierarchies of equations. But first, it is necessary to distinguish three cases of hierarchies of shape invariant Schrödinger equations (also referred as shape-invariant potentials). These are well known [1,4–6], but here we will briefly introduce them in order to set the notation.

- *Increasing hierarchy (or I-hierarchy)*
  Consider the Schrödinger hierarchy of second order Hamiltonians H_n which can be factorized as follows:

$$H_n = -\partial_{xx} + V_n(x) = a_n^+ a_n^- + \mu_n^2 = a_{n-1}^- a_{n-1}^+ + \mu_{n-1}^2, \qquad n = 0, 1, \ldots \quad (2.1)$$

  where $\partial_{xx} := \frac{d^2}{dx^2}$. Let us explain the standard notation here used. The first order operators $a_n^\pm$ are defined by

$$a_n^{\pm}(x) = \mp \partial_x + w_n(x) \tag{2.2}$$

Here, ∂_x is the derivative with respect to x, $\partial_x := \frac{d}{dx}$; the functions $w_n(x)$ are called super-potentials and, according to eq. (2.1), satisfy the following relations with the potential $V_n(x)$,

$$V_n(x) = w_n(x)^2 - w_n'(x) + \mu_n^2 = w_{n-1}(x)^2 + w_{n-1}'(x) + \mu_{n-1}^2 \tag{2.3}$$

where $w_n'(x) := \frac{dw_n(x)}{dx}$. The factorization energies μ_n^2 are assumed to be positive, but the following results can be also adapted to other factorization energies, as we will see later.

In all the hierarchies, in particular in this I-hierarchy, we will also assume that a_n^- are annihilation operators, in other words the equation

$$a_n^- \psi_n^0 = 0, \quad \implies \quad \psi_n^0(x) = K\,e^{-\int w_n(x)dx} \tag{2.4}$$

(where K is an integration constant) determines a square-integrable function $\psi_n^0(x)$. The function $\psi_n^0(x)$ corresponds to the ground state of H_n, while μ_n^2 is the ground state energy (this is consistent with the factorization (2.1)):

$$E_n^0 = \mu_n^2 \tag{2.5}$$

This sequence of Hamiltonians H_n has increasing ground energies. The key factorization property (2.1) implies that the operators a_n^- intertwine the consecutive Hamiltonians H_n and H_{n+1}:

$$a_n^- H_n = H_{n+1} a_n^-, \qquad H_n a_n^+ = a_n^+ H_{n+1} \tag{2.6}$$

The annihilation operators a_n^- go towards the Hamiltonians on the right (index n) but decrease the energy level (index k). The creation operators a_n^+ act in the opposite way:

$$a_n^- : \psi_n^k \;\to\; \psi_{n+1}^{k-1}, \qquad a_n^+ : \psi_{n+1}^{k-1} \;\to\; \psi_n^k \tag{2.7}$$

In fact,

$$a_n^- \psi_n^k = \sqrt{\mu_{n+k}^2 - \mu_n^2}\,\psi_{n+1}^{k-1}, \qquad a_n^+ \psi_{n+1}^{k-1} = \sqrt{\mu_{n+k}^2 - \mu_n^2}\,\psi_n^k \tag{2.8}$$

The k-excited eigenfunction ψ_n^k of H_n can be obtained from the ground eigenfunction ψ_{n+k}^0 of H_{n+k}, by means of creation operators:

$$\psi_n^k \propto a_n^+ a_{n+1}^+ \dots a_{n+k-1}^+ \psi_{n+k}^0, \qquad E_n^k = \mu_{n+k}^2 \tag{2.9}$$

An example of this increasing hierarchy type is the trigonometric Pöschl-Teller (PT) Hamiltonians as we will see later.

- **Decreasing hierarchy (or D-hierarchy)** Next, we start from a Schrödinger hierarchy of second order Hamiltonians H_n satisfying the following factorization property:

$$H_n = -\partial_{xx} + V_n(x) = a_n^+ a_n^- - \mu_n^2 = a_{n+1}^- a_{n+1}^+ - \mu_{n+1}^2, \qquad n = 0, 1, \dots \tag{2.10}$$

The first order operators $a_n^{\pm}$ are defined in the same way as (2.2). The superpotentials $w_n(x)$ and the potentials $V_n(x)$ are here related by

$$V_n(x) = w_n(x)^2 - w_n'(x) - \mu_n^2 = w_{n+1}(x)^2 + w_{n+1}'(x) - \mu_{n+1}^2 \tag{2.11}$$

The factorization energies are given by $-\mu_n^2$, where μ_n^2 are real numbers which we will assume to be positive. The annihilation operator a_n^- determines the ground state wavefunction $\psi_n^0(x)$ of H_n,

$$a_n^- \psi_n^0 = 0, \qquad E_n^0 = -\mu_n^2$$

where $-\mu_n^2$ is the negative ground state energy, in a similar way as in the previous positive case (2.4)–(2.5).

Here the sequence of Hamiltonians H_n have decreasing ground energies. In the D-hierarchies, the factorization property (2.10) implies that the operators $a_n^\pm$ intertwine the consecutive Hamiltonians, H_n and H_{n-1}, as follows:

$$a_n^- H_n = H_{n-1} a_n^-, \qquad H_n a_n^+ = a_n^+ H_{n-1} \tag{2.12}$$

The annihilation operators a_n^- go towards the Hamiltonians to the left (in index n) and decrease the energy level (index k). The creation operators a_n^+ go in the opposite way:

$$a_n^- : \psi_n^k \rightarrow \psi_{n-1}^{k-1}, \qquad a_n^+ : \psi_{n-1}^{k-1} \rightarrow \psi_n^k$$

$$a_n^- \psi_n^k = \sqrt{\mu_n^2 - \mu_{n-k}^2}\, \psi_{n-1}^{k-1}, \qquad a_n^+ \psi_{n-1}^{k-1} = \sqrt{\mu_n^2 - \mu_{n-k}^2}\, \psi_n^k \tag{2.13}$$

The k-excited eigenfunction ψ_n^k of H_n can be obtained from the ground eigenfunction ψ_{n-k}^0 of Hamiltonians to the left H_{n-k}, by means of creation operators:

$$\psi_n^k \propto a_n^+ a_{n-1}^+ \ldots a_{n-k+1}^+ \psi_{n-k}^0, \qquad E_n^k = E_{n-k}^0 = -\mu_{n-k}^2, \qquad k = 1, 2 \ldots \tag{2.14}$$

An example of this decreasing hierarchy type is the sequence of hyperbolic Pöschl-Teller Hamiltonians.

- **Equal hierarchy (or E-hierarchy)**.
  This is a hierarchy where all the factorizacion energies are equal:

$$\mu^2 = \mu_n^2, \qquad \forall n \tag{2.15}$$

In this case the spectrum of all the Hamiltonians H_n are the same (they are isospectral). The operators $a_n^\pm$ are neither annihilation nor creation, they could be called iso-operators. This case will not be considered in this work.

3 Hierarchies of 1D Dirac Hamiltonians: increasing case

Consider an increasing hierarchy of Schrödinger Hamiltonians H_n characterized by the factorization (2.1). Then, we can form a 2×2 matrix Hamiltonian hierarchy h_n defined by

$$h_n = \begin{pmatrix} \mu_n & i a_n^+ \\ -i a_n^- & -\mu_n \end{pmatrix} \tag{3.16}$$

We will refer to this first order matrix operator h_n as a Dirac-like Hamiltonian. This type of Dirac-like Hamiltonians have been obtained in a number of problems related with Dirac-Weyl systems in planar graphene, spherical surfaces or for other cases with cylindrical symmetry [9–15].

The matrix Hamiltonian h_n in (3.16) is explicitly Hermitian from a formal point of view. The square of h_n gives a pair of Schrödinger Hamiltonians of the scalar I-hierarchy in the

diagonal, therefore in this sense it mimics the Dirac Hamiltonians:

$$h_n^2 = \begin{pmatrix} H_n & 0 \\ 0 & H_{n+1} \end{pmatrix} := \boldsymbol{H}_n \tag{3.17}$$

Next, we will look for a pair of first order matrix differential (anti)intertwining operators $A_n^{\pm}$ between the matrix Hamiltonians h_n and h_{n+1} (similar to the operators $a_n^{\pm}$ between H_n and H_{n+1} in (2.6) for the I-hierarchy):

$$A_n^- h_n = \gamma \, h_{n+1} A_n^-, \qquad h_n A_n^+ = \gamma \, A_n^+ h_{n+1} \tag{3.18}$$

where γ is for a sign. As a consequence of (3.18), the operators $A_n^{\pm}$ will also intertwine h_n^2 and h_{n+1}^2:

$$A_n^- h_n^2 = h_{n+1}^2 A_n^-, \qquad h_n^2 A_n^+ = A_n^+ h_{n+1}^2, \tag{3.19}$$

where $A_n^{\pm} = (A_n^{\mp})^{\dagger}$. There are two solutions to this problem (up to a multiplicative constant):
a) True intertwining operators ($\gamma = 1$)

$$R_n^- = \begin{pmatrix} a_n^- & i(\mu_{n+1} - \mu_n) \\ 0 & a_{n+1}^- \end{pmatrix}, \qquad R_n^- h_n = h_{n+1} R_n^- \tag{3.20}$$

b) Anti-intertwining operators ($\gamma = -1$)

$$T_n^- = \begin{pmatrix} a_n^- & -i(\mu_{n+1} + \mu_n) \\ 0 & -a_{n+1}^- \end{pmatrix}, \qquad T_n^- h_n = -h_{n+1} T_n^- \tag{3.21}$$

One couple, $R_n^{\pm}$, plays the role of the usual intertwining operators; but there is a new pair $T_n^{\pm}$ carrying a minus sign that we call anti-intertwining. This possibility is specific of Dirac-like Hamiltonians where the spectrum includes a negative sector, as we will see in the next subsection.

3.1 Spectrum of the Dirac hierarchy $\boldsymbol{h_n}$

We will compute the eigenfunctions and eigenvalues of each Dirac Hamiltonian (3.16),

$$h_n \Psi_n^k = \varepsilon_n^k \Psi_n^k \tag{3.22}$$

where ε_n^k designs the kth excited level of the energy eigenvalues of Hamiltonian h_n and Ψ_n^k is a corresponding spinor eigenfunction. According to the square of h_n given in (3.17), we will express the spectrum and eigen-spinors of h_n (ε_n^k and Ψ_n^k) in terms of the eigenfunctions of the scalar Schrödinger Hamiltonians H_n (μ_{n+k}^2, ψ_n^k) and H_{n+1} (μ_{n+k+1}^2, ψ_{n+1}^k), respectively. We make the natural ansatz

$$\Psi_n^k = \begin{pmatrix} \alpha \, \psi_n^k \\ i\beta \, \psi_{n+1}^{k-1} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C} \tag{3.23}$$

where α, β must be computed by substituting in the eigenvalue equation for h_n defined in (3.16). There are two signs for the energy. Then, to specify the energy sign $\pm$ we will write $\varepsilon_n^{k\pm}$ and $\Psi_n^{k\pm}$ for the respective eigenfunctions. After some computations we get the following solutions (they are not normalized).

- Positive spectrum

$$\varepsilon_n^{k+} = \mu_{n+k}, \quad \Psi_n^{k+} = \begin{pmatrix} \sqrt{\mu_{n+k} + \mu_n}\, \psi_n^k \\ -i\sqrt{\mu_{n+k} - \mu_n}\, \psi_{n+1}^{k-1} \end{pmatrix}, \qquad k = 1, 2, \ldots$$

Ground level :   (3.24)

$$\varepsilon_n^{0+} = \mu_n, \qquad \Psi_n^{0+} = \begin{pmatrix} \psi_n^0 \\ 0 \end{pmatrix}, \qquad k = 0$$

- Negative spectrum

$$\varepsilon_n^{k-} = -\mu_{n+k}, \qquad \Psi_n^{k-} = \begin{pmatrix} \sqrt{\mu_{n+k} - \mu_n}\, \psi_n^k \\ i\sqrt{\mu_{n+k} + \mu_n}\, \psi_{n+1}^{k-1} \end{pmatrix}, \qquad k = 1, 2, \ldots \quad (3.25)$$

Notice some properties of the spectrum.

1. The positive and negative solutions $\Psi_n^{k\pm}$ are solutions of the square of h_n with the same positive eigenvalue:

$$(h_n)^2 \Psi_n^{k\pm} = \begin{pmatrix} H_n & 0 \\ 0 & H_{n+1} \end{pmatrix} \begin{pmatrix} \sqrt{\mu_{n+k} \pm \mu_n}\, \psi_n^k \\ \mp i\sqrt{\mu_{n+k} \mp \mu_n}\, \psi_{n+1}^{k-1} \end{pmatrix} = \mu_{n+k}^2\, \Psi_n^{k\pm} \qquad (3.26)$$

2. The ground level $k = 0$ exists only for positive energies, while the "ground negative" energy level is reached for $k = 1$.
3. The eigenfunctions for positive energy have the upper component with bigger norm than the lower one, while eigenfunctions of negative energy have oposite character: the norm of the lower component is bigger than the upper.
4. Eigenfunctions of positive and negative energies are orthogonal.

3.2 The spectrum for a null factorization energy (Dirac–Weyl like equation)

Up to now we have assumed that each Dirac Hamiltonian h_n, as it is defined in (3.16), has the factorization energy (it may play the role of mass) given by μ_n. However, we could be interested in the spectrum of a particular h_{n_0} with $n = n_0$, having a different value of the factorizacion energy, for instance $\mu_{n_0} = 0$, (it would correspond to a null mass case associated to a Dirac–Weyl like equation).

   In principle, the Schrödinger Hamiltonian hierarchy (increasing (2.1) or decreasing (2.10)) will have $\mu_{n_0}^2 \neq 0$. Then, we will subtract from all the Hamiltonian hierarchy $\{H_n = a_n^+ a_n^- + \mu_n^2\}$ the same value: $\mu_{n_0}^2 = E_{n_0}^0$. Hence, we will have the new sequence $\{\tilde{H}_n = a_n^+ a_n^- + \tilde{\mu}_n^2\}$ where

$$\tilde{\mu}_n^2 = \mu_n^2 - \mu_{n_0}^2$$

This new hierarchy has the same properties, intertwining, eigenfunctions, etc. except the spectrum that now will be displaced the same amount:

$$\tilde{E}_n^k = \tilde{E}_{n+k}^0 = \tilde{\mu}_{n+k}^2 = E_{n+k}^0 - E_{n_0}^0, \qquad \tilde{E}_{n_0}^0 = 0$$

In other words, the ground energy of the displaced Hamiltonian $\tilde{H}_{n_0}$ is zero and the corresponding Dirac Hamiltonian $\tilde{h}_{n_0}$ will have zero mass:

$$\tilde{h}_n = \begin{pmatrix} \tilde{\mu}_n & ia_n^+ \\ -ia_n^- & -\tilde{\mu}_n \end{pmatrix}, \qquad \tilde{h}_{n_0} = \begin{pmatrix} 0 & ia_{n_0}^+ \\ -ia_{n_0}^- & 0 \end{pmatrix}, \qquad \tilde{\mu}_{n_0} = 0 \qquad (3.27)$$

The formulas for the positive (3.24) or negative (3.25) spectrum $\tilde{\varepsilon}_n^{k\pm}$ and wave functions $\tilde{\Psi}_n^{k\pm}$ will still be valid, with the replacement of the new values of $\tilde{\mu}_n$ in their respective expressions.

3.3 Eigenfunctions annihilated by the intertwining operators

Next, we want to know what kind of eigenfunctions annihilate the intertwining operators R_n^- and T_n^-. Since they are first order linear 2×2-matrix differential operators, they must annihilate two linearly independent spinor functions.

- Eigenfunctions annihilated by R_n^-
  1) The ground state of positive energies:

$$\Psi_n^{0+} = \begin{pmatrix} \psi_n^0 \\ 0 \end{pmatrix}, \qquad \varepsilon_n^{0+} = \mu_n$$

  2) The "ground state" of negative energies:

$$\Psi_n^{1-} = \begin{pmatrix} \sqrt{\mu_{n+1} - \mu_n}\, \psi_n^1 \\ i\sqrt{\mu_{n+1} + \mu_n}\, \psi_{n+1}^0 \end{pmatrix}, \qquad \varepsilon_n^{1-} = -\mu_{n+1}$$

- Eigenfunctions annihilated by T_n^-
  1) The ground state of positive energies:

$$\Psi_n^{0+} = \begin{pmatrix} \psi_n^0 \\ 0 \end{pmatrix}, \qquad \varepsilon_n^{0+} = \mu_n$$

  2) The first excited state of positive energies:

$$\Psi_n^{1+} = \begin{pmatrix} \sqrt{\mu_{n+1} + \mu_n}\, \psi_n^1 \\ -i\sqrt{\mu_{n+1} - \mu_n}\, \psi_{n+1}^0 \end{pmatrix}, \qquad \varepsilon_n^{1+} = \mu_{n+1}$$

In conclusion, the matrix intertwining operators R_n^- and T_n^- annihilate some of lowest energy states of the Hamiltonians h_n, in a similar way as the factor operators a_n^- do for the ground states of H_n. However, in the matrix case there are more options due to the two sectors (positive and negative) of the spectrum and the fact that each intertwining operator annihilates two independent states.

3.4 Symmetries from intertwining operators

As we have seen in a precedent subsection, the spectrum of this type of Dirac Hamiltonians h_n has simple degeneracy: the eigenspaces are one-dimensional. This means that there are no additional independent symmetry, or in other words, any symmetry must be a function of the Hamiltonian h_n.

A method to build symmetries is by means of the product of two intertwining operators in opposite direction; for instance S_n obtained by the product $R_n^+ R_n^-$ will be a symmetry of h_n:

$$R_n^- h_n = h_{n+1} R_n^-, \qquad h_n R_n^+ = R_n^+ h_{n+1} \implies [S_n, h_n] = 0$$

Therefore, this symmetry must be a function of h_n. After a computation we find:

$$S_n = R_n^+ R_n^- = (h_n - \mu_n)(h_n + \mu_{n+1})$$

and the commutator is

$$R_n^+ R_n^- - R_{n-1}^- R_{n-1}^+ = h_n(\mu_{n+1} - 2\mu_n + \mu_{n-1}) + \mu_n(\mu_{n-1} - \mu_{n+1})$$

This result seems reasonable since we have seen before that the operator R_n^- annihilates eigenvectors with eigenvalues μ_n and $-\mu_{n+1}$ and the same happens with the operators on the right hand side of the last equality.

The symmetry obtained from the other intertwining (anti-intertwining) operator:

$$S_n' = T_n^+ T_n^-$$

will have a similar expression,

$$S_n' = (h_n - \mu_n)(h_n - \mu_{n+1})$$

and in this case the commutator has the form

$$T_n^+ T_n^- - T_{n-1}^- T_{n-1}^+ = -h_n(\mu_{n+1} + 2\mu_n + \mu_{n-1}) - \mu_n(\mu_{n-1} - \mu_{n+1})$$

This is consistent with the eigenvectors (and their eigenvalues) annihilated by T_n^- (see the previous subsection).

3.5 Intertwining operators for m_0-mass Dirac-like Hamiltonians

In this subsection, we will mention briefly how our approach to Dirac-like matrix Hamiltonians is compatible with a constant mass m_0.

Firstly, we define the 4×4 Dirac-like Hamiltonian $\mathcal{H}_n$ with mass m_0 from the previous h_n as follows [9,10,17]:

$$\mathcal{H}_n = \begin{pmatrix} m_0 c^2 & h_n \\ h_n & -m_0 c^2 \end{pmatrix} \tag{3.28}$$

Then, we can also introduce 4×4 intertwining operators for the hierarchy $\mathcal{H}_n$, in terms of the previous 2×2 operators, $R_n^\pm = (R_n^\mp)^\dagger$ and $T_n^\pm = (T_n^\mp)^\dagger$. In fact, we have two extensions for each pair having the following form (together with their adjoint operators and taking $c = 1$):

$$\mathcal{R}_n^- = \begin{pmatrix} R_n^- & 0 \\ 0 & R_n^- \end{pmatrix}, \qquad \tilde{\mathcal{R}}_n^- = \begin{pmatrix} -m_0 M^- & R_n^- \\ R_n^- & m_0 M^- \end{pmatrix} \tag{3.29}$$

$$\mathcal{T}_n^- = \begin{pmatrix} -T_n^- & 0 \\ 0 & T_n^- \end{pmatrix}, \qquad \tilde{\mathcal{T}}_n^- = \begin{pmatrix} m_0 M^- & -T_n^- \\ T_n^- & m_0 M^- \end{pmatrix} \tag{3.30}$$

where

$$M^- := -i\sigma^+ = -i(\sigma_1 + i\sigma_2)$$

is a constant matrix which satisfies the following (anti)commutation relations with the 2×2 Hamiltonians h_n:

$$M^- h_n + h_{n+1} M^- = -2 R_n^-, \qquad M^- h_n - h_{n+1} M^- = -2 T_n^-$$

It can be checked that all these global intertwining operators fulfil the usual type of intertwining relations, for example:

$$\mathcal{R}_n^- \mathcal{H}_n = \mathcal{H}_{n+1} \mathcal{R}_n^-, \qquad \mathcal{T}_n^- \mathcal{H}_n = \mathcal{H}_{n+1} \mathcal{T}_n^-$$

The adjoint operators $\mathcal{R}_n^+ = (\mathcal{R}_n^-)^\dagger$ and $\mathcal{T}_n^+ = (\mathcal{T}_n^-)^\dagger$ will satisfy the opposite intertwining relations. Finally, we can say that in fact we have obtained a four dimensional vector space of global intertwining operators for the same hierarchy of 4×4 Hamiltonians $\mathcal{H}_n$. In principle, this is surprising because in scalar Hamiltonian hierarchies there is only one set of factor intertwining operators for the same hierarchy. In our case, the degeneracy of the positive and negative energy levels of each Hamiltonian $\mathcal{H}_n$ allows that the intertwining operators be non unique.

Let us obtain the eigenfunctions and the spectrum of $\mathcal{H}_n$. The eigenvalue equation is

$$\mathcal{H}_n \, \Xi_n^{ks} = \mathcal{E}_{n\pm}^k \, \Xi_n^{ks} \tag{3.31}$$

where the energy is $\mathcal{E}_{n\pm}^k$ and each state Ξ_n^{ks} is composed of two eigen-spinors of h_n of same type:

$$\Xi_n^{ks} = \begin{pmatrix} \Psi_n^{ks} \\ \Phi_n^{ks} \end{pmatrix} \tag{3.32}$$

For instance, Ψ_n^{ks} corresponds to the eigenvalue ε_n^{ks} of the n-Hamiltonian h_n, k-energy level, and energy sign $s = \pm$. Using $\mathcal{H}_n$ in the eigenvalue equation, we get two coupled equations:

$$m_0 c^2 \Psi_n^{ks} + h_n \Phi_n^{ks} = \mathcal{E}_{n\pm}^k \Psi_n^{ks}, \qquad -m_0 c^2 \Phi_n^{ks} + h_n \Psi_n^{ks} = \mathcal{E}_{n\pm}^k \Phi_n^{ks} \tag{3.33}$$

From these equations, we obtain the following relations

$$\Phi_{n\pm}^{ks} = \frac{h_n \Psi_n^{ks}}{\mathcal{E}_{n\pm}^k + m_0 c^2}, \qquad \Psi_{n\pm}^{ks} = \frac{h_n \Phi_n^{ks}}{\mathcal{E}_{n\pm}^k - m_0 c^2} \tag{3.34}$$

Substituting these in (3.33), we get the equations for the spinors Ψ and Φ

$$h_n^2 \Psi_n^{ks} = ((\mathcal{E}_n^k)^2 - m_0^2 c^4) \Psi_n^{ks}, \qquad h_n^2 \Phi_n^{ks} = ((\mathcal{E}_n^k)^2 - m_0^2 c^4) \Phi_n^{ks} \tag{3.35}$$

taking into account $h_n^2 \Psi_n^{ks} = (\varepsilon_n^{ks})^2 \Psi_n^{ks}$, we get the eigenvalues

$$\mathcal{E}_{n\pm}^k = \pm \sqrt{(\varepsilon_n^{ks})^2 + m_0^2 c^4} = \pm \sqrt{\mu_{n+k}^2 + m_0^2 c^4} \tag{3.36}$$

where $E_n^k = (\varepsilon_n^k)^2 = \mu_{n+k}^2$, as shown in (3.26). Then, we find the eigenfunctions:

$$\Xi_{n+}^{ks} = \begin{pmatrix} \Psi_n^{ks} \\ \Phi_{n+}^{ks} \end{pmatrix}, \qquad \Phi_{n+}^{ks} = \frac{s \, \mu_{n+k} \Psi_n^{ks}}{\sqrt{\mu_{n+k}^2 + m_0^2 c^4} + m_0 c^2} \tag{3.37}$$

or

$$\Xi_{n-}^{ks} = \begin{pmatrix} \Psi_{n-}^{ks} \\ \Phi_n^{ks} \end{pmatrix}, \qquad \Psi_{n-}^{ks} = -\frac{s \, \mu_{n+k} \Phi_n^{ks}}{\sqrt{\mu_{n+k}^2 + m_0^2 c^4} + m_0 c^2} \tag{3.38}$$

We notice that the spectrum (3.36) for a non-vanishing m_0 is symmetric in the positive and negative eigenvalue sectors, contrary to h_n. The eigenvalues have double degeneracy $s = \pm$, except for the ground energy of each sector which has simple degeneracy.

4 Example: the Trigonometric Pöschl–Teller Potential

An example of a hierarchy of increasing type is the following sequence of PT Schrödinger Hamiltonians

$$H_n(x) = -\partial_{xx} + \frac{(n + 1/2)(n - 1/2)}{\sin^2 x}, \qquad x \in (0, \pi), \qquad n \in \mathbb{N} \qquad (4.39)$$

$H_n(x)$ satisfy the factorization given by (2.1). The factor operators and factor energies are

$$a_n^\pm(x) = \mp \partial_x - (n + 1/2) \cot x, \qquad \mu_n^2 = (n + 1/2)^2, \qquad n = 0, 1, 2, \dots \quad (4.40)$$

Therefore, the associated Dirac Hamiltonians h_n are defined, according to (3.16), by

$$h_n = \begin{pmatrix} n + 1/2 & i(-\partial_x - (n + 1/2) \cot x) \\ -i(+\partial_x - (n + 1/2) \cot x) & -n - 1/2 \end{pmatrix} \qquad (4.41)$$

Their square reproduce the Hamiltonians H_n and H_{n+1}, as it is given in (3.17). The explicit intertwining operators, given by (3.20) and (3.21), take here the expressions

$$R_n^- = \begin{pmatrix} \partial_x - (n + 1/2) \cot x & i \\ 0 & \partial_x - (n + 3/2) \cot x \end{pmatrix} \qquad (4.42)$$

and

$$T_n^- = \begin{pmatrix} \partial_x - (n + 1/2) \cot x & -i(2n + 2) \\ 0 & -(\partial_x - (n + 3/2) \cot x) \end{pmatrix} \qquad (4.43)$$

In the next subsection, we will explain the origin of the scalar and matrix hierarchies together with the (anti)intertwining operators.

4.1 Dirac-like equations for systems on the $\mathcal{S}^2$ sphere

Here we want to show how these two Hamiltonians, the scalar PT Schrödinger one (4.39) and the matrix PT Dirac (4.41), can be obtained by reduction of a scalar and a spinor particle systems respectively, defined on the $\mathcal{S}^2$ sphere. We will start with the scalar case, which is more or less well known, and then we will introduce the one of Dirac type, which is new up to our knowledge.

(a) Scalar system

Let us consider a system on the two-dimensional sphere $\mathcal{S}^2$ defined by the points $(x, y, z) \in \mathbb{R}^3$ satisfying

$$x^2 + y^2 + z^2 = 1$$

The $SO(3)$ group will act in the usual way on the scalar wavefunctions $\tilde{\psi}(\mathbf{x})$ defined on $\mathcal{S}^2$,

$$U(g)\tilde{\psi}(\mathbf{x}) = \tilde{\psi}(g^{-1}\mathbf{x}), \qquad g \in SO(3) \qquad (4.44)$$

If we use Cartesian (x, y, z) or spherical (r, θ, ϕ) coordinates

$$x = r \sin \theta \cos \phi, \qquad y = r \sin \theta \sin \phi, \qquad z = r \cos \theta$$

the Hermitian generators L_i, $i = 1, 2, 3$ (or L_x, L_y, L_z) of $SO(3)$ represented by (4.44) are given by:

$$L_x = -i(y\partial_z - z\partial_y) = i(\sin \phi \, \partial_\theta + \cos \phi \cot \theta \, \partial_\phi)$$

$$L_y = -i(z\partial_x - x\partial_z) = i(-\cos \phi \, \partial_\theta + \sin \phi \cot \theta \, \partial_\phi) \qquad (4.45)$$

$$L_z = -i(x\partial_y - y\partial_x) = i(-\partial_\phi)$$

We recall the notation of the lowering and raising angular momentum operators:

$$L_\pm = L_x \pm iL_y, \qquad L_\pm = e^{\pm i\phi}(\pm\partial_\theta + i \cot \theta \partial_\phi) \qquad (4.46)$$

such that

$$[L_z, L^\pm] = \pm L^\pm, \qquad [L^-, L^+] = -2L_z$$

The quadratic Casimir operator $\tilde{H} := \mathcal{C} = L_x^2 + L_y^2 + L_z^2$ plays the role of a free two-dimensional Hamiltonian in the space of wavefunctions defined on the sphere, with eigenvalues $\ell(\ell+1)$ corresponding to the eigenfunctions of each irreducible representation:

$$\tilde{H}\tilde{\psi}_\ell = \left(-\partial_{\theta\theta} - \cot \theta \partial_\theta - \frac{\partial_{\phi\phi}}{\sin^2 \theta}\right) \tilde{\psi}_\ell = \ell(\ell+1)\tilde{\psi}_\ell, \qquad \ell = 0, 1, \ldots \quad (4.47)$$

We will eliminate the term with the first partial derivative by means of the transformation

$$\tilde{\psi} = \frac{1}{\sqrt{\sin \theta}} \psi \quad \implies \quad \partial_\theta \rightarrow \partial_\theta - \frac{1}{2} \cot \theta; \quad \tilde{H} \rightarrow H - 1/4 \qquad (4.48)$$

After applying this transformation, we get the following Hamiltonian equation (with $H := \tilde{H} + 1/4$) in a explicitly Hermitian equivalent form

$$H\psi_\ell = \left(-\partial_{\theta\theta} - \frac{1/4 + \partial_{\phi\phi}}{\sin^2 \theta}\right) \psi_\ell = (\ell + 1/2)^2 \, \psi_\ell, \qquad \ell = 0, 1, \ldots \quad (4.49)$$

In summary, the above Schrödinger Hamiltonian can be identified, in the realization (4.48), with the $SO(3)$ Casimir operator (plus the constant $1/4$)

$$H = \boldsymbol{L}^2 + 1/4 \qquad (4.50)$$

If we take separated solutions, which are also eigenfunctions of the generator L_z defined in (4.45),

$$\psi_{\ell n}(\theta, \phi) = \psi_{\ell n}(\theta)e^{in\phi}, \qquad L_z\psi_{\ell n}(\theta, \phi) = n \, \psi_{\ell n}(\phi, \theta), \qquad |n| \leq \ell \qquad (4.51)$$

the component $\psi_{\ell n}(\theta)$ will satisfy the reduced equation

$$H_n\psi_{\ell n} = \left(-\partial_{\theta\theta} + \frac{(n + 1/2)(n - 1/2)}{\sin^2 \theta}\right) \psi_{\ell n} = (\ell + 1/2)^2 \, \psi_{\ell n}, \quad n = 0, \pm 1, \cdots \pm \ell$$

$$(4.52)$$

If we fix the eigenvalue by means of the parameter ℓ then, the Hamiltonians H_n having this eigenvalue are those parametrised by $n = 0, \pm 1, \ldots, \pm \ell$. The states span the $2\ell + 1$–dimensional representation "ℓ" of $SO(3)$ with eigenvalue $\varepsilon_{\ell n} = (\ell + 1/2)^2$.

On the other hand, once selected a value n, the spectrum of the reduced Hamiltonian H_n is given by the eigenvalues

$$E_n^k = (n + k + 1/2)^2, \quad k = 0, 1, \ldots, \quad n + k := \ell$$

Equation (4.52) is the same as the initial one (4.39) of the PT potential (with the variable θ instead of x) and the factorization energies μ_n^2 coincide with E_n^0. Therefore, we have the following connection between the eigenvalue notation $(E_n^k, \psi_n^k(\theta))$ and the notation of $SO(3)$ representations $(\varepsilon_{\ell n} = (\ell + 1/2)^2, \psi_{\ell n})$:

$$\begin{cases} H_n \psi_n^k = E_n^k \psi_n^k, \quad E_n^k = (n + k + 1/2)^2 \\ H_n \psi_{\ell n} = \varepsilon_{\ell n} \psi_{\ell n}, \quad \varepsilon_{\ell n} = (\ell + 1/2)^2 \end{cases} \implies n + k = \ell \implies \begin{cases} \psi_n^k = \psi_{(n+k)n} \\ \psi_{\ell n} = \psi_n^{\ell - n} \end{cases}$$

This relationship tell us that the eigenfunctions of this hierarchy H_n are essentially spherical harmonics (subjected to the above transformation):

$$\psi_n^k(\theta) e^{in\phi} \propto \sqrt{\sin\theta} \, Y_{\ell n}(\theta, \phi), \quad \ell = n + k \tag{4.53}$$

In particular, for $k = 0$ have the form:

$$\psi_n^0(\theta) e^{in\phi} \propto \sqrt{\sin\theta} \, Y_{nn}(\theta, \phi)$$

If we apply the transformation (4.48) to the orbital angular operators (4.46), they become

$$L_+ = e^{i\phi}\Big(\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\Big), \quad L_- = \Big(-\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\Big)e^{-i\phi} \tag{4.54}$$

where now it is explicit the adjoint relation $(L_+)^\dagger = L_-$. If we allow them to act on the separated solutions (4.51) we find the reduced expressions

$$L_n^+ = \partial_\theta - \cot\theta(1/2 + n), \quad L_n^- = -\partial_\theta - \cot\theta(1/2 + n) \tag{4.55}$$

which coincide with the intertwining operators a_n^- (corresponds to L_n^+) and a_n^+ (to L_n^-), given in (4.40).

Next, we want to carry out a similar construction of the Dirac-like Hamiltonian (4.41) by means of a spinor realization of $SO(3)$ (or better, its universal covering $SU(2)$) on the sphere $\mathcal{S}^2$.

(b) Spinor system

We will consider an spinor system defined on the unit sphere by means of two-component wavefunctions $\Psi(\mathbf{x})$, $\mathbf{x} \in \mathcal{S}^2$. The action of the group $SU(2)$ on a function of this space has the usual form

$$U(g)\Psi(\mathbf{x}) = D(g)\Psi(g^{-1}\mathbf{x}), \quad g \in SU(2)$$

where D is for the spin $1/2$ fundamental representation of $SU(2)$, and the action of $g \in SU(2)$ on $\mathcal{S}^2$ is through the homomorphism of $SU(2)$ onto $SO(3)$. The generators corresponding to this action are given by the total rotations (orbital plus spin):

$$J_i = L_i(\theta, \phi) + \frac{1}{2}\sigma_i := L_i(\theta, \phi) + S_i, \quad i = 1, 2, 3$$

with σ_i ($\sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z$) being the three 2×2 Pauli spin matrices. These operators close the $su(2) \approx so(3)$ Lie algebra. In this case, the Casimir operator is

$$\mathcal{C}_s = \mathbf{J}^2 = \mathbf{L}^2 + 2\mathbf{L}\mathbf{S} + \mathbf{S}^2 \tag{4.56}$$

It is clear that L^2 and S^2 are also Casimir operators for the spinor system (both commute with all the generators J_i). Therefore, another Casimir operator commuting with the operators J is

$$\tilde{h} := \mathcal{C}_{so} = 2LS = 2\left(\sum_k L_k S_k\right) \tag{4.57}$$

which has the form of a spin-orbit coupling term. Notice that this is a first order Casimir in the momentum operators and it has a matrix character due to S; by these reasons we will identify this operator as a Dirac-like Hamiltonian $\tilde{h}$.

In order to simplify some calculations, we will make use of the lowering and raising spin operators:

$$S_\pm = S_x \pm i S_y, \quad S_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then, we have another useful expression of $\tilde{h}$ defined by (4.57) in terms of $L_\pm$ (given in (4.54)) and $S_\pm$:

$$\tilde{h} = L_- S_+ + L_+ S_- + 2L_z S_z \tag{4.58}$$

After replacement of the operators in (4.58), the final explicit expression for $\tilde{h}$ is

$$\tilde{h} = \begin{pmatrix} -i\partial_\phi & \left(-\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right)e^{-i\phi} \\ e^{i\phi}\left(\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right) & i\partial_\phi \end{pmatrix} \tag{4.59}$$

Since $\tilde{h}$ commutes with the generators J, the eigenfunctions Ψ_{jm} can be labeled by the parameters j and m corresponding to the eigenvalues of J^2 and J_z:

$$J^2\Psi_{jm} = j(j+1)\Psi_{jm}, \quad J_z\Psi_{jm} = m\Psi_{jm} \tag{4.60}$$

If we apply the matrix Hamiltonian (4.59) on the eigenfunctions of the generator J_z, with undetermined value of J^2,

$$J_z\Psi_m = m\Psi_m \implies \Psi_m(\theta, \phi) = \begin{pmatrix} \varphi_m(\theta)e^{i(m-1/2)\phi} \\ \chi_m(\theta)e^{i(m+1/2)\phi} \end{pmatrix} \tag{4.61}$$

then, we get the reduced Hamiltonian in the θ variable

$$\tilde{h}_m = \begin{pmatrix} m - 1/2 & -(\partial_\theta + m\cot\theta) \\ -(-\partial_\theta + m\cot\theta) & -m - 1/2 \end{pmatrix} \tag{4.62}$$

which is similar to h_n proposed at the beginning of this section in (4.41); in fact,

$$\tilde{h}_m + 1/2 = h_n, \quad m = n + 1/2 \tag{4.63}$$

We must have in mind that, according to its definition m is a pure half-integer, i.e. $m-1/2 = n$ and $m+1/2 = n+1$, are integer values of the orbital angular momentum L_z corresponding to the spinor components in (4.61). We will also use the notation of (4.63) for non reduced Hamiltonians: $\tilde{h} + 1/2 = h$.

We can also compute the square of $\tilde{h}$ with the aim of recovering the Schrödinger equation (4.49):

$$(2\boldsymbol{LS})^2 = \boldsymbol{L}^2 - 2\boldsymbol{LS} \implies \left(2\boldsymbol{LS} + \frac{1}{2}\right)^2 = \boldsymbol{L}^2 + \frac{1}{4} := \boldsymbol{H} \tag{4.64}$$

In other words, we must choose $h = \tilde{h} + 1/2$, where $\tilde{h}$ is defined in (4.59) and square it as in (4.64) to get

$$h^2 = \boldsymbol{H} \tag{4.65}$$

Therefore, by taking the square (in this modified way) of the Dirac operator h, we get the scalar Hamiltonian (4.49), just in a similar manner as from the Dirac equation we can get the Klein-Gordon equation.

In particular, if we apply this formula to the Hamiltonian $\tilde{h}_m$ in (4.62), with $m = n + 1/2$, we get the scalar reduced Hamiltonians H_n and H_{n+1} in a diagonal matrix, as it was shown in (4.52),

$$\left(\tilde{h}_m + \frac{1}{2}\right)^2 = \begin{pmatrix} H_n & 0 \\ 0 & H_{n+1} \end{pmatrix}, \qquad m = n + 1/2 \tag{4.66}$$

We are always making use of the relation $n = m - 1/2$ and $n + 1 = m + 1/2$ for the spinor components.

In this way, we have explained the origin of the first order Dirac-like matrix Hamiltonians h_n corresponding to the scalar Hamiltonians H_n from the factorization method [1]. Next, we will determine the intertwining operators of the sequence h_n by means of the symmetries of h.

(c) Symmetries and "anti-symmetries"

1) Symmetries $J_\pm$, J_z.

   In the following we will compute the symmetries corresponding to the lowering and raising operators $J_\pm$ for the eigenfunctions Ψ_{jm} of J_z and $\boldsymbol{J}^2$. Taking into account the expressions of (4.54), we get

$$J_+ = L_+ + S_+ = \begin{pmatrix} e^{i\phi}\left(\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right) & 1 \\ 0 & e^{i\phi}\left(\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right) \end{pmatrix}$$

$$J_- = L_- + S_-$$
$$= \begin{pmatrix} \left(-\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right)e^{-i\phi} & 0 \\ 1 & \left(-\partial_\theta + \cot\theta(-1/2 + i\partial_\phi)\right)e^{-i\phi} \end{pmatrix}$$

When J_+ act on Ψ_{jm} and J_- on Ψ_{jm+1} their effect on the components $\varphi_m(\theta)$ and $\chi_m(\theta)$ of (4.61) turns into

$$J_m^+ = \begin{pmatrix} \partial_\theta - m\cot\theta & 1 \\ 0 & \partial_\theta - (m+1)\cot\theta \end{pmatrix} ;$$

$$J_m^- = \begin{pmatrix} -\partial_\theta - m\cot\theta & 0 \\ 1 & -\partial_\theta - (m+1)\cot\theta \end{pmatrix}$$

These expressions are related with the previous ones shown in (4.42) by means of a simple equivalence ($R_n^\pm \approx J_m^\mp$, $m = n + 1/2$). We should also include the symmetry $J_z = L_z + S_z$ which is used to define the basis of eigenvectors Ψ_{jm} (4.60).

2) Anti-symmetries $T^\pm$, T_3. We will also find the anti-symmetries $T^\pm$ which anti-commute with the Hamiltonian h and behave like shift operators of J_z:

$$T^\pm(\tilde{h} + 1/2) = -(\tilde{h} + 1/2)\, T^\pm, \qquad [J_z, T^\pm] = \pm T^\pm$$

Due to their anti-commutation with $\tilde{h} + 1/2$, the operators $T^\pm$ change a positive energy state of $\tilde{h} + 1/2$ into another one with the same absolute but opposite sign energy. It can be checked that such anti-symmetries are given by

$$T^+ = L_+ S_3 - L_3 S_+, \qquad (T^+)^\dagger = T^- = L_- S_3 - L_3 S_-$$

where $S_3 = S_z$ and $L_3 = L_z$. When they act on the basis Ψ_{jm}, $T^\pm$ acquire an equivalent expression to that given in (4.43). Besides the above anti-symmetry operators, we must include a third one, called T_3 defined by:

$$T_3 = -\frac{1}{2}(L_+ S_- - L_- S_+)$$

We can also check that the set of anti-symmetries $T_\pm$, T_3 constitute a vector operator under the action of $\boldsymbol{J}$; in other words, they satisfy the commutation rules

$$[J_+, T^+] = 0, \quad [J_+, T^-] = 2T_3, \quad [J_\pm, T_3]$$
$$= \mp T^\pm, \quad [J_3, T^\pm] = \pm T^\pm, \quad [J_3, T_3] = 0$$

The operators $T^\pm$, T_3 are "odd" operators in the sense that connect the spaces of positive and negative energies states. The pair $T^\pm$ besides changing the energy sign, they also change the $J_3 = J_z$ eigenvalue m, but T_3 keeps m invariant. All of them change the eigenvalue j of $\boldsymbol{J}^2$ to $j - 1$ or to $j + 1$ if the initial space was of positive or negative energies, respectively. This is consistent with the following commutation rule,

$$[\boldsymbol{J}^2, T^+] = 2(\tilde{h} + 1/2)T^+$$

The existence of anti-symmetry operators does not imply that the positive and negative eigenspaces are completely symmetric, as we will see later. Notice that this type of nontrivial anti-symmetry operators is a remarkable property of this kind of Dirac-like Hamiltonians.

(d) Spectrum and eigenfunctions

In this subsection, we will compute the discrete spectrum and eigenfunctions in the same way as we have seen above. Later on we will identify the eigenfunctions in relation with the symmetries.

First of all we choose the eigenfunctions with a definite value of the momentum J_z given by (4.61). As we have seen in (4.66), the square of $\tilde{h}_m + 1/2$ is a diagonal matrix with H_n and H_{n+1} in the diagonal. These two scalar Hamiltonians have the same eigenvalues $E_n^k = (n + k + 1/2)^2$ for the respective eigenfunctions ψ_n^k and ψ_{n+1}^{k-1}. Therefore, this suggest the following form of the spinor eigenfunctions of $\tilde{h}_m + 1/2$:

$$\Psi_n^k = \begin{pmatrix} \alpha\, \psi_n^k(\theta)e^{i(m-1/2)\phi} \\ \beta\, \psi_{n+1}^{k-1}(\theta)e^{i(m+1/2)\phi} \end{pmatrix}, \qquad m = n + 1/2, \quad k = 0, 1, 2 \ldots$$

where the constants $\alpha, \beta \in \mathbb{C}$ must be found from the eigenvalue equation:

$$(\tilde{h}_m + 1/2)\Psi_n^k = \tilde{\varepsilon}\,\Psi_n^k, \qquad m = n + 1/2$$

We have positive energy solutions

$$\tilde{\varepsilon}_n^{k+} = +(n + k + 1/2), \quad \Psi_n^{k+}$$
$$= \begin{pmatrix} \alpha_n^{k+}\,\psi_n^k(\theta)e^{i(m-1/2)\phi} \\ \beta_n^{k+}\,\psi_{n+1}^{k-1}(\theta)e^{i(m+1/2)\phi} \end{pmatrix}, \qquad m = n + 1/2, \quad k = 0, 1, 2 \ldots$$

and negative energy solutions

$$\tilde{\varepsilon}_n^{k-} = -(n + k + 1/2), \quad \Psi_n^{k+} = \begin{pmatrix} \alpha_n^{k-}\,\psi_n^k(\theta)e^{i(m-1/2)\phi} \\ \beta_n^{k-}\,\psi_{n+1}^{k-1}(\theta)e^{i(m+1/2)\phi} \end{pmatrix},$$
$$m = n + 1/2, \quad k = 1, 2 \ldots$$

with eigenfunctions determined by the coefficients

$$\alpha_n^{k+} = \sqrt{2n + k + 1}, \ \beta_n^{k+} = -\sqrt{k}, \qquad \alpha_n^{k-} = \sqrt{k}, \ \beta_n^{k-} = \sqrt{2n + k + 1}$$

The zero ground energy level is positive:

$$\tilde{\varepsilon}_n^{0+} = +(n + 1/2)$$

and its wave function is

$$\Psi_n^{0+}(\theta, \phi) = \begin{pmatrix} \psi_n^0(\theta)e^{i(m-1/2)\phi} \\ 0 \end{pmatrix}$$

Next, we will interpret the eigenfunctions $\Psi_n^{k\pm}$ and their spinor components in terms of the commuting symmetries of $\tilde{h}$: J_z, L^2 and J^2.

J_z: The label n of the eigenfunctions $\Psi_n^{k\pm}$ means that, in fact they are eigenfunctions of J_z with eigenvalue $m = n + 1/2$:

$$J_z\Psi_n^{k\pm} = m\Psi_n^{k\pm}, \qquad m = n + 1/2$$

L^2: We know from the previous scalar case that the scalar functions $\psi_n^k(\theta)e^{i(m-1/2)}$ and $\psi_{n+1}^{k-1}(\theta)e^{i(m+1/2)}$ (which are de components of the spinors $\Psi_n^{k\pm}$) are eigenfunctions of L^2 with eigenvalue "$\ell(\ell + 1)$".

Therefore, the spinor eigenfunctions of $\tilde{h}$ are also eigenfunctions of L^2 with the same eigenvalue

$$(L^2 + 1/4)\Psi_n^{k\pm} = (\ell + 1/2)^2\Psi_n^{k\pm}, \qquad \ell = n + k, \quad m = n + 1/2$$

J^2: The above eigenfunctions of $\tilde{h}$ are also eigenfunctions of J^2. We will make use of expression (4.56) in order to see this:

$$J^2\Psi_n^{k+} = \left(L^2 + \tilde{h} + S^2\right)\Psi_n^{k+} = (\ell(\ell + 1) + \ell + 3/4)\Psi_n^{k+} = (\ell + 1/2)(\ell + 3/2)\Psi_n^{k+}$$

In other words, it can be written as

$$J^2\Psi_n^{k+} = (j^+)(j^+ + 1)\Psi_n^{k+}, \qquad j^+ = \ell + 1/2$$

In the same way, we check that

$$\boldsymbol{J}^2 \Psi_n^{k-} = (j^-)(j^- + 1)\Psi_n^{k+}, \qquad j^- = \ell - 1/2$$

We could explain these results from the fact that the spinor wave functions Ψ belong to tensor products of the representations "ℓ" from the orbital part and $s = 1/2$ from spin: $\Psi \in \ell \otimes 1/2$. As we know (from the composition of angular momenta):

$$\ell \otimes 1/2 = j^+ \oplus j^-, \qquad j^{\pm} = \ell \pm 1/2$$

Then, the positive energy spinors Ψ_n^{k+} belong to the representation of the total angular momentum j^+, while the negative spinors Ψ_n^{k-} to j^-.

In summary, the eigenfunctions of $\tilde{h} + 1/2$ could be labeled in two equivalent ways with the notations $\Psi_n^{k\pm}$ and $\Psi_{j\pm\ell m}$:

$$\Psi_n^{k+} = \Psi_{j^+\ell m}; \; \tilde{\varepsilon}_n^{k+} = \ell + 1/2, \qquad \ell = n + k, \quad m = n + 1/2, \quad j^+ = \ell + 1/2$$

$$\Psi_n^{k-} = \Psi_{j^-\ell m}; \; \tilde{\varepsilon}_n^{k-} = -(\ell + 1/2), \; \ell = n + k, \quad m = n + 1/2, \quad j^- = \ell - 1/2$$

As a consequence, this result confirms us that the degeneracy of positive energy $\tilde{\varepsilon}_n^{k+}$ levels is greater than the corresponding negative level $\tilde{\varepsilon}_n^{k-}$. This is due to the fact that the positive eigenspace is the support of the representation $j^+ = \ell + 1/2$, whose dimension is $2\ell + 2$, while the negative eigenspace supports j^- of dimension 2ℓ.

5 Hierarchies of 1D Dirac Hamiltonians: decreasing case

Next, we will carry out this program for a decreasing Hamiltonian hierarchy H_n characterized by the factorization

$$H_{n-1} = -\partial_{xx} + V_n(x) = a_{n-1}^+ a_{n-1}^- - \mu_{n-1}^2 = a_n^- a_n^+ - \mu_n^2, \qquad n = 1, 2, \ldots \quad (5.67)$$

which was given in (2.10). Then, we can form a Dirac-like Hamiltonian hierarchy h_n defined by

$$h_n = \begin{pmatrix} \mu_n & ia_n^- \\ ia_n^+ & -\mu_n \end{pmatrix} \qquad (5.68)$$

The square of h_n gives a pair of Schrödinger Hamiltonians of the D-hierarchy, with an extra minus sign

$$h_n^2 = -\begin{pmatrix} H_{n-1} & 0 \\ 0 & H_n \end{pmatrix} := -\boldsymbol{H}_n \qquad (5.69)$$

Thus, we see that the Dirac-like Hamiltonians (5.68) in this case have a relevant difference with the increasing hierarchy: they are non-Hermitian. In fact, we could say that they are σ_3-Hermitian [25,26] in the sense that

$$h_n = \sigma_3 h_n^{\dagger} \sigma_3 := h_n^{\sharp} \qquad (5.70)$$

However, since σ_3 commutes with h_n^2 given in (5.69) we find that this is Hermitian: $(h_n^2)^{\dagger} = h_n^2$.

Next, we are looking for a pair of intertwining operators $A_n^{\pm}$ between the matrix Hamiltonians h_n and h_{n-1} in the same way as (2.12) for the scalar D-hierarchy:

$$A_n^- h_n = \gamma h_{n-1} A_n^-, \qquad h_n A_n^+ = \gamma A_n^+ h_{n-1} \tag{5.71}$$

where $A_n^{\pm} = \sigma_3 (A_n^{\mp})^{\dagger} \sigma_3$ and γ is for a sign. As a consequence, the operators $A_n^{\pm}$ will intertwine h_n^2 and h_{n-1}^2:

$$A_n^- h_n^2 = h_{n-1}^2 A_n^-, \qquad h_n^2 A_n^+ = A_n^+ h_{n-1}^2 \tag{5.72}$$

The solution for the case of intertwining operators ($\gamma = 1$) is

$$R_n^- = \begin{pmatrix} a_{n-1}^- & 0 \\ i(\mu_{n-1} - \mu_n) & a_n^- \end{pmatrix}, \qquad R_n^- h_n = h_{n-1} R_n^- \tag{5.73}$$

While the anti-intertwining operators ($\gamma = -1$) are $T_n^{\pm}$:

$$T_n^- = \begin{pmatrix} a_{n-1}^- & 0 \\ i(\mu_{n-1} + \mu_n) & -a_n^- \end{pmatrix}, \qquad T_n^- h_n = -h_{n-1} T_n^- \tag{5.74}$$

5.1 Spectrum of the Dirac hierarchy for decreasing case

The spectrum and spinor eigenfunctions of the Hamiltonian h_n (5.68) can also be found following the same procedure as before. We get the solutions:

- Positive spectrum

$$\varepsilon_n^{k+} = \mu_{n-k}, \quad \Psi_n^{k+} = \begin{pmatrix} \sqrt{\mu_n + \mu_{n-k}}\, \psi_{n-1}^{k-1} \\ i\sqrt{\mu_n - \mu_{n-k}}\, \psi_n^k \end{pmatrix}, \qquad k = 1, 2, \ldots \tag{5.75}$$

- Negative spectrum

$$\varepsilon_n^{k-} = -\mu_{n-k}, \quad \Psi_n^{k-} = \begin{pmatrix} \sqrt{\mu_n - \mu_{n-k}}\, \psi_{n-1}^{k-1} \\ i\sqrt{\mu_n + \mu_{n-k}}\, \psi_n^k \end{pmatrix}, \qquad k = 1, 2, \ldots$$

Ground level : $\tag{5.76}$

$$\varepsilon_n^{0-} = -\mu_n, \quad \Psi_n^{0-} = \begin{pmatrix} 0 \\ \psi_n^0 \end{pmatrix}, \qquad k = 0$$

Remark that in this Dirac-like version of the scalar D-hierarchies the "absolute" ground state has negative energy. Now, the role of antiparticle states are played by the positive energy spectrum. This is an example of non-Hermitian matrix Hamiltonian, as it is explicit in (5.68), where the discrete energy levels are real and the eigenfunctions square integrable.

5.2 Eigenfunctions annihilated by the intertwining operators

Next, we want to know what kind of eigenfunctions annihilate the intertwining operators R_n^- and T_n^-.

- Eigenfunctions annihilated by R_n^-

1) The ground state of negative energies:

$$\Psi_n^{0-} = \begin{pmatrix} 0 \\ \psi_n^0 \end{pmatrix}, \qquad \varepsilon_n^{0-} = -\mu_n$$

2) The "ground state" of positive energies:

$$\Psi_n^{1+} = \begin{pmatrix} \sqrt{\mu_n + \mu_{n-1}}\,\psi_{n-1}^0 \\ i\,\sqrt{\mu_n - \mu_{n-1}}\,\psi_n^1 \end{pmatrix}, \qquad \varepsilon_n^{1+} = \mu_{n-1}$$

- Eigenfunctions annihilated by T_n^-
  1) The ground state of negative energies:

$$\Psi_n^{0-} = \begin{pmatrix} 0 \\ \psi_n^0 \end{pmatrix}, \qquad \varepsilon_n^{0-} = -\mu_n$$

2) The first excited state of negative energies:

$$\Psi_n^{1-} = \begin{pmatrix} \sqrt{\mu_n - \mu_{n-1}}\,\psi_{n-1}^0 \\ i\,\sqrt{\mu_n + \mu_{n-1}}\,\psi_n^1 \end{pmatrix}, \qquad \varepsilon_n^{1-} = -\mu_{n-1}$$

5.3 Example: the hyperbolic Pöschl–Teller potential

The following sequence of hyperbolic PT Schrödinger Hamiltonians

$$H_n(x) = -\partial_{xx} - \frac{(n+1/2)(n-1/2)}{\cosh^2 x}, \qquad x \in (-\infty, \infty), \qquad n \in \mathbb{N} \qquad (5.77)$$

are of decreasing type and satisfy the factorization (2.10). The factor operators and factor energies are

$$a_n^\pm(x) = \mp\partial_x + (n-1/2)\tanh x, \qquad \mu_n^2 = (n-1/2)^2, \qquad n = 1, 2, \dots \qquad (5.78)$$

Therefore, the associated Dirac Hamiltonians h_n are defined by

$$h_n = \begin{pmatrix} (n-1/2) & i\,(\partial_x + (n-1/2)\tanh x) \\ i\,(-\partial_x + (n-1/2)\tanh x) & -(n-1/2) \end{pmatrix} \qquad (5.79)$$

and the intertwining operators by

$$R_n^+ = \begin{pmatrix} -\partial_x + (n-3/2)\tanh x & -i \\ 0 & -\partial_x + (n-1/2)\tanh x \end{pmatrix} \qquad (5.80)$$

$$T_n^+ = \begin{pmatrix} -\partial_x + (n-3/2)\tanh x & i\,(2n-2) \\ 0 & -(-\partial_x + (n-1/2)\tanh x) \end{pmatrix} \qquad (5.81)$$

5.4 Dirac-like equations for systems on the $\mathcal{H}^2$ hyperboloid

We can get the hyperbolic PT scalar Hamiltonian (5.77) and the matrix PT Hamiltonian corresponding to (5.79) from separation of variables of a system defined on a hyperboloid in a similar way as that employed in the trigonometric case in the space of functions on the sphere. In order to get the potential in (5.77), we start with one sheet hyperboloid (this choice is different from the two sheets hyperboloid and it has some peculiarities). We have made

this election in order to work with representations of eigenfunctions in the angular ϕ variable and to obtain the specific form of the hyperbolic potential (5.77).

(a) The scalar system

Consider the hyperbolic surface $\mathcal{H}^2$,

$$x^2 + y^2 - z^2 = 1 \tag{5.82}$$

parametrised with pseudo-spherical coordinates:

$$x = \cosh \chi \cos \phi, \qquad y = \cosh \chi \sin \phi, \qquad z = \sinh \chi \tag{5.83}$$

The surface (5.82) is invariant under the group $SO(2, 1)$ and the scalar wavefunctions defined on $\mathcal{H}^2$ constitute the support space of a quasi-regular representation U,

$$U(g)\tilde{\psi}(\mathbf{x}) = \tilde{\psi}(g^{-1}\mathbf{x}), \qquad g \in SO(2, 1) \tag{5.84}$$

The generators of the group $SO(2, 1)$ are given by: one of them, L_z, generating rotations around the z-axis and two, L_x and L_y, of hyperbolic transformations around the x and y axes. A differential realization in Cartesian coordinates is

$$L_x = y\partial_z + z\partial_y, \qquad L_y = -x\partial_z - z\partial_x, \qquad L_z = -x\partial_y + y\partial_x$$

If we pass to the pseudo-spherical coordinates (5.83) and multiply by the imaginary unit i, they take the form

$$\begin{aligned}
L_x &= i(\sin\phi\,\partial_\chi + \tanh\chi\cos\phi\,\partial_\phi), \\
L_y &= i(-\cos\phi\,\partial_\chi + \tanh\chi\sin\phi\,\partial_\phi), \quad L_z = i(-\partial_\phi)
\end{aligned} \tag{5.85}$$

Their commutation rules close the $so(2, 1) \approx su(1, 1)$ Lie algebra

$$[L_z, L_x] = i\,L_y, \qquad [L_z, L_y] = -i\,L_x, \qquad [L_x, L_y] = -i\,L_z$$

We build the lowering and raising operators from the realization (5.85),

$$L_\pm = L_x \pm iL_y = e^{\pm i\phi}\Big(\pm\partial_\chi + i\,\tanh\chi\,\partial_\phi\Big) \tag{5.86}$$

In this basis the $so(2, 1) \approx su(1, 1)$ Lie algebra is

$$[L_z, L_\pm] = \pm L_\pm, \qquad [L_-, L_+] = 2L_z$$

The Casimir operator, which we will identify with the scalar Schrödinger Hamiltonian, has the form

$$\mathcal{C} := \tilde{H} = L_x^2 + L_y^2 - L_z^2 = L_+L_- - L_z(L_z - 1) \tag{5.87}$$

The discrete series of unitary irreducible representations (uir's) of $SO(2, 1)$ are characterized by the Casimir eigenvalues [22–24]

$$\tilde{H}\tilde{\psi} = -\lambda(\lambda - 1)\tilde{\psi}, \qquad \lambda = 1, 2, 3, \ldots \tag{5.88}$$

In terms of the pseudo-spherical coordinates $\mathcal{C}$ $(= \tilde{H})$ takes the expression

$$\tilde{H} = -\partial_{\chi\chi} - \tanh\chi\,\partial_\chi + \frac{\partial_{\phi\phi}}{\cosh^2\chi}$$

There is a first order term in ∂_χ that can be eliminated by means of the following transformation:

$$\tilde{\psi} = \frac{1}{\sqrt{\cosh \chi}} \psi \implies \partial_\chi \to \partial_\chi - \frac{1}{2} \tanh \chi \tag{5.89}$$

Then, the displaced Hamiltonian $H := \tilde{H} - \frac{1}{4}$ becomes

$$H = \tilde{H} - \frac{1}{4} = -\partial_{\chi\chi} + \frac{1/4 + \partial_{\phi\phi}}{\cosh^2 \chi} \tag{5.90}$$

The eigenvalues of this Hamiltonian corresponding to the uir's of $SO(2, 1)$ are, according to (5.88),

$$H\psi = -(\lambda - 1/2)^2 \psi, \qquad \lambda = 1, 2, \ldots$$

If we consider a basis in this representation of common eigenfunctions of $\mathcal{C}$ and L_z,

$$\psi(\chi, \phi) = \psi_n(\chi) e^{in\phi}, \qquad n \in \mathbb{N}$$

we find

$$H_n \psi_n = \left(-\partial_{\chi\chi} - \frac{(n + 1/2)(n - 1/2)}{\cosh^2 \chi} \right) \psi_n = -(\lambda - 1/2)^2 \psi_n, \qquad \lambda = 1, 2, 3, \ldots, n \tag{5.91}$$

where $n \geq \lambda \geq 1$. Therefore, once fixed n then the number of bound states is finite: $k = 0, 1, \ldots, n - 1$. In case we fixed the eigenvalue λ, the potentials that may support such a negative level are labeled by $n \geq \lambda$.

We will use the notation ψ_n^k for the eigenfunction corresponding to:

$$\psi_n^k = \begin{cases} \text{Potential} & V_n = -\frac{(n+1/2)(n-1/2)}{\cosh^2 \chi} \\ \text{Energy level} & E_n^k = -(n - k - 1/2)^2, \qquad k = 0, 1 \ldots n - 1 \\ \text{Representation } \lambda = n - k, & L_z \psi_n^k = n\psi_n^k, \qquad \lambda = n, n - 1, \ldots, 1 \end{cases}$$

We can check that indeed this equation coincides with the initial hyperbolic PT equation (5.77) as well as the eigenvalues (5.78).

On the other hand, the lowering and raising operators, when they act between the Hamiltonians H_n and H_{n-1}, after the transformation (5.89), become

$$\begin{aligned} L_+ = L_x + iL_y = \left(+\partial_\chi - (n - 1/2) \tanh \chi \right) \\ L_- = L_x - iL_y = \left(-\partial_\chi - (n - 1/2) \tanh \chi \right) \end{aligned} \tag{5.92}$$

These operators coincide with the intertwining operators $a_n^\pm$ given before in (5.78).

(b) The spinor system

Next, we will consider the spinor representation of $so(2, 1) \approx su(1, 1)$ defined in the space of spinor wavefunctions $\Psi(\mathbf{x})$ on the hyperboloid (5.82). In this space, the generators of $SU(1, 1)$, which will be called K_i, are defined by

$$K_i = L_i + S_i, \qquad i = 1, 2, 3 \tag{5.93}$$

where the orbital generators $(L_1 = L_x, L_2 = L_y, L_3 = L_z)$ have been given in (5.85), while the spinor generators $(S_1 = S_x, S_2 = S_y, S_3 = S_z)$ have been chosen as

$$S_x = i\tfrac{1}{2}\sigma_x, \qquad S_y = i\tfrac{1}{2}\sigma_y, \qquad S_z = \tfrac{1}{2}\sigma_z$$

and therefore, $S_\pm = S_x \pm i S_y$ are

$$S_+ = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad S_- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \tag{5.94}$$

The matrices S_x, S_y are anti-Hermitian, while S_z is Hermitian. They satisfy the $su(1, 1)$ Lie algebra:

$$[S_z, S_x] = i\, S_y, \qquad [S_z, S_y] = -i\, S_x, \qquad [S_x, S_y] = -i\, S_z$$

or

$$[S_z, S_\pm] = \pm S_\pm, \qquad [S_-, S_+] = 2 S_z$$

Then, the Casimir operator of this realization is

$$\mathcal{C}_s = \mathbf{K}^2 = (L_x + S_x)^2 + (L_y + S_y)^2 - (L_z + S_z)^2 = \mathbf{L}^2 + 2\mathbf{L}\mathbf{S} + \mathbf{S}^2 = -\nu(\nu - 1) \tag{5.95}$$

where the cross terms constitute the matrix Hamiltonian h,

$$h = 2\mathbf{L}\mathbf{S} = 2(L_x S_x + L_y S_y - L_z S_z) = L_+ S_- + L_- S_+ - 2L_z S_z \tag{5.96}$$

and $\mathbf{S}^2 = -3/4$. We will use the spinor expressions (5.94) and the orbital operators $L_\pm$ of (5.86) after the transformation (5.89),

$$L_\pm = e^{\pm i\phi}\big(\pm\, \partial_\chi + (i\partial_\phi \mp 1/2)\tanh\chi\big)$$

to be replaced in the expression (5.96) of h in order to find the explicit matrix Hamiltonian

$$h = \begin{pmatrix} i\partial_\phi & ie^{-i\phi}(-\partial_\chi + (i\partial_\phi + 1/2)\tanh\chi) \\ i(+\partial_\chi + (i\partial_\phi + 1/2)\tanh\chi)e^{i\phi} & -i\partial_\phi \end{pmatrix} \tag{5.97}$$

As we know from the beginning this Hamiltonian is not explicitly Hermitian since the matrix generators S_x and S_y were not Hermitian. Let us write the form of h when it acts on simultaneous eigenfunctions $\Psi_{\nu m}$ of the Casimir $\mathcal{C}$ (5.95) and K_z:

$$K_z \Psi_{\nu m} = m\Psi_{\nu m} \implies \Psi_{\nu m}(\chi, \phi) = \begin{pmatrix} \varphi_m(\chi) e^{i(m-1/2)\phi} \\ \xi_m(\chi) e^{i(m+1/2)\phi} \end{pmatrix} \tag{5.98}$$

Then, the reduced Hamiltonian in the variable χ, using this basis, is

$$h_m = \begin{pmatrix} -m + 1/2 & i(-\partial_\chi - m\tanh\chi) \\ i(\partial_\chi - m\tanh\chi) & m + 1/2 \end{pmatrix} \tag{5.99}$$

The next step is to compute the square of h:

$$h^2 = -\mathbf{L}^2 + h \tag{5.100}$$

According to (5.90) $H = \boldsymbol{L}^2 - 1/4$ so that finally we get

$$(h - 1/2)^2 = -H = -\left(-\partial_{\chi\chi} + \frac{1/4 + \partial_{\phi\phi}}{\cosh^2 \chi}\right) \tag{5.101}$$

If we take the square $(h_m - 1/2)^2$ of the reduced Hamiltonian (5.99), we find

$$(h_m - 1/2)^2 = -\begin{pmatrix} H_{n-1} & 0 \\ 0 & H_n \end{pmatrix} \tag{5.102}$$

where H_m coincides with H_n of (5.91), provided $m = n - 1/2$. Let us make some remarks.
(a) These two formulas for the square of h are reasonable since the square of a real spectrum of h will always produce a positive spectrum, that is, this square must have oposite sign than the scalar Hamiltonians H, which have a negative spectrum as shown in (5.91).

(b) The matrix hyperbolic Hamiltonian h given in (5.97), or its reduced form (5.99), are not explicitly Hermitian, therefore this seems an inconsistency. However, h although non-Hermitian it has a real discrete spectrum with square integrable solutions. We will show this property in the following subsection by computing the eigenvalues and eigenvectors.
(c) Let us notice that, as a consequence of the previous remark, this spinor representation of $SU(1, 1)$ is not unitary. However, there is a Hermitian invariant product given by

$$\langle \Phi, \Psi \rangle = \int \Phi^\dagger(\chi, \phi) \sigma_3 \Psi(\chi, \phi) \, d\chi d\phi$$

where the invariant measure on the hyperboloid is hidden in the wavefunctions by the transformation (5.89). One can check that indeed, the reduced Hamiltonian (5.97) is Hermitian with respect to σ_3: $h^\dagger \sigma_3 = \sigma_3 h$. This is a non positive definite product which is consistent with the non-unitary character.

(c) Spectrum and eigenfunctions

If we have in mind the diagonal form of the square $(h - 1/2)^2$, and at the same time we write the operators in the entries of the matrix h_m given in (5.99) we guess that the eigenfunctions will have the following form, making use of the factor operators $a_m^\pm$ of (5.78):

$$\begin{pmatrix} -m & -ia_m^- \\ -ia_m^+ & m \end{pmatrix} \begin{pmatrix} \alpha \psi_{m-1}^{k-1} \\ i\beta \psi_m^k \end{pmatrix} = \varepsilon_n^k \begin{pmatrix} \alpha \psi_{m-1}^{k-1} \\ i\beta \psi_m^k \end{pmatrix} \tag{5.103}$$

Taking into account that the action of these factor operators is

$$a_m^- \psi_m^k = \sqrt{\mu_m^2 - \mu_{m-k}^2} \, \psi_{m-1}^{k-1}, \qquad a_m^+ \psi_{m-1}^{k-1} = \sqrt{\mu_m^2 - \mu_{m-k}^2} \, \psi_m^k, \qquad \mu_{m-k}^2 = (m-k)^2$$

Then, we get the eigenvalues of h:

$$\varepsilon_n^{k\pm} = \pm(m - k), \qquad m = n - 1/2 \tag{5.104}$$

The corresponding eigenspinors $\Psi_n^{k\pm}$ are given by their coefficients $\alpha^\pm$ and $\beta^\pm$:

$$\alpha^- = \sqrt{k}, \quad \beta^- = \sqrt{2m - k}, \quad \alpha^+ = \sqrt{2m - k}, \quad \beta^+ = \sqrt{k}$$

The ground state of these eigenfunctions is negative:

$$\varepsilon_n^{0-} = -m, \qquad \Psi_n^{0-} = \begin{pmatrix} 0 \\ \psi_m^0 \end{pmatrix}$$

We notice that the greater spinor component of the negative spectrum eigenfunctions is the lower one with spin $-1/2$. However, the greater component of the positive spectrum spinor eigenfunctions is the upper corresponding to spin $1/2$. The degeneracy is infinite for each positive or negative eigenvalue. In the following we will determine the $SU(1, 1)$ representations corresponding to the eigenvalues of the same absolute value but opposite sign: $\varepsilon_n^{k\pm} = \pm(n - k - 1/2)$. In order to find these representations, we will characterize the eigenfunctions $\Psi_n^{k\pm}$ and their spinor components in terms of the commuting symmetries of h: K_z, $\boldsymbol{L}^2$ and $\boldsymbol{K}^2$.

K_z: The label n of the spinor functions $\Psi_n^{k\pm}$ means that they are eigenfunctions of K_z with eigenvalue $m = n - 1/2$:

$$K_z \Psi_n^{k\pm} = m\Psi_n^{k\pm}, \qquad m = n - 1/2$$

$\boldsymbol{L}^2$: We know that the scalar functions $\psi_n^k(\theta)e^{i(m+1/2)\phi}$ and $\psi_{n-1}^{k-1}(\theta)e^{i(m-1/2)\phi}$ (which are de components of the spinors $\Psi_n^{k\pm}$) are eigenfunctions of $\boldsymbol{L}^2$ with eigenvalue $\lambda = n - k$. Therefore, the spinor eigenfunctions of h are also eigenfunctions of $\boldsymbol{L}^2$ with the same eigenvalue

$$(\boldsymbol{L}^2 - 1/4)\Psi_n^{k\pm} = -(\lambda - 1/2)^2\Psi_n^{k\pm}, \qquad \lambda = n - k, \qquad m = n - 1/2$$

$\boldsymbol{K}^2$: The above eigenfunctions of h are also eigenfunctions of $\boldsymbol{K}^2$. We will make use of expression (4.56) in order to see this:

$$\boldsymbol{K}^2\Psi_n^{k-} = \left(\boldsymbol{L}^2 + \tilde{h} + \boldsymbol{S}^2\right)\Psi_n^{k-}$$
$$= (-\lambda(\lambda - 1) - \lambda + 1/4)\Psi_n^{k-} = -(\lambda + 1/2)(\lambda - 1/2)\Psi_n^{k-}$$

In other words, we can write as

$$\boldsymbol{K}^2\Psi_n^{k-} = (\nu^+)(\nu^+ - 1)\Psi_n^{k-}, \qquad \nu^+ = \lambda + 1/2$$

and in a similar way

$$\boldsymbol{K}^2\Psi_n^{k+} = (\nu^-)(\nu^- - 1)\Psi_n^{k+}, \qquad \nu^- = \lambda - 1/2$$

We could explain these results from the fact that the spinor wave functions Ψ belong to tensor products of the representations λ from the orbital part and $s = 1/2$ from spin: $\Psi \in \lambda \otimes 1/2$. As we know from the composition of "angular momenta":

$$\lambda \otimes 1/2 = \nu^+ \oplus \nu^-, \qquad \nu^\pm = \lambda \pm 1/2$$

Then, the negative energy spinors Ψ_n^{k-} belong to the representation of the total "angular momentum" ν^+, while the positive spinors Ψ_n^{k+} to ν^-.

In conclusion, we have obtained a real spectrum for a non-Hermitian matrix Hamiltonian.

(d) Symmetries and "anti-symmetries"

1) Symmetries $K_\pm$, K_3. In the same way as for $su(2)$ the spinor symmetries in the case of $su(1, 1)$ are given by

$$K_+ = L_+ + S_+, \qquad K_- = L_- + S_-, \qquad K_3 = L_3 + S_3$$

Their restriction to the eigenfunctions of $K_3 = K_z$ give rise to the intertwining operators $R_n^\pm$.

2) Anti-symmetries

We will also find the anti-symmetries $T^{\pm}$ which anti-commute with the Hamiltonian $h_n - 1/2$,

$$T^{\pm}(\tilde{h} - 1/2) = -(\tilde{h} - 1/2)\, T^{\pm}, \qquad [K_3, T^{\pm}] = \pm T^{\pm}$$

and have the same form as in the trigonometric case:

$$T^+ = L_+ S_3 - L_3 S_+, \qquad T^- = L_- S_3 - L_3 S_-, \qquad T_3 = \frac{1}{2}(L_+ S_- - L_- S_+)$$

The spectrum of Dirac-Weyl like equations, symmetries in terms of intertwining operators and global intertwining operators for 4×4 Dirac-like Hamiltonians can also be obtained following the same procedure as above. However, in order to shorten the length of the paper we have not discussed them here.

6 Conclusions and remarks

We have introduced a kind of matrix Dirac-like Hamiltonians which are in close correspondence with factorizable Schrödinger Hamiltonians. The scalar intertwining operators, $a_n^{\pm}$ and the factorization energies $\pm \mu_n^2$, are used as ingredients to construct the matrix Hamiltonian and the intertwining matrix operators. This type of Dirac Hamiltonians have appeared in many previous references taking part in different problems. Here, we have tried to explain them by a reduction process of spinor systems defined on curved spaces. We have called h_n (the reduced cases in one variable) or h (in two coordinates of the surface) to these Dirac-like Hamiltonians. In the case of spherical symmetry they are called Dirac operators [27]. In [8], L.M. Nieto et al showed that the usual properties for scalar differential intertwining operators have correspondence to the matrix case. In fact, the matrix intertwining operators used in this current work coincide with some of the operators introduced in [8]. On the other hand, our interest here is on the hierarchies of shape invariant Dirac Hamiltonians and its relation with superintegrable systems; a problem which was not touched in that reference.

We have worked out two simple examples. One of them defined on a sphere $\mathcal{S}^2$, giving rise to a matrix version of a trigonometric Pöschl-Teller potential. The symmetries and anti-symmetries were characterized as well as the degeneracy of the energy levels. In particular the anti-symmetries interchanged the sign of energy. The second example which was defined in a two dimensional hyperboloid $\mathcal{H}^2$ gave rise to a matrix non-Hermitian Hamiltonian with a real spectrum, which is related to a scalar hyperbolic Pöschl-Teller potential. The scalar systems were obtained as the square of the Dirac-like systems, just in a similar way as the Dirac and Klein-Gordon equations are related. We expect to complete this picture in the near future by finding the Dirac-like cases of all the scalar factorizable potentials with or without the mass m_0. In particular they will need higher space dimensions and higher gamma matrices. Another point that we pursue is to identify the role of this kind of Dirac-like Hamiltonians; it is reasonable that they be related with the symmetries of different classes of Dirac systems, since our building relies only on symmetry considerations. We hope that this kind of Dirac-like Hamiltonians be also consistent with some symmetric interactions with external fields [28,29].

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