Article

Heisenberg-Weyl groups and generalized Hermite functions

E. Celeghini[1,2,‡], M. Gadella [2,3,‡] and M. A. del Olmo [2,3,‡*]

[1] Dipartimento di Fisica, Università di Firenze and INFN-Sezione di Firenze, 150019 Sesto Fiorentino, Firenze, Italy; celeghini@fi.infn.it
[2] Departamento de Física Teórica, Atómica y Optica and IMUVA, Universidad de Valladolid, 47011 Valladolid, Spain
[3] IMUVA – Mathematical Research institute, Universidad de Valladolid, 47011 Valladolid, Spain ; gadella@fta.uva.es
* Correspondence: marianoantonio.olmo@uva.es
‡ These authors contributed equally to this work.

Abstract: We introduce a multi-parameter family of bases in the Hilbert space $L^2(\mathbb{R})$, which are associated to the set of Hermite functions, which also serve as a basis for $L^2(\mathbb{R})$. The Hermite functions are eigenfunctions of the Fourier transform, a property which is in some sense shared by these "generalized Hermite functions". The construction of these new bases is grounded on some symmetry properties of the real line under translations, dilations and reflexions and some properties of the Fourier transform. We show how these generalized Hermite functions are transformed under the unitary representations of a series of groups including the Weyl-Heisenberg group and some of their extensions.

Keywords: Hermite functions; Wey-Heisenberg groups; group representations; Fourier transform; bases in Hilbert space $L^2(\mathbb{R})$; rigged Hilbert spaces

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1. Introduction

The present paper studies the relations between some physical relevant low-dimensional Lie groups, in connection to affine transformations on the whole real line ($\mathbb{R}$), their representations on the Hilbert space $L^2(\mathbb{R})$ as well as to some other notions as the Hermite functions, other bases in $L^2(\mathbb{R})$ and the eigenfunctions of the Fourier transform. As a consequence of these relations, some invariance properties are disclosed.

These invariance properties come from the option to choose between four types of freedom. These are: (i) the freedom to choose between coordinate and momentum representations and the respective bases determined by each of the representations; (ii) the freedom to choose an origin on the real line when using any of these two representations; (iii) the freedom to choose the units of length on $\mathbb{R}$ and (iv) the freedom to choose an orientation on the line. We span one dimensional wave functions in terms of bases in either coordinate or momentum representation. The family of bases on a parameter covering the whole set of real numbers $\mathbb{R}$ is a homogeneous self-similar and not oriented space, as is well known. The Fourier transform, which is an invertible correspondence between coordinate and momentum representations [1], implies some restrictions on self-similarity and orientation.

This invariance suggests a principle of relativity: Assume that two observers are located at different points of the line and that, furthermore, they use different length and/or momentum units. These observers would perceive the same physical state as exactly the same description of the reality. This means that under these invariances the one-dimensional physical world may be equivalently described by the coordinate x and the momentum p or by the coordinate $x' = kx + a$ and the momentum $p' = k^{-1}p + b$ with $a, b \in \mathbb{R}$ and $k \in \mathbb{R}^* \equiv \mathbb{R} - \{0\}$.

Likewise other well-known situations showing invariance properties, this type of invariance is described by a Lie group, which is usually denoted by $\widetilde{H}(1)$. This is a twofold version of the affine Weyl-Heisenberg group $\widetilde{H}_o(1)$ [2–8] since it includes the discrete symmetry associated to the reflection or Parity operator $\mathcal{P}:(x,p)\to(-x,-p)$. The Lie algebra of the affine Weyl-Heisenberg group, $\widetilde{\mathfrak{h}}(1)$ has four infinitesimal generators: D, X, P and I that correspond to dilations, position operator, momentum operator and a central operator commuting with the others, respectively. As we shall show later, the Lie group $\widetilde{H}(1)$ is isomorphic to the the central extension of the Poincaré group in 1+1 dimensions [9] enlarged with the discrete symmetry $\mathcal{PT}$, where $\mathcal{P}$ is the parity and $\mathcal{T}$ the time-reversal.

From now on, when we speak about symmetry or invariance on the real line we refer to the existence of properties of spaces constructed over $\mathbb{R}$, as for example $L^2(\mathbb{R})$. This includes many others depending on a unique continuous parameter.

The Hermite functions are all real and determine a basis of the (complex) space of functions $L^2(\mathbb{R})$. Self-similarity transformations do not change this property. In addition, it is rather simple to construct additional bases of $L^2(\mathbb{R})$ after some transformations on Hermite functions, as for instance under the action of the group $\widetilde{H}(1)$. The result are the so called generalized Hermite functions, to be defined later (Section 4). Contrary to the basis of Hermite functions, these bases of generalized Hermite functions are not sets of real functions as they usually have a complex phase.

As is well known, the real line $\mathbb{R}$ as one dimensional Euclidean space is the homogeneous space $E_o(1)/\{0\}$, where $E_o(1)$ is the group of translations on the line and $\{0\}$ is the isotropy group of an arbitrary point of the line, for instance the origin. The real line supports two important continuous bases for $L^2(\mathbb{R})$: $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$. As is well known, each of these bases is transformed into the other by the Fourier transform. The meaning of continuous bases will be clarified later, although it is nonetheless explained in [10].

One consequence of the homogeneity is that the continuous basis in the coordinate representation given by $\{|x\rangle\}$, where x runs out the set of real numbers, is equivalent to the continuous basis $\{|x+a\rangle\}$, where $x \xrightarrow{T_a} x+a$, for each fixed $a\in\mathbb{R}$, with $T_a\in E_o(1)$. Analogously, the continuous basis in the momentum representation, $\{|p\rangle\}$, is equivalent for the continuous basis $\{|p+b\rangle\}$, where p runs out the set of real numbers and b is an arbitrary, although fixed, real number.

If we consider the position (X) and momentum (P) operators acting on their generalized eigenvectors, which are $|x\rangle$ and $|p\rangle$, respectively, we have that

$$
\begin{aligned}
X\,|x\rangle = x|x\rangle \quad &\Rightarrow \quad e^{-iXa}|x\rangle = e^{-iax}|x\rangle\,, \\
P\,|p\rangle = p|p\rangle \quad &\Rightarrow \quad e^{-iPb}|p\rangle = e^{-ibp}|p\rangle\,.
\end{aligned}
\tag{1}
$$

The Fourier transform and its inverse produce the following relations [10] :

$$
|p\rangle = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ipx}|x\rangle\,dx\,, \qquad |x\rangle = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{-ipx}|p\rangle\,dp\,. \tag{2}
$$

We also have the following relations:

$$
\begin{aligned}
e^{-iXa}\,|p\rangle &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ipx}e^{-iXa}|x\rangle\,dx = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ix(p-a)}|x\rangle\,dx = |p-a\rangle \\
e^{-iPb}\,|x\rangle &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{-ipx}e^{-iPb}|p\rangle\,dp = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{-i(x+b)p}|x\rangle\,dp = |x+b\rangle
\end{aligned}
\tag{3}
$$

The conclusion is that X and P together with the central operator I determine the Lie algebra for the Heisenberg-Weyl group $H(1)$. In this context, we say that the real line (meaning the space $L^2(\mathbb{R})$) supports a unitary representation of $H(1)$.

However, the group $H(1)$ does not exhaust self-similarity invariances on the real line and for our purposes is "not oriented", in the sense that it is equivalent to consider the direction on the line either from left to right or from right to left. Moreover, as commented earlier, the continuous basis $\{|x\rangle\}$ is equivalent to the continuous basis $\{|kx\rangle\}$ with $k \in \mathbb{R}^*$. This suggest the use of the *dilatation operator*, D, which may be defined by the action of its exponential on the continuous basis as $e^{-idD}|x\rangle = e^{-d/2}\,|e^d x\rangle$ (d real) and then extended as a self-adjoint operator on $L^2(\mathbb{R})$. This action considers positive dilatations only as $e^d > 0$ for any real d. Note that if $\langle x|y\rangle = \delta(x-y)$ then $\langle e^d x|e^d y\rangle = \delta(e^d(x-y)) = e^{-d}\,\delta(x-y)$, this is the reason to introduce the factor $e^{-d/2}$ in the definition of the action of e^{-idD} in $|x\rangle$ in order that $\langle x|\left(e^{-idD}\right)^\dagger e^{-idD}|y\rangle = \langle x|y\rangle$.

Analogously, the continuous basis $\{|p\rangle\}$ is equivalent to the continuous basis $\{|k'p\rangle\}$, with $k' \in \mathbb{R}$. Consistency with Fourier transform invariance implies that $k' = k^{-1}$. This suggest a result that shall become evident soon, that the algebra describing the invariance in the real line should be $\widetilde{H}_o(1)$, i.e., the Weyl-Heisenberg group enlarged with dilations.

Nevertheless, we need to introduce orientation invariance and negative numbers k for dilatations in our picture. This is performed by the parity operator $\mathcal{P}$. As is well known, the action of $\mathcal{P}$ on the continuous bases are given by $\mathcal{P}\,|x\rangle = |-x\rangle$ and $\mathcal{P}\,|p\rangle = |-p\rangle$. If we add this parity operator to the connected group $\widetilde{H}_o(1)$, we obtain the general group of invariance of the real line $\widetilde{H}(1)$. The the space $L^2(\mathbb{R})$ supports a unitary representation U of $\widetilde{H}(1)$.

This representation U can be well studied using the *generalized* Hermite functions, we mentioned earlier. For our purposes, we need two families of bases constructed as follows. Choose the basis of the normalized Hermite functions $\{\psi_n(x)\}$ and their Fourier transforms $\{\widetilde{\psi}_n(p)\}$. Then, $U(\tilde{g})$ with $\tilde{g} \in \widetilde{H}(1)$ being the unitary representation, these families are $\{U(\tilde{g})\psi_n(x)\}_{x\in\mathbb{R}}^{\tilde{g}\in\widetilde{H}(1)}$ and $\{U(\tilde{g})\hat{\psi}_n(p)\}_{x\in\mathbb{R}}^{\tilde{g}\in\widetilde{H}(1)}$. These two families of generalized Hermite functions are transformed into each other by the Fourier transform (FT) and its inverse (IFT), in similarity with the behaviour of the Hermite functions [10].

The present article is organized as follows: In the next Section 2, we arrive to the Weyl-Heisenberg group $H(1)$, starting from the translations groups and supposing some more symmetries for the line, provided that we also implement the symmetry under Fourier Transform for the Hermite functions. In Section 3 we present some general properties of the Weyl-Heisenberg group and its extension to $\widetilde{H}(1)$. This group is connected to the general symmetry on the real line. We deal with local structures, exhibited by the Lie algebra of $\widetilde{H}(1)$, which is presented in its more familiar form including the parity operator. In Section 4, we construct the unitary representations of the Weyl-Heisenberg group and its generalisations defined in the previous Section. Considering the behaviour of the Hermite functions under the group $\widetilde{H}(1)$, we introduce in Section 5 a generalization of such Hermite functions: We obtain a 3-parameter family of "generalized Hermite functions" that are bases of $L^2(\mathbb{R})$. We study properties of these generalized Hermite functions as well as their behaviour under the Fourier transform. Also, we construct Rigged Hilbert space structures associated to these generalized Hermite functions. We give some concluding remarks in the final Section 6. For the benefit of the reader, we have added some Appendices with some known material about of group representation.

2. From Translation group to the Weyl-Heisenberg group

Let us consider the group of the translations of the real line, $E_o(1)$, that can be considered as the connected part of the isometries of the line (translations and reflexions in a point, the origin for instance) that constitute the Euclidean group on one dimension $E(1)$.

The group $E_o(1)$ is isomorphic to the group $(\mathbb{R}, +)$. Under a translation T_a the point x of the real line is transformed as

$$x \xrightarrow[a\in\mathbb{R}]{T_a} x + a\,. \tag{4}$$

The action of $E_o(1)$ on the space of square integrable functions defined on $\mathbb{R}$ ($L^2(\mathbb{R})$) is given by

$$(U(T_a)f)(x) = f(x-a)\,, \tag{5}$$

where we have taking into account that if a group G acts on a space X from the left (i.e., $\forall x \in X \xrightarrow{g\in G} g\,x \in X$ such that $e\,x = x$, being e the identity element of G, and $g'(g\,x) = (g'\,g)\,x\,,\ \forall g, g' \in G$) then there is a representation of this group in the space of functions defined in X as

$$(U(g)f)(x) = f(g^{-1}x)\,. \tag{6}$$

Let P be the infinitesimal generator of the translation group, hence $U(T_a) = e^{-iaP}$ and from (6) we get that

$$P = -i\frac{d}{dx}\,. \tag{7}$$

2.1. The group $E_o(1)$ extended by dilations: a matrix realization

If we consider also transformations like dilations acting as

$$x \xrightarrow[k\in\mathbb{R}^*]{D_k} kx\,, \tag{8}$$

the composition of both transformations $T_a \cdot D_k$ acts as

$$x \xrightarrow[k\in\mathbb{R}^*]{D_k} kx \xrightarrow[a\in\mathbb{R}]{T_a} kx + a\,. \tag{9}$$

We can realize the group spanned by both transformations as the group of matrices

$$M_{[k,a]} = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}, \qquad k \neq 0, b \in \mathbb{R} \tag{10}$$

acting on the real line as follows

$$M_{[k,a]}x = \begin{pmatrix} k & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} k\,x + a \\ 1 \end{pmatrix}. \tag{11}$$

in agreement with (9). Henceforth, we shall denote this group as $\widetilde{E}(1)$. It is non-connected and shows two connected components: the connected component of the unit characterized by $k > 0$ and and a second component for which $k < 0$.

2.2. The connected component of $\widetilde{E}(1)$: $\widetilde{E}_o(1)$

Let us start by restricting ourselves to the connected component of the unit of $\widetilde{E}(1)$ that we denote for $\widetilde{E}_o(1)$. The infinitesimal generators in the matrix representation (10) are

$$P = \left.\frac{dM_{[k,a]}}{da}\right|_{a=0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D = \left.\frac{dM_{[k,a]}}{dk}\right|_{k=1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12}$$

The commutation relation of P and D is

$$[D, P] = P\,. \tag{13}$$

We see that under exponentiation (i.e. e^{aP} and e^{kK}), we only recover $\widetilde{E}_o(1)$

$$e^{aP}\,e^{kD} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^k & 0 \\ 0 & 1 \end{pmatrix} = M_{[a,e^k]} \tag{14}$$

Let us denote by $g = (a,k) = e^{aP}\, e^{kD}$ an arbitrary element of $\widetilde{E}_o(1)$ with $a,k \in \mathbb{R}$. The group law is given by

$$g' \cdot g = (a',k')(a,k) = (a' + e^{k'} a, k' + k)\,. \tag{15}$$

Moreover

$$g = (a,1)(0,k)\,, \qquad g^{-1} = (-e^{-k} a, -k)\,. \tag{16}$$

The action of g on the functions $f(x)$ is given by (see (6))

$$(U(a,k) f)(x) = e^{-k/2}\, f(e^{-k}(x-a))\,, \tag{17}$$

where the term $e^{-k/2}$ has been added so as to assure the unitarity of this representation [11,12]. In particular, the Hermite functions $\psi_n(x)$ are functions in $L^2(\mathbb{R})$. In addition, Hermite functions are a basis of $L^2(\mathbb{R})$. Consequently, they support the representation of $\widetilde{E}_o(1)$, so that,

$$(U(a,k)\psi_n)(x) = e^{-k/2}\, \psi_n(e^{-k}(x-a))\,. \tag{18}$$

After (17) ($U(a,k) = e^{-iaP}\, e^{-ikD}$), the infinitesimal generators take the explicit form

$$P = -i\frac{d}{dx}\,, \qquad D = -i\frac{1}{2}\left(x\frac{d}{dx} + \frac{d}{dx}\, x \right), \tag{19}$$

and its Lie commutator is given by

$$[D,P] = iP\,. \tag{20}$$

2.3. The group $\widetilde{E}(1)$

In order to take into account the orientation invariance of the real line or, in other words, to consider the other connect component of the group $\widetilde{E}(1)$, we must include the parity or reflexion operator around the origin $\mathcal{P}$, that act on $\mathbb{R}$ as

$$x \ \xrightarrow{\mathcal{P}}\ -x\,. \tag{21}$$

The infinitesimal generators P and D transform under $\mathcal{P}$ as

$$(P,D) \ \xrightarrow{\mathcal{P}}\ (-P,D) \tag{22}$$

and the elements of $\widetilde{E}_o(1)$ transform under parity as

$$g = (a,k) \in \widetilde{E}_o(1) \ \xrightarrow{\mathcal{P}}\ (a,k)^{\mathcal{P}} = (a^{\mathcal{P}}, k^{\mathcal{P}}) = (-a,k)\,. \tag{23}$$

Each of the $\tilde{g} \in \widetilde{E}(1)$ can be parametrized by

$$\tilde{g} = (a,k,\alpha)\,, \qquad \alpha \in \mathcal{V} = \{\mathcal{I}, \mathcal{P}\} \tag{24}$$

where $\mathcal{I}$ is the identity transformation.

The group law is

$$\tilde{g}' \cdot \tilde{g} = (a',k',\alpha')(a,k,\alpha) = (a' + e^{k'} a^{\alpha'}, k'+k, \alpha'\alpha)\,, \tag{25}$$

where, obviously,

$$a^{\alpha} = \begin{cases} a & \text{if } \alpha = \mathcal{I} \\ -a & \text{if } \alpha = \mathcal{P} \end{cases}. \tag{26}$$

Thus, $\widetilde{E}(1)$ is a semidirect product, i.e., $\widetilde{E}(1) = \widetilde{E}_o(1) \odot \mathcal{V} = (E_o(1) \odot \mathcal{V}) \odot \mathcal{D}$, where $\mathcal{D}$ is the dilations group $\{(0, k, \mathcal{I})\}_{k \in \mathbb{R}}$, since

$$\tilde{g} = (a, k, \alpha) = (a, k, \mathcal{I})\,(0, 0, \alpha) = (a, k, \mathcal{I})\,(0, 0, \alpha) = (a, 0, \mathcal{I})\,(0, 0, \alpha)\;\,(0, k, \mathcal{I})\,. \tag{27}$$

On the given representation of $\widetilde{E}(1)$, the operator $\mathcal{P}$ is realized as a linear operator, so that the representation is unitary. It has the form [13]

$$\begin{aligned} (U(a,k,\alpha)f)(x) &= e^{-k/2}\, f(e^{-k}(x^\alpha - a))\,. \\ (U(a,k,\alpha)\psi_n)(x) &= e^{-k/2}\, \psi_n(e^{-k}(x^\alpha - a))\,. \end{aligned} \tag{28}$$

128 *2.4. The Weyl-Heisenberg group $H(1)$*

An important fact of the Hermite functions is that they are eigenfunctions of the Fourier transform [10]

$$FT\,[\psi_n(x), x, p] \;=\; i^n\, \psi_n(p)\,, \qquad IFT\,[\psi_n(p), p, x] \;=\; (-i)^n\, \psi_n(x)\,, \tag{29}$$

where $(I)FT\;\;[\psi_n(x), x, p]$ means the Inverse Fourier transform of the function $\psi_n(x)$ integrated on the variable x as a function of the variable p, i.e.

$$\begin{aligned} FT[f(x), x, p] &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{ipx}\, f(x)\, dx = \hat{f}(p)\,, \\ IFT[\hat{f}(p), p, x] &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-ipx}\, \hat{f}(p)\, dp = f(x)\,. \end{aligned} \tag{30}$$

129 Henceforth, we shall use this notation.

All we have previously mentioned for the Hermite functions $\psi_n(x)$ in this section is valid for their FTs $\psi_n(p)$. Hence

$$\begin{aligned} \left(e^{-iPa}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{ipx} \left(e^{-iPa}\, f\right)(x)\, dx = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{ipx}\, f(x-a)\, dx \\ &= \frac{1}{\sqrt{2}}\, e^{ipa} \int_{\mathbb{R}} e^{iup}\, f(u)\, du = e^{ipa}\, \hat{f}(p)\,, \\ \left(e^{-iDk}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{ipx} \left(e^{-iDk}\, f\right)(x)\, dx = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{ipx}\, e^{-k/2}\, f(e^{-k}x)\, dx \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{k/2}\, e^{ie^k vp}\, f(v)\, dv = e^{k/2}\, \hat{f}(e^k p)\,. \end{aligned} \tag{31}$$

In the above relations, we have proceed with the change of variables $u = x - a$ and $v = e^{-k}x$. We need to have a translation operator acting on the real line in the p representation. First of all, we recall some properties of the FT such as:

$$x f(x) \;\xrightarrow{FT[\bullet,x,p]}\; -i\frac{d}{dp}\hat{f}(p)\,, \qquad \frac{d}{dx} f(x) \;\xrightarrow{FT[\bullet,x,p]}\; -ip\hat{f}(p)\,. \tag{32}$$

Hence, we define a new operator X acting on the space of square integrable functions on the line in the following manner:

$$(Xf)(x) = x\, f(x)\,, \qquad (e^{iX} f)(x) = e^{ix}\, f(x)\,. \tag{33}$$

Then

$$\begin{aligned} \left(e^{iXb}\,\hat{f}\right)(p) &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ipx}\left(e^{iXb}\,f\right)(x)\,dx = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ipx}\,e^{ibx}\,f(x)\,dx \\ &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{ix(p+b)}\,f(x)\,dx = \hat{f}(p+b)\,. \end{aligned} \tag{34}$$

Thus, X is the infinitesimal generator of translations on the p-real line.

From (29) and taking into account the isomorphism between the real x-line and the real p-line, we can identify up to a phase the Hermite functions $\psi_n(x)$ and their FT, i.e.

$$\psi_n(x) \quad \xrightarrow{TF} \quad \hat{\psi}_n(p) = i^n\,\psi_n(p) \equiv i^n\,\psi_n(x)\,. \tag{35}$$

Hence, we have properly determined the generators X (33) and P (20) acting on $L^2(\mathbb{R})$ being $\mathbb{R}$ the x-line. From (33) and (5), we note that X produces a phase and P a translation, respectively. Obviously from (32) the roles of X and P interchange when $\mathbb{R}$ is the p-line. Both operators along to the central operator I determine the Weyl-Heisenberg group since they verify the Lie commutators

$$[X,P] = i\,I\,, \qquad [I,\bullet\,] = 0\,. \tag{36}$$

In the next section, we study the Weyl-Heisenberg group as well some of its extensions in detail.

3. The Weyl-Heisenberg group and its extensions

In this section, we start presenting a review of the Weyl-Heisenberg (WH) group as well as one of its extensions. Also we revisite their Lie algebras. Finally, we provide the isomorphism between the extended WH group and the a central extension of the Poincaré (1+1) group enlarged by the discrete symmetry $\mathcal{PT}$ (parity-time inversion).

3.1. The Weyl-Heisenberg group: a matrix realization

The Weyl-Heisenberg group $H(1)$ shows as the most common commutation relation in ordinary relativistic Quantum Physics appears, i.e., $[x,p] \equiv [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar$. This group admits a representation by real 3×3 upper unitriangular matrices [8] such as:

$$A = \begin{bmatrix} 1 & a & \theta \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,\theta \in \mathbb{R}\,. \tag{37}$$

These matrices form a group with the usual matrix multiplication as one readily sees:

$$A'\cdot A = \begin{bmatrix} 1 & a'+a & ac'+\theta'+\theta \\ 0 & 1 & c'+c \\ 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The identity element is the identitiy matrix, i.e. $Id = A\big|_{a,b,\theta=0}$, and the inverse of A (37) is given by

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-\theta \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

Note that $H(1)$ is a subgroup of the group of all upper triangular matrices 3×3, $M_3(\mathbb{R})$, see [14].

3.2. The extended Weyl-Heisenberg group

In order to include self-similarity on the real line, one needs to look at a more general subgroup of $M_3(\mathbb{R})$, which is the set of all 3×3 matrices of the form:

$$B \;=\; \begin{bmatrix} 1 & a & \theta \\ 0 & k & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a, b, \theta \in \mathbb{R},\; k \in \mathbb{R}^* \,. \tag{40}$$

The group law is given by

$$B' \cdot B \;=\; \begin{bmatrix} 1 & ka' + a & \theta' + \theta + a'b \\ 0 & k'k & k'b + b' \\ 0 & 0 & 1 \end{bmatrix}. \tag{41}$$

The identity element is $Id = B\big|_{a,b,\theta=0,k=1}$ and the inverse of B (40) is

$$B^{-1} \;=\; \begin{bmatrix} 1 & -a/k & \theta + ab/k \\ 0 & 1/k & -b/k \\ 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

Obviously, this group reduces to $H(1)$ if and only if $k = 1$. In other words $H(1)$ is a subgroup of this extended Weyl-Heisenberg group. Consequently, we denote the extended group as $\widetilde{H}(1)$.

Th group $\widetilde{H}(1)$ has two connected components: the connected component of the identity characterized for $k > 0$, which is a subgroup of $\widetilde{H}(1)$, here denoted as $\widetilde{H}_o(1)$, and a second component containing the elements elements caracterized by $k < 0$. It can be obtained multiplying the elements of $\widetilde{H}_o(1)$ by the "parity" matrix $\mathcal{P} = \text{Diagonal}[1, -1, 1]$.

3.3. The Weyl-Heisenberg algebras

Let us go back to the group $H(1)$ of matrices of the form (37). It depends on three real parameters a, θ and b related to the generators X, I and P, respectively, of the Lie algebra $\mathfrak{h}(1)$. In addition, the Lie algebra $\widetilde{\mathfrak{h}}(1)$ contains another generator, D, which is associated with the real parameter k in the group of matrices (40). The explicit form of these generators is given by

$$\begin{aligned} X = \frac{\partial B}{\partial a}\Big|_{Id} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & I = \frac{\partial B}{\partial \theta}\Big|_{Id} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ P = \frac{\partial B}{\partial c}\Big|_{Id} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & D = \frac{\partial B}{\partial k}\Big|_{Id} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{43}$$

The commutation relations are

$$[X, P] \;=\; I\,, \quad [D, X] \;=\; -X\,, \quad [D, P] \;=\; P, \quad [I, \bullet\,] \;=\; 0\,. \tag{44}$$

It is noteworthy that the action of the parity matrix, $\mathcal{P} = \text{Diagonal}[1, -1, 1]$, on the generators is given by $\mathcal{P}\,Y\,\mathcal{P}^{-1}$ (with $Y = X, P, I, D$), so that

$$\mathcal{P}\,X\,\mathcal{P}^{-1} = -X\,, \quad \mathcal{P}\,P\,\mathcal{P}^{-1} = -P\,, \quad \mathcal{P}\,I\,\mathcal{P}^{-1} = I\,, \quad \mathcal{P}\,D\,\mathcal{P}^{-1} = D\,. \tag{45}$$

Due to the fact that for arbitrary $\mathfrak{g} \in h(1)$ one has that $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$, we conclude that $h(1) \equiv \langle X, P, I \rangle$ is nilpotent. On the other hand, this is not the case for $\widetilde{\mathfrak{h}}_o(1) \equiv \langle X, P, D, I \rangle$, which is not nilpotent, although solvable.

The four one-parametric subgroups of $\widetilde{\mathfrak{h}}_o(1)$, corresponding to its four independent real parameters, are constructed by direct exponentiation of the matrices in (43). They are

$$e^{aX}=\begin{pmatrix}1&a&0\\0&1&0\\0&0&1\end{pmatrix},\; e^{\theta I}=\begin{pmatrix}1&0&\theta\\0&1&0\\0&0&1\end{pmatrix},\; e^{bP}=\begin{pmatrix}1&0&0\\0&1&b\\0&0&1\end{pmatrix},\; e^{dD}=\begin{pmatrix}1&0&0\\0&e^d&0\\0&0&1\end{pmatrix}$$

with $a,\theta,b,d\in\mathbb{R}$. Note that $e^d>0$, because by exponentiation we only obtain the elements of the connected component of the unit, i.e, $\widetilde{H}_o(1)$.

We can factorize the group $\widetilde{H}_o(1)$ as product of its four one-dimensional groups as

$$\begin{aligned}
g(\theta,a,b,d)&=e^{\theta I}\,e^{bP}\,e^{dD}\,e^{aX}=\begin{pmatrix}1&a&\theta\\0&e^d&b\\0&0&1\end{pmatrix},\\
g(\theta,a,b,d)&=e^{\theta I}\,e^{bP}\,e^{aX}\,e^{dD}=\begin{pmatrix}1&e^da&\theta\\0&e^d&b\\0&0&1\end{pmatrix},\\
g(\theta,a,b,d)&=e^{\theta I}\,e^{aX}\,e^{bP}\,e^{dD}=\begin{pmatrix}1&e^da&\theta+ac\\0&e^d&c\\0&0&1\end{pmatrix},
\end{aligned}\tag{46}$$

or also as

$$g(\theta,a,b,d)=e^{\theta I}\,e^{aX+bP}\,e^{dD}=\begin{pmatrix}1&e^da&\theta+ab/2\\0&e^d&b\\0&0&1\end{pmatrix}.\tag{47}$$

In the following, any $g\in\widetilde{H}_o(1)$ will be written as a product of the four one-parametric groups according to the second factorization displayed in (46), i.e.,

$$g\equiv(\theta,b,a,d)=e^{\theta I}\,e^{bP}\,e^{iaX}\,e^{dD}\,,\qquad \theta,\,b,\,a,\,d\in\mathbb{R}\,.\tag{48}$$

In this parametrization the group law is

$$g'g=(\theta',b',a',d')\,(\theta,b,a,d)=(\theta'+\theta+a'\,e^{d'}b,b'+e^{d'}b,e^{-d'}a+a',d'+d)\tag{49}$$

and the inverse element of $g=(\theta,b,a,d)$ is

$$g^{-1}=(-\theta+ab,-e^{-d}b,-e^{d}a,-d)\,.\tag{50}$$

It is simple to compute the adjoint action of the four one-parameter subgroups on the four generators of the Lie algebra $\widetilde{\mathfrak{h}}_o(1)$, which is given by

$$\begin{aligned}
e^{aX}Pe^{-aX}&=P+aI\,,& e^{aX}De^{-aX}&=D+aX\,,\\
e^{bP}Xe^{-bP}&=X-bI\,,& e^{bP}De^{-bP}&=D-bP\,,\\
e^{dD}Xe^{-dD}&=e^{-d}X\,,& e^{dD}Pe^{-dD}&=e^{d}P\,.
\end{aligned}\tag{51}$$

Since I is a central generator for the algebra, we conclude that

$$e^{bI}Ye^{-bI}=Y\,,\qquad e^{tY}Ie^{-tY}=I\,,\qquad \forall Y\in\widetilde{\mathfrak{h}}_o(1)\,.$$

Also, $e^{tY}Ye^{tY}=Y$ for any $Y\in\widetilde{\mathfrak{h}}_o(1)$.

From (48) and (51) we can easily compute the adjoint action of the group $\widetilde{H}_o(1)$ on its Lie algebra $\widetilde{\mathfrak{h}}_o(1)$. Thus,

$$\begin{aligned} gXg^{-1} &= e^{-d}X - e^{-d}bI\,, \\ gPg^{-1} &= e^{d}P + e^{d}aI\,, \qquad\qquad g = (\theta, b, a, d) \\ gDg^{-1} &= D + a\,X - b\,P - a\,b\,I\,. \end{aligned} \tag{52}$$

Hence, equations (52) show that, under the action of the elements of $\widetilde{H}_o(1)$, the position and the momentum operators are transformed as $X' = e^{-d}X - e^{-d}bI$ and $P' = e^{d}P + e^{d}aI$, respectively.. Therefore, the whole group describing the invariances in the *oriented* real line should be $\widetilde{H}(1)$, as $e^{\pm d}$ is always positive, so that it does not change the orientation of X and P. However, the real line is not properly speaking an oriented space as can be seen equally well from left to the right or from right to the left. The conclusion is that, we have to add to $\widetilde{H}_o(1)$ a parity operator $\mathcal{P}$ acting like the parity matrix Diagonal$[1, -1, 1]$ (45). Hence

$$\widetilde{H}(1) = \mathcal{V}_2 \otimes \widetilde{H}_o(1)\,, \tag{53}$$

where $\mathcal{V}_2$ is the group of the discrete symmetries $\{\mathcal{I}, \mathcal{P}\}$.

3.4. The extended WH group versus an extension of the Poincaré (1+1) group

The group $\widetilde{H}(1)$ is isomorphic to an extension of the Poincaré (1+1) group, which we denote by $\widetilde{P}(1,1)$. More especifically, it is the connected component of the identity of the extended Poincaré group in (1+1) dimensions [7,9]. The group $\widetilde{P}_o(1,1)$, enlarged with the symmetry $\mathcal{P}\,\mathcal{T}$, gives

$$\widetilde{P}(1,1) = \widetilde{P}_o(1,1) \cup \mathcal{P}\,\mathcal{T} \cdot P(1,1) = \mathcal{V}_2 \otimes P_o(1,1)\,. \tag{54}$$

Here, $\mathcal{V}_2$ is the group of the discrete symmetries $\{\mathcal{I}, \mathcal{P}\mathcal{T}\}$. As a matter of fact, the group $\widetilde{P}_o(1,1)$ is spanned by H, P, K, C. These are the infinitesimal generators of the time-translations, space-translations, boots and the central extension, respectively. Their Lie commutators are

$$[P, H] = C, \quad [K, H] = P, \quad [K, P] = H, \quad [\cdot, C] = 0\,. \tag{55}$$

Under the discrete symmetry $\mathcal{P}\,\mathcal{T}$, the infinitesimal generators transform as

$$(H, P, K, C) \xrightarrow{\mathcal{P}\,\mathcal{T}} (-H, -P, K, C)\,. \tag{56}$$

Now, let us consider the new generators

$$X_\pm = H \pm P\,, \qquad I = 2C \tag{57}$$

together with K. Their commutations relations are

$$[X_+, X_-] = I, \quad [K, X_+] = X_+, \quad [K, X_-] = -X_-, \quad [\bullet, I] = 0\,. \tag{58}$$

From (56) the behaviour of $X_\pm$ under the symmetry $\mathcal{P}\,\mathcal{T}$ is $(\mathcal{P}\,\mathcal{T})\,X_\pm\,(\mathcal{P}\,\mathcal{T})^{-1} = -X_\pm$. Hence, the identification

$$(X_+, X_-, K, I) \quad \Longleftrightarrow \quad (X, P, D, I) \tag{59}$$

along to the symmetry $(\mathcal{P}\,\mathcal{T}) \Leftrightarrow \mathcal{P}$ allow us to show the existence of an isomorphism between the Lie algebras Lie$[\widetilde{P}(1,1)]$ and Lie$[\widetilde{H}(1)]$ and their Lie groups.

4. Unitary representations of the WH groups

In this section, we are going to review the unitary representations (UR) and or the unitary irreducible representations (UIR) the of the different HW groups described in the previous section.

4.1. UIR of the Weyl-Heisenberg group $H(1)$

It is noteworthy that we may consider the WH group as a central extension of the abelian group of the translations on the 2-dimensional euclidean plane. The elements of the WH group are parametrized by [7,15,16]

$$g = (\theta, a, b)\,, \qquad \theta \in \mathbb{R}\,,\ (a,b) \in \mathbb{R}^2\,, \tag{60}$$

with the multiplication law

$$\begin{aligned} g_1 \cdot g_2 &= (\theta_1, a_1, b_1)(\theta_2, a_2, b_2) \\ &= (\theta_1 + \theta_2 + \xi((a_1,b_1),(a_2,b_2)), a_1 + a_2, b_1 + b_2)\,, \end{aligned} \tag{61}$$

where the exponent ξ is [16]

$$\xi((a_1,b_1),(a_2,b_2)) = \frac{1}{2}\,(a_1\,b_2 - a_2\,b_1)\,. \tag{62}$$

For the sake of simplicity we write $\vec{a} = (a, b, 0)$ so that after (62), we have

$$\xi(\vec{a}_1, \vec{a}_2) = \frac{1}{2}\,\vec{a}_1 \wedge \vec{a}_2\,, \qquad \vec{a}_i = (a_i, b_i, 0),\ i = 1,2\,. \tag{63}$$

Note that (61) is related with the more usual factorization

$$g = (\theta, a, b) = e^{\theta I}\, e^{aX + bP}\,. \tag{64}$$

The latter formula can be easily checked:

$$\begin{aligned} g_1 \cdot g_2 &= (\theta_1, a_1, b_1)(\theta_2, a_2, b_2) \\ &= e^{\theta_1 I}\, e^{a_1 X + b_1 P}\, e^{\theta_2 I}\, e^{a_2 X + b_2 P} = e^{(\theta_1+\theta_2) I}\, e^{a_1 X + b_1 P}\, e^{a_2 X + b_2 P} \\ &= e^{(\theta_1 + \theta_2 + \frac{1}{2}(a_1 b_2 - a_2 b_1)) I}\, e^{(a_1 + a_2) X + (b_1 + b_2) P} \\ &= (\theta_1 + \theta_2 + \frac{1}{2}(a_1 b_2 - a_2 b_1), a_1 + a_2, b_1 + b_2)\,. \end{aligned} \tag{65}$$

Here, we have made use of the Glauber formula [1,7], which is a particular case of the Baker-Campbell-Hausdorff formula, which states that if A and B are two operators such that $[A,[A,B]] = [B,[A,B]] = 0$, then $e^A\, e^B = e^{A+B}\, e^{\frac{1}{2}[A,B]}$ or equivalently $e^A\, e^B = e^B\, e^A\, e^{[A,B]}$.

It is noteworthy that the Glauber formula relates the different parametrizations of the group

$$g = (\theta, a, b) = e^{\theta I}\, e^{aX + bP} = e^{(\theta - \frac{1}{2}\,ab) I}\, e^{bP}\, e^{aX} = (\theta - \frac{1}{2}\,ab, 0, 0)(0, b, 0)(0, 0, a)\,. \tag{66}$$

The UIR's of the WH group on the space of square integrable functions on the real line $L^2(\mathbb{R})$ are well known after their applications in Quantum Mechanics. Here, we can distinguish two types or classes thereof:

I.- The infinite-dimensional representations labeled by a real parameter $h \in \mathbb{R}^*$ given by the product of operators [7,15]

$$U_h(g) \equiv U_h(\theta, a, b) = e^{ih\theta}\, e^{ih(aX - bP)} = e^{ih(\theta - ab/2)}\, e^{ihaX}\, e^{-ihbP}\,. \tag{67}$$

for which its explicit expression acting on the functions $f(x) \in L^2(\mathbb{R})$ is given by

$$(U_h(g)f)(x) = e^{ih\theta}\, e^{iha(x-b/2)}\, f(x-b)\,. \tag{68}$$

Note that $U_{h'}$ and U_h with $h' \neq h$ are non-equivalent.

II.- The one-dimensional and trivial UIR with $h = 0$, so that $(U_0(g)f)(x) = f(x)$. These are not relevant in our discussion.

Under the representations of class I, see (68), the infinitesimal generators X, P, I take the form

$$(X\,f)(x) = x\,f(x)\,, \quad (P\,f)(x) = -\frac{i}{h}\frac{df}{dx}(x)\,, \quad [X, P] = \frac{i}{h}\,I \Rightarrow I = h\,. \tag{69}$$

If $h = 1/\hbar$ we recover well-known results in Quantum Mechanics..

We may say that the real line, we recall that we here mean the space of square integrable functions on the real line $L^2(\mathbb{R})$, supports a UIR U_h of the Weyl-Heisenberg group $H(1)$.

4.2. UIR of the Weyl-Heisenberg group with dilations $\widetilde{H}_o(1)$

As mentioned in Section 1 the group $H(1)$ does not exhaust self-similarity invariances on the real line that for our purposes should be considered as "non oriented". By non-orientation, we refer to the equivalence of both directions to left to right or to right to left. Since the Lie algebra describing the invariance in the real line is $\tilde{\mathfrak{h}}(1)$, with generators fulfilling the commutation relations (44) and taking into account the realization of the infinitesimal generators of the WH group (69) and Subsection 2.2 (in particular expression (20)), we obtain the following expression for the infinitesimal generator D:

$$(D\,f)(x) = -\frac{i}{2h}\left(x\frac{d}{dx} + \frac{d}{dx}\,x\right)f(x) = -\frac{i}{2h}\cdot\left(2x\frac{df(x)}{dx} + f(x)\right) \tag{70}$$

Hence

$$\left(e^{-ihdD}f\right)(x) = e^{-d/2}\,f(e^{-d}x)\,. \tag{71}$$

Another interesting fact is that this group has two Casimir elements: I (central charge) and the quadratic Casimir

$$\mathcal{C} = X\,P - I\,D\,. \tag{72}$$

The eigenvalues of these central elements $(h, \mathcal{C}) \in \mathbb{R}^2$ label the UIR's of $\widetilde{H}_o(1)$. For the sake of our purposes, the suitable UIRs of $\widetilde{H}_o(1)$ are characterized by $(h \neq 0, \mathcal{C})$ and given by

$$(U_{h,\mathcal{C}}(\hat{g})\,f)(x) = e^{-d/2}\,e^{ih(\theta+\mathcal{C})}\,e^{iha(x-b/2)}\,f(e^{-d}(x-b))\,, \tag{73}$$

where according to (64), we have

$$\hat{g} = (g, d) = (\theta, a, b, d) = e^{\theta I}\,e^{aX+bP}\,e^{dD}\,, \quad g \in H(1)\,,\ d \in \mathbb{R}\,. \tag{74}$$

Now, the group law is given by

$$\hat{g}_1\,\hat{g}_2 = (\theta_1 + \theta_2 + \frac{1}{2}\,\xi((a_1, b_1), (e^{d_1}a_2, e^{-d_1}b_2)), a_1 + e^{d_1}a_2, b_1 + e^{-d_1}b_2, d_1 + d_2)\,, \tag{75}$$

where we have taken into account (61) and (74). The inverse of the element $\hat{g} = (\theta, a, b, d)$ is given by

$$\hat{g}^{-1} = (-\theta, -e^{-d}a, -e^{d}b, -d)\,. \tag{76}$$

With the notation used in (62), we can rewrite the exponent ξ of (75) as

$$\xi(\hat{g}_1\,\hat{g}_2) = \xi((a_1, b_1), (e^{d_1}a_2, e^{-d_1}b_2)) = \xi\left(\vec{a}_1, \vec{a}_2^{\,d_1}\right), \qquad \vec{a}^{\,d} = (e^{d}a, e^{-d}b)\,. \tag{77}$$

The factor systems [17] $\omega^{\widetilde{H}_o(1)} = e^{ih\xi}$ of the group $\widetilde{H}_o(1)$ are

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1\,\hat{g}_2) = e^{ih\xi\left(\vec{a}_1, \vec{a}_2^{\,d_1}\right)}\,. \tag{78}$$

In Reference [9] the UIRs of the Poincaré (1+1) group are constructed. Taking into account the relationship between this group and $\widetilde{H}_o(1)$ as displayed in paragraph 3.4, it is straightforward to rewrite these representations in relation to our results obtained for $\widetilde{H}_o(1)$.

4.3. UR of the extended Weyl-Heisenberg group $\widetilde{H}(1)$

The invariance under orientation, or invariance under the change $x \leftrightarrow -x$ suggest the need for the use of the parity operator, $\mathcal{P}$. The connected group $\widetilde{H}_o(1)$ plus the parity operator provide the general group of invariance of the real line as a semidirect product of the group of the discrete symmetries $\mathcal{V}_2 = \{\mathcal{I}, \mathcal{P}\}$, where $\mathcal{I}$ is the identity operator, and the affine Weyl-Heisenberg group (53). This semidirect group is

$$\widetilde{H}(1) = \mathcal{V}_2 \odot \widetilde{H}_o(1)\,. \tag{79}$$

The action of the parity into $\widetilde{H}_o(1)$ is given by

$$(\theta, a, b, d) \;\; \stackrel{\mathcal{P}}{\rightarrow} \;\; (\theta, -a, -b, d)\,. \tag{80}$$

The elements of the group $\widetilde{H}(1)$ can be written as

$$\tilde{g} = (\hat{g}, \alpha)\,, \qquad \hat{g} = (\theta, a, b, d) \in \widetilde{H}_o(1)\,,\; \alpha \in \{\mathcal{I}, \mathcal{P}\}\,. \tag{81}$$

The law group of $\widetilde{H}(1)$ is given by

$$\tilde{g}_1 \cdot \tilde{g}_2 = (\hat{g}_1, \alpha_1)(\hat{g}_2, \alpha_2) = (\hat{g}_1 \cdot \hat{g}_2^{\alpha_1}, \alpha_1\alpha_2)\,, \tag{82}$$

where

$$\hat{g}^{\alpha} = \left\{ \begin{array}{ll} \hat{g} & \text{if } \; \alpha = \mathcal{I} \\ \hat{g}^{\mathcal{P}} = (\theta, -a, -b, d) & \text{if } \; \alpha = \mathcal{P} \end{array} \right. , \qquad \hat{g} = (\theta, a, b, d)\,. \tag{83}$$

Following (82) and (76), the inverse of $\tilde{g}$ is

$$\tilde{g}^{-1} = (\hat{g}, \alpha)^{-1} = (\left(\hat{g}^{-1}\right)^{\alpha}, \alpha) = (-\theta, -e^{-d}a^{\alpha}, -e^{d}b^{\alpha}, -d, \alpha)\,. \tag{84}$$

Coming back to the parity operator $\mathcal{P}$, we may select two choices for its representation $U(\mathcal{P})$, either by a linear or by an antilinear operator. In the sequel, we analyze both possibilities [13]:

I) If we look at $\mathcal{P}$ as a linear operator, then $H = \widetilde{H}(1)$. The factor systems [13,17] of this group can be written as

$$\omega^{\widetilde{H}(1)}(\hat{g}_1, \alpha_1, \hat{g}_2, \alpha_2) = \omega^{\widetilde{H}_o(1)}(\hat{g}_1, \hat{g}_2^{\alpha_1})\, \omega^{\mathcal{V}_2}(\alpha_1, \alpha_2)\, \Lambda(\hat{g}_2, \alpha_1)\,, \tag{85}$$

where the factors $\omega^{\widetilde{H}_o(1)}$, $\omega^{\mathcal{V}_2}$ and Λ, with $\Lambda : \widetilde{H}_o(1) \times \mathcal{V}_2 \rightarrow U(1)$ fulfil equations (A155) and (A156) in Appendix G. Note that the action ${}^{*}H\big|_{\mathcal{V}_2}(\mathcal{P})$ is here trivial after the linearity of $\mathcal{P}$ and the form of (A155) and (A156)

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^{\alpha}, \hat{g}_2^{\alpha}) \quad = \quad \omega^{\widetilde{H}_o(1)}(\hat{g}_1, \hat{g}_2)\, \Lambda(\hat{g}_1\hat{g}_2, \alpha)(\Lambda(\hat{g}_1, \alpha)\, \Lambda(\hat{g}_2, \alpha))^{-1}\,, \tag{86}$$

$$\Lambda(\hat{g}, \alpha_1\alpha_2) \quad = \quad \Lambda(\hat{g}^{\alpha_2}, \alpha_1)\, \Lambda(\hat{g}, \alpha_2)\,. \tag{87}$$

In this case, we take the factors (78) for $\widetilde{H}_o(1)$. Then,

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^{\alpha}, \hat{g}_2^{\alpha}) = \omega^{\widetilde{H}_o(1)}(\hat{g}_1, \hat{g}_2)\,, \qquad \alpha \in \{\mathcal{I}, \mathcal{P}\}\,. \tag{88}$$

Hence Λ, is 2-coboundary (A148) so that we may dismiss it. The factor $\omega^{\mathcal{V}_2}(\alpha_1, \alpha_2)$ is trivial in this case as we easily show. As we easily chaeck $\omega^{\mathcal{V}_2}(\mathcal{P}, \mathcal{P}) = m \in U(1)$ while all the others from (A145) are trivial, i.e.,

$$\omega(\mathcal{I}, \mathcal{I}) = \omega(\mathcal{I}, \mathcal{P}) = \omega(\mathcal{P}, \mathcal{I}) = 1\,. \tag{89}$$

Now from (A148), we can write

$$\omega_1(\mathcal{P}, \mathcal{P}) = m = \lambda(\mathcal{P})\, \lambda(\mathcal{P})\, \lambda(\mathcal{P}^2)^{-1} = \lambda(\mathcal{P})^2 \quad \Rightarrow \quad \lambda(\mathcal{P}) = m^{1/2} \tag{90}$$

because $\lambda(\mathcal{I}) = 1$. Thus, the UIRs are given by

$$(U_{h,\mathcal{C}}(\hat{g}, \alpha) f)(x) = e^{-d/2}\, e^{ih(\theta + C)}\, e^{iha(x - b/2)^{\alpha}}\, f(e^{-d}(x - b)^{\alpha})\,. \tag{91}$$

II) The second option is that $\mathcal{P}$ be an antilinear operator and, then, $H = \widetilde{H}_o(1)$. The factors for $\widetilde{H}(1)$ satisfy the relation (85). Also $\omega^{\widetilde{H}_o(1)}$, $\omega^{\mathcal{V}_2}$ and Λ verify equations (A155) and (A156). Since $\mathcal{P}$ is an antilinear operator, the action ${}^{*}H\big|_{\mathcal{V}_2}(\mathcal{P})$ is the complex conjugation. Hence,

$$\omega^{\widetilde{H}_o(1)}(\hat{g}_1^{\alpha}, \hat{g}_2^{\alpha}) \quad = \quad \omega^{\widetilde{H}_o(1)}(\hat{g}_1, \hat{g}_2)^{{}^{*}H|_{\mathcal{V}_2}(\alpha)}\, \Lambda(\hat{g}_1\hat{g}_2, \alpha)(\Lambda(\hat{g}_1, \alpha)\, \Lambda(\hat{g}_2, \alpha))^{-1}\,, \tag{92}$$

$$\Lambda(\hat{g}, \alpha_1\alpha_2) \quad = \quad \Lambda(\hat{g}^{\alpha_2}, \alpha_1)\, \Lambda(\hat{g}, \alpha_2)^{{}^{*}H|_{\mathcal{V}_2}(\alpha)}\,. \tag{93}$$

From (78) and (88), we conclude that (92) have no solutions for Λ unless $h = 0$. Moreover, $\omega^{\mathcal{V}_2}$ is not trivial now and we have $\omega_m^{\mathcal{V}_2}(\mathcal{P}, \mathcal{P}) = m$ with $m = \pm 1$. Then Λ becomes trivial. Its factor system is now

$$\omega^{\widetilde{H}(1)}(\hat{g}_1^{\alpha}, \hat{g}_2^{\alpha}) = \omega_m^{\mathcal{V}_2}(\alpha_1, \alpha_2)\,. \tag{94}$$

We have obtained a semi-unitary representation of the whole group such that its restriction to the connected component is a realization with $h = 0$. We have now

$$(U_{0,\mathcal{C}}(\hat{g}, \alpha) f)(x) = \Delta(\alpha)\, f(x^{\alpha})\,, \tag{95}$$

where $\Delta(\mathcal{I}) =$ Identity and $\Delta(\mathcal{P}) = \mathbf{K}$, the conjugation operator.

In the following, we shall focus our attention in the representations of class I, i.e., in the unitary representations (91) or (100) as they are the only non-trivial.

198 *4.4. Unitary representations of $\widetilde{H}(1)$ and Fourier Transform*

The above unitary representations can be translated to functions $\hat{f}(p)$ via the Fourier transform. Thus, for the representation (91) we have

$$\begin{aligned}\left(U_{h,\mathcal{C}}(\tilde{g})\hat{f}\right)(p) &= \int_{\mathbb{R}} e^{ihpx}\,(U_{h,\mathcal{C}}(\tilde{g})f)(x)\,dx \\ &= \int_{\mathbb{R}} e^{-d/2}\,e^{ihp^{\alpha}x^{\alpha}}\,e^{ih(\theta+\mathcal{C})}\,e^{iha(x-b/2)^{\alpha}}\,f(e^{-d}(x^{\alpha}-b))\,dx \\ &= e^{d/2}\,e^{ih(\theta+\mathcal{C})}\,e^{ihpb}\int_{\mathbb{R}} e^{ihe^{d}(p+a)^{\alpha}\,y^{\alpha}}\,f(y^{\alpha})\,dy\,. \\ &= e^{d/2}\,e^{ih(\theta+\mathcal{C})}\,e^{ihpb}\,\hat{f}(e^{d}(p+a)^{\alpha})\,dy\,.\end{aligned} \tag{96}$$

In the third identity in (96), we have used a new variable, defined as $y^{\alpha}=e^{-d}(x-b)^{\alpha}$. For the representation (95), we have used

$$\begin{aligned}\left(U_{0,\mathcal{C}}(\hat{g},\alpha)\hat{f}\right)(p) &= \int_{\mathbb{R}} e^{\varepsilon_{\alpha}ihpx}\,(U_{0,\mathcal{C}}(\hat{g},\alpha)f)(x)\,dx \\ &= \int_{\mathbb{R}} e^{\varepsilon_{\alpha}ihpx}\,\Delta(\alpha)\,f(x^{\alpha})\,dx = \Delta(\alpha)\int_{\mathbb{R}} e^{ihp^{\alpha}x^{\alpha}}\,f(x^{\alpha})\,dx \\ &= \Delta(\alpha)\,f(p^{\alpha})\,,\end{aligned} \tag{97}$$

199 where $\varepsilon_{\alpha}=\mathrm{sign}(\Delta(\alpha)i)$.

200 5. A generalization of the Hermite functions

The most used orthonormal basis for the Hilbert space $L^2(\mathbb{R})$ is the basis of the normalized Hermite functions, $\{\psi_n(x)\}$, defined as [18,19]

$$\psi_n(x):=\frac{e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}\,H_n(x)\,,\quad H_n(x):=(-1)^n\,e^{x^2}\,\frac{d^n}{dx^n}e^{-x^2}\,,\quad n=0,1,2,\dots\,, \tag{98}$$

where the $H_n(x)$ are the so called the (physicists) Hermite polynomials [10,20,21]. We recall the following well known relations that assure that the normalized Hermite functions are a basis for $L^2(\mathbb{R})$:

$$\int_{-\infty}^{+\infty}\psi_n(x)\,\psi_{n'}(x)^*\,dx=\delta_{nn'}\,,\qquad \sum_{n=0}^{\infty}\psi_n(x)\,\psi_n(y)^*=\delta(x-y)\,. \tag{99}$$

201 The basis of Hermite functions (98) has two interesting properties: (i) nonetheless
202 the complex character of the functions in the Hilbert space $L^2(\mathbb{R})$, all Hermite functions
203 are real and (ii) they are eigenfunctions of the FT and also of the IFT (29) [10] .

We can restrict the UIR of $\widetilde{H}(1)$ (91) to those elements $\tilde{g}=(\hat{g},\alpha)$ with $\theta=0$, recall that $\tilde{g}=(\theta,a,b,d,\alpha)$. Let us denote $\tilde{g}_0=(0,a,b,d,\alpha)$ and take $\mathcal{C}=0$. The action of $\tilde{g}_0$ on the Hermite functions is given by

$$(U_{h,0}(\tilde{g}_0)\psi_n)(x)=e^{-d/2}\,e^{iha(x-b/2)^{\alpha}}\,\psi_n(e^{-d}(x-b)^{\alpha})\,. \tag{100}$$

Next, let us consider the inner product (99) and compute

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} [U_{h,0}(\hat{g}_0,\alpha)\psi_n](x)\, [U_{h,0}(\hat{g}_0,\alpha)\psi_{n'}](x)^*\, dx \\
&\qquad = e^{-d} \int_{-\infty}^{+\infty} \psi_n(e^{-d}(x-b)^\alpha)\, \psi_{n'}(e^{-d}(x-b)^\alpha)\, dx \\
&\qquad = \int_{-\infty}^{+\infty} \psi_n(y^\alpha)\, \psi_{n'}(y^\alpha)\, dy = \int_{-\infty}^{+\infty} \psi_n(y)\, \psi_{n'}(y)\, dy \\
&\qquad = \delta_{nn'}\,,
\end{aligned}
\tag{101}
$$

where we have used the change of variables $y^\alpha = e^{-d}(x-b)^\alpha$ in the third equality. In addition, we have that

$$
\begin{aligned}
&\sum_{n=0}^{+\infty} [U_{h,0}(\hat{g}_0,\alpha)\psi_n](x)^*\, [U_{h,0}(\hat{g}_0,\alpha)\psi_n](y)^* \\
&\qquad = e^{-d}\, e^{-iha(x^\alpha - y^\alpha)} \sum_{n=0}^{\infty} \psi_n(e^{-d}(x-b)^\alpha)\, \psi_{n'}(e^{-d}(y-b)^\alpha) \qquad (102) \\
&\qquad = e^{-d}\, e^{-iha(x^\alpha - y^\alpha)}\, \delta(e^{-d}(x^\alpha - y^\alpha)) \\
&\qquad = \delta(x^\alpha - y^\alpha) = \delta(x-y)\,. \qquad (103)
\end{aligned}
$$

204 If we split (102) into its real and imaginary parts, we arrive to the following pair of
205 equations, both together equivalent to (103):

$$
\begin{aligned}
\sum_{n=0}^{\infty} \cos[ha(x-y)]\, \psi_n(kx+b)\, \psi_n(ky+b) &= \delta(x-y)\,, \\
\sum_{n=0}^{\infty} \sin[a(x-y)]\, \psi_n(kx+b)\, \psi_n(ky+b) &= 0\,.
\end{aligned}
\tag{104}
$$

Now, let us consider an element $\tilde{g}_0 = (0,a,b,d) \in \widetilde{H}(1)$ and its inverse given by (84), i.e., $\tilde{g}_0^{-1} = (0, -e^{-d}a^\alpha, -e^d b^\alpha, -d, \alpha)$. Then (100) becomes

$$
(U_{h,0}(\hat{g}_0,\alpha)\psi_n)(x) = e^{-d/2}\, e^{iha(x-b/2)^\alpha}\, \psi_n(e^{-d}(x-b)^\alpha)\,. \tag{105}
$$

After (26) and (105), it becomes obvious that the Parity induces a particular case of dilatation, since

$$
e^{-d}\, x^\alpha = \begin{cases} e^{-d}\, x = kx & \text{with } k>0 \quad \text{if } \alpha = \mathcal{I} \\ -e^{-d}\, x = kx & \text{with } k<0 \quad \text{if } \alpha = \mathcal{P} \end{cases} \tag{106}
$$

206 In the sequel, we shall introduce a generalization of the Hermite functions and
207 study some of their properties.

208 *5.1. Generalized Hermite Functions*

Let us define a three-parameter family of square integrable functions based on the Hermite functions as follows:

$$
\chi_n(x,k,a,b) := \sqrt{|k|}\, e^{-iax}\, \psi_n(kx+b)\,, \qquad a,b \in \mathbb{R};, k \neq 0 \in \mathbb{R}^*\,. \tag{107}
$$

From the two expression in (99), we readily obtain, respectively, the following relations valid for $n, n' = 0, 1, 2, \ldots$:

$$\begin{aligned} \int_{-\infty}^{+\infty} \chi_n(x,k,a,b)\, \chi_{n'}(x,k,a,b)^*\, dx &= \delta_{nn'}\,, \\ \sum_{n=0}^{\infty} \chi_n(x,k,a,b)\, \chi_n(y,k,a,b)^* &= \delta(x-y)\,, \end{aligned} \tag{108}$$

which show that for fixed a, b and $k \neq 0$, the functions $\chi_n(x,k,a,b)$, $n = 0, 1, 2, \ldots$, form a basis for $L^2(\mathbb{R})$. Thus, we have constructed a family of bases for this Hilbert space, which under transformations by the FT and the IFT becomes,

$$\begin{aligned} FT[\chi_n(x,k,a,b), x, y] &= i^n \chi_n(y, k^{-1}, b, -a)\,, \\ IFT[\chi_n(y,k,a,b), y, x] &= (-i)^n \chi_n(x, k^{-1}, -b, a)\,. \end{aligned} \tag{109}$$

This is a generalization of (29), which shows that the Fourier transform and its inverse are symmetry transformations of the representations of the Weyl-Heisenberg group $H(1)$. After (109) we realize that both are symmetry transformations of the $\widetilde{H}(1)$ group as well. Obviously, both expressions of (109) are written in terms of the coordinate representation. Their explicit forms in terms of the momentum representation can be easily obtain. We see that under the FT (IFT) transform, the basis $\{\chi_n(x,k,a,b)\}$ changes into $\{\chi_n(x,k^{-1},b,-a)\}$ (or viceversa). Thus, the generalized Hermite functions are not eigenvectors of the FT (IFT) contrarily to the Hermite functions (29). On the other hand, if

$$k = k^{-1},\ a = b,\ b = -a \qquad \Longrightarrow \qquad k = \pm 1,\ a = 0,\ b = 0 \tag{110}$$

the corresponding generalized Hermite functions are eigenvalues of the FT (IFT). This only happens for the standard Hermite functions.

Consequently, the Fourier transform and its inverse, transform bases into bases of $L^2(\mathbb{R})$, which are relevant for symmetry transformations after the action of groups like $H(1)$ and $\widetilde{H}(1) \simeq \widetilde{P}(1+1)$. In the first case, the FT and the IFT transform bases into bases. In the second, they transform any basis of the family into another basis of the same family, although having with different parameters as we see in (110). Furthermore, we find another difference between the two approaches: while the Hermite functions are real, the generalized Hermite functions are not real and only they are real for the particular choice $a = 0$, where the three-parameter family of bases becomes restricted to a two-parameters family.

Finally, we may disregard translational invariance and consider self-similarity and invalid orientation only. Then, the three-parameter family of bases (107) reduces to a one-parameter family, depending only on $k \in \mathbb{R}^*$. This is

$$\{\chi_n(x,k)\}_{\in\mathbb{R}^*}^{n\in\mathbb{N}} \equiv \{\chi_n(x,k,0,0)\}_{\in\mathbb{R}^*}^{n\in\mathbb{N}} \equiv \{\sqrt{|k|}\ \psi_n(kx)\}_{\in\mathbb{R}^*}^{n\in\mathbb{N}}\,. \tag{111}$$

We shall discuss the importance of these basis in the sequel.

5.2. $\widetilde{P}(1,1)$ *and the "classical" real line*

In Section 3, we have extended the group $H(1)$ so as to include non-commutativity and self-similarity. Thus, we arrived to $\widetilde{H}(1)$ which is isomorphic to an extension of the Poincaré group in 1+1 dimensions, $\widetilde{P}(1,1)$, see Subsection 3.4. Nevertheless, its is always possible to start from symmetries of "classical physics" given by $P_o(1,1)$, which is the connected component of the Poincaré group in $(1+1)$ dimensions to arrive again to $\widetilde{P}(1+1)$ using the central extension and the $\mathcal{P}\,\mathcal{T}$ symmetry as a tool.

In order to implement this programme, we start withwith the connected algebra $\mathrm{Lie}(P_o(1+1)) = \mathcal{P}_o(1+1)$ with basis $\{H, P, K\}$ [9]. Here, H and P are the infinitesimal generators of the time and space translations, respectively, and K is the infinitesimal generator of the Lorentz transformations. Their commutation relations are

$$[H,P]=0, \quad [H,K]=P, \quad [P,K]=H\,. \tag{112}$$

The action of an arbitrary element $(a^0, a^1, \Lambda(\eta)) \in P_o(1+1)$ on the space-time is given by

$$(a,b,\Lambda(\eta))\mathbf{x} \equiv \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} + \begin{pmatrix} a^0 \\ a^1 \end{pmatrix}, \tag{113}$$

where $\mathbf{x} = (x^0, x^1)^T$. Using relations (57) and (59), we obtain a new basis $\{X, P, K\}$ such that $[X,P]=0$. These new basis elements are related to to the light-cone coordinates:

$$x_\pm = x^0 \pm x^1 \quad \Leftrightarrow \quad x^0 = \frac{x_+ + x_-}{2}\,, \quad x^1 = \frac{x_+ - x_-}{2}\,. \tag{114}$$

The commutator $[X,P]=0$ justifies the label "classicality" for the symmetry with group of invariance $P_o(1,1)$. As previously remarked, the group $P(1,1)$ is the result of the addition of the operator $\mathcal{PT}$ to $P_o(1,1)$. The action of each $g=(a,b,d,\alpha) \in P(1,1)$ on any square integrable function in the coordinate and the momentum representation is ($x_+ = x$, $x_- = p$), respectively according to (105) and (106):

$$\begin{aligned} U(g)\, f(x) &= |k|^{-1/2}\, f(k^{-1}(x-b)) \\ U(g)\, f(p) &= |k|^{1/2}\, f(k(p+a)) \end{aligned} \qquad k = \left(e^d\right)^\alpha. \tag{115}$$

Now, let us consider self-similarity and parity transformations on the line, performing the operations $x \Longrightarrow kx$ and $p \Longrightarrow k^{-1}p$, along the symmetries induced by these transformations. The translation invariance introduced in Quantum Physics by the non-commutativity is not relevant here. For $k \neq 0$ and real, equation (111) yields to

$$\chi_n(x,k) = \sqrt{|k|}\, \frac{e^{-k^2x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}\, H_n(kx)\,. \tag{116}$$

From (108), we readily obtain for any $k \in \mathbb{R}^*$

$$\int_{-\infty}^{+\infty} \chi_n(x,k)\, \chi_{n'}(x,k)^*\, dx = \delta_{nn'}\,, \qquad \sum_{n=0}^{\infty} \chi_n(x,k)\, \chi_n(y,k)^* = \delta(x-y)\,. \tag{117}$$

This shows that $\{\chi_n(x,k)\}$ is a one-parameter family of orthonormal bases for $L^2(\mathbb{R})$. Under the Fourier transform and its inverse, these bases become

$$FT[\chi_n(x,k),x,y] = i^n \chi_n(y,k^{-1})\,, \qquad IFT[\chi_n(p,k^{-1}),y,x] = (-i)^n \chi_n(x,k)\,. \tag{118}$$

The functions belonging to the family of bases $\{\chi_n(x,k)\}$ are all real for all $k \in \mathbb{R}^*$, a property shared by the basis of Hermite functions $\{\psi_n(x)\}$. This means that both set of bases are equally appropriate for the Hilbert space $L^2(\mathbb{R})$, no matter if this is a Hilbert space on the set of either the complex or the real field. This property is in general false if we choose $\{\chi_n(x,k,a,b)\}$ as a basis, which for most values of the parameters is solely a basis for $L^2(\mathbb{R})$ as a Hilbert space on the complex field.

On the other hand, all the bases $\{\psi_n(x)\}$, $\{\chi_n(x,k,a,b)\}$ and $\{\chi_n(x,k)\}$ have a similar behaviour under Fourier transform and its inverse, so that all serve as bases in the momentum representation (29), (109) and (118).

5.3. Generalized Hermite polynomials

Some comments on the functions $\{\chi_n(x,k)\}$ are in order here. For each value of $n = 0, 1, 2, \ldots$, these functions include the factor $H_n(kx)$, which is nothing else that the

n-th Hermite polynomial (98) with a dilation on its argument. The Rodrigues formula for $H_n(kx)$ follows straightforwardly from (98) and gives

$$H_n(kx) = (-1)^n\, e^{k^2x^2}\,\frac{d^n}{k^n dx^n} e^{-k^2x^2} = \left(2kx - \frac{1}{k}\frac{d}{dx}\right)^n * 1\,, \tag{119}$$

with generating function

$$e^{2kxt-t^2} = \sum_{n=0}^{\infty} H_n(kx)\frac{t^n}{n!}\,. \tag{120}$$

Other relevant formulas or recurrence relations of the Hermite polynomials $H_n(x)$ are are straightforwardly obtained from $H_n(kx)$. As for instance, the differential equation for $H_n(kx)$:

$$H''_n(kx) - 2k^2xH'_n(kx) + 2k^2nH_n(kx) = 0\,. \tag{121}$$

5.4. The set of functions $\{\chi_n(x,k)\}$ as basis for representations of the WH algebra $h(1)$

As is well known, $\{\psi_n(x)\} \equiv \{\chi_n(x,1)\}$ is a basis for representations of the WH algebra $h(1)$ [22], which are supported on $L^2(\mathbb{R})$. In addition, following previous experiences with the use of ladder operators, we may also here construct a set of operators, $\{H, A_+, A_-\}$, for $h(1)$ such that the basis functions $\{\chi_n(x,k)\}$ are eigenfunctions of H, and are transformed into each other using the others, $A_\pm$, as ladder operators. The explicit form of these operators for $h(1)$ is

$$H := \frac{1}{2}(k^2X^2 + k^{-1}P^2)\,, \qquad A_\pm := \frac{k}{\sqrt{2}}\,x \mp \frac{1}{\sqrt{2}\,k}\,\frac{d}{dx}\,. \tag{122}$$

Thesy fulfil the following commutation relations in $h(1)$:

$$[H, A_\pm] = \pm A_\pm\,, \qquad\qquad [A_+, A_-] = -1\,. \tag{123}$$

It is quite simple to show that the operators $A_\pm$ act as ladder operators with respect to the family of bases $\{\chi_n(x,k)\}$:

$$A_+\,\chi_n(x,k) = \sqrt{n+1}\,\chi_{n+1}(x,k)\,, \quad A_-\,\chi_n(x,k) = \sqrt{n}\,\chi_{n-1}(x,k)\,. \tag{124}$$

Then, we may define the number operator $N := A_+A_-$ so that from (124) we have

$$N\chi_n(x,k) = n\,\chi_n(x,k)\,, \tag{125}$$

as we may have expected. Note that $H = N + 1/2$ and that relations (123) and (124) are independent on k. This representation of $h(1)$ has the zero operator as Casimir [22,23]:

$$\left[H - \frac{1}{2}\{A_+, A_-\}\right]\chi_n(x,k) = 0\,, \tag{126}$$

This relation may be extended to the common domain of the operators $\{H, A_+, A_-\}$. This domain is dense in $L^2(\mathbb{R})$, since it contains the Schwartz space. We also may write the Casimir in terms of the basis $\{X, P, H\}$. Needless to say that, in this explicit realization (122) the Casimir is also zero, i.e.,

$$\left[H\; - \;\frac{1}{2}(k^2X^2 + k^{-2}P^2)\right]\chi_n(x,k) = 0\,. \tag{127}$$

Observe that the formal expression for the Casimir depends now on k. This is also the case of the kinetic energy operator, which on each member of the basis $\{\chi_n(x,k)\}$ acts as

$$\frac{P^2}{2}\,\chi_n(x,k) = k^2\left[(N+1/2)\; - \;\frac{k^2X^2}{2}\right]\chi_n(x,k)\,. \tag{128}$$

Note that the right hand side of (128) goes to the free particle of zero energy in the limit $k \to 0$. This exhibits a limiting connection between the Harmonic Oscillator and the free particle within the context of Quantum Mechanics.

5.5. Representation on rigged Hilbert space

Thus far, we have discussed representations of some Lie algebras as operators on the Hilbert space $L^2(\mathbb{R})$. These operators, although self-adjoint, are unbounded. It would have been interesting to represent these algebras of operators as *continuous* operators on some topological vector space. The formalism of *rigged Hilbert spaces* (RHS), or Gelfand triplets is very suitable in achieving this goal. A rigged Hilbert space is a triplet of spaces [24].

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{129}$$

such that $\mathcal{H}$ is a complex separable infinite dimensional Hilbert space. The locally convex space Φ is endowed with a strictly finer topology than the inherited by Φ from $\mathcal{H}$, so that the canonical injection $\Phi \longmapsto \mathcal{H}$ is continuous. Finally, the space of all continuous *antilinear* functionals on Φ is $\Phi^\times$, which is the *antidual* space of Φ. It may have any topology compatible with the dual pair $\{\Phi, \Phi^\times\}$, i.e., weak, strong or MacKey. We usually choose this antiduality instead of duality for notational convenience [25,26]. See also [10,27–30].

The simplest example for Φ is the Schwartz space $\mathcal{S}$ of all complex indefinitely differentiable functions on the real line, such that they and their derivatives go to zero at the infinity faster than the inverse of any polynomial. A good discussion on the Schwartz space may be found in [31]. The Schwartz space contains all the basis $\{\chi_n(x,k,a,b)\}$ and

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times \tag{130}$$

is a RHS. In the sequel, we shall see why this RHS is suitable for our purposes. We should note first that if A is a symmetric (Hermitian) continuous operator [31] on $\mathcal{S}$, then, it may be extended to a continuous operator on $\mathcal{S}^\times$ by using the *duality formula*:

$$\langle A\varphi|F\rangle = \langle \varphi|AF\rangle \,, \qquad \forall\, \varphi \in \mathcal{S} \,, \qquad \forall\, F \in \mathcal{S}^\times \,, \tag{131}$$

and $\langle \varphi|F\rangle$ is the action of $F \in \mathcal{S}^\times$ on $\varphi \in \mathcal{S}$.

The usual Frèchet topology on $\mathcal{S}$ is given by a countable set of norms. There are several countable families of norms given the same topology on $\mathcal{S}$, although the most convenient for our purposes in the following [31]: A square integrable function $f(x) \in L^2(\mathbb{R})$ with

$$f(x) = \sum_{n=0}^{\infty} a_n\, \psi_n(x) \tag{132}$$

is in $\mathcal{S}$ if and only if

$$\sum_{n=0}^{\infty} |a_n|^2\, (n+1)^{2r} < \infty \,, \qquad r = 0,1,2,\ldots\,. \tag{133}$$

Then, for any $f \equiv f(x) \in \mathcal{S}$, we define the following countable family of norms, $p_r(f)$, as:

$$p_r(f) := \sqrt{\sum_{n=0}^{\infty} |a_n|^2\, (n+1)^{2r}} \,, \qquad r = 0,1,2,\ldots\,. \tag{134}$$

It is worthy noticing that for $r = 0$, we have the Hilbert space norm, so that the canonical injection $i : \mathcal{S} \longmapsto L^2(\mathbb{R})$ is continuous.

What happens if we use the other families of bases such as $\{\chi_n(x,k)\}$ or $\{\chi_n(x,k,a,b)\}$? Note that for fixed real numbers a, b and $k \neq 0$, we have

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} b_n\, \chi_n(x,k,a,b) = \sum_{n=0}^{\infty} b_n\, \sqrt{k}\, e^{-iax}\, \psi_n(kx+b) \\
&= \sum_{n=0}^{\infty} b_n\, \sqrt{k}\, e^{-i(y/k-b/k)}\, \psi_n(y)\,,
\end{aligned} \tag{135}
$$

so that for all $r = 0,1,2,\dots,$

$$
p_r^2(f) = k \sum_{n=0}^{\infty} |b_n|^2\, (n+1)^{2r}\,, \tag{136}
$$

and hence, $|a_n|^2 = k\,|b_n|^2$, $n = 0,1,2,\dots$, for k fixed. Same for the span of $f(x)$ in terms of the family of basis $\{\chi_n(x,k)\}$.

With these ideas in mind, it is rather trivial to prove that the operators $A_\pm$, H and N, defined in (122)-(124) are continuous operators on $\mathcal{S}$ and, therefore, continuously extensible to $\mathcal{S}^\times$. This comes from the following result [31]:
Theorem.- Let Φ a locally convex space for which the topology is defined by the family of seminorms $\{p_i(\cdot)\}_{i\in I}$. A *linear* operator $A : \Phi \longmapsto \Phi$ is continuous on Φ if and only if for each seminorm p_j of the previous family, there exist a positive constant $K > 0$ and k fixed seminorms of the same collection $p_{n_1}, p_{n_2}, \dots, p_{n_k}$ such that *for all* $\varphi \in \Phi$, we have

$$
p_i(\varphi) \leq K\{p_{n_1}(\varphi) + p_{n_2}(\varphi) + \cdots + p_{n_k}(\varphi)\}\,. \tag{137}
$$

The constant K, the seminorms $p_{n_1}, p_{n_2}, \dots, p_{n_k}$ and its number k may depend on p_j.

Proof.- In order to prove our claim, let us first show that for any $f(x) \in \mathcal{S}$, then $A_\pm f(x) \in \mathcal{S}$ and same property is true for H and N. Take,

$$
[A_+ f](x) = \sum_{n=0}^{\infty} a_n\, \sqrt{n+1}\, \chi_{n+1}(x,k)\,, \tag{138}
$$

so that for any norm, p_r, in (134), one has for $r = 0,1,2,\dots$:

$$
\begin{aligned}
p_r(A_+ f) &= \sqrt{k}\sqrt{\sum_{k=0}^{\infty} |a_n|^2\, (n+1)\, (n+1)^{2r}} \leq \sqrt{k}\sqrt{\sum_{k=0}^{\infty} |a_n|^2\, (n+1)^{2(r+1)}} \\
&\leq \sqrt{k}\, p_{r+1}(f)\,.
\end{aligned} \tag{139}
$$

This proves both that $A_+ f \in \mathcal{S}$ for any $f \in \mathcal{S}$ and that, according to the previous Theorem, A_+ is continuous on $\mathcal{S}$. Similar proofs can be used for A_-, H and N. Since,

$$
X = \frac{1}{\sqrt{2}\,k}\,(A_+ + A_-)\,, \qquad P = \frac{ik}{\sqrt{2}}\,(A_- - A_+)\,, \tag{140}
$$

it comes that X and P are also continuous operators on $\mathcal{S}$. The same property holds for the parity operator $\mathbb{P}$. All these operators are continuously extensible to $\mathcal{S}^\times$.

6. Concluding remarks

We have studied how invariance properties on the real line under geometric transformations like translations, dilations and inversions can be represented as unitary mappings on $L^2(\mathbb{R})$. This representation transforms the basis of Hermite functions in new basis of functions, which generalize the notion of Hermite functions. In the process, we arrive to the Euclidean group on the line $E(1)$.

The properties of the Fourier transform and, in particular, that one that transform coordinates into momenta and viceversa, $\mathrm{FT}[f(x),x,p] = \widehat{f}(p)$, have forced us to introduce an enlarged group adding a new generator, so as to extend the Weyl-Heisenberg group $H(1)$ to the group $\widetilde{H}(1)$. This group is isomorphic to the central extension of the Poincaré group in (1+1) dimensions enlarged with the $\mathcal{PT}$ transformation. Analogously, $\widetilde{H}(1)$ is isomorphic to the central extension group of isometries of the two dimensional space $\mathbb{R}^2$ with signature $(+,-)$. This extension is denoted as $\widetilde{P}(1,1)$ or also $\widetilde{E}(1,1)$.

One representation of the infinitesimal generators of $\widetilde{E}(1,1)$ as operators on $L^2(\mathbb{R})$ is explicitly given by $X = x,\; P = -(\mathbf{i}/h)\,\partial_x,\; D = -\frac{i}{2h}(x\partial_x + \partial_x\, x),\; I = h$. While X and P algebraically express the connection between configuration and momenta representation described analytically by the Fourier transform, the dilatation operator is given so as to obtain the factor $e^{\mp d/2}$. This factor is necessary in order to normalize the representation (73), (96) and (101). Finally, if we choose for h the value $1/\hbar$, we recover all the well known results of Quantum Mechanics.

We have introduced a generalisation of the Hermite functions, which are quite appropriate to our discussion due to their behaviour under transformations by the group $\widetilde{H}(1)$. These new generalized Hermite functions also provide a 3-parameter family of bases of $L^2(\mathbb{R})$. However, these generalized Hermite functions are not eigenvectors of the Fourier transform on $L^2(\mathbb{R})$, no matter if the Fourier transform maps orthonormal basis into orthonormal basis. We may say that, from this point of view, the usual Hermite functions are those with better properties among all types of generalized Hermite functions.

As a final remark, let us mention that the generalized Hermite functions are discrete bases in a rigged Hilbert space on which the generators of $H(1)$ or $\widetilde{H}(1)$ are continuous.

Appendix G Factor systems of semidirect products

Let G be a connected Lie group acting transitively on a differentiable manifold X. A unitary realization of G on the vector space of functions $f : X \to \mathbb{C}$ can be defined as [32,33]

$$(U(g)f)(gx) = \eta(g,x)\, f(x)\,, \tag{A141}$$

where η is a function $\eta : G \times X \to U(1)$ verifying

$$\eta(g',gx)\,\eta(g,x) = \omega(g',g)\,\eta(g'gx)\,, \tag{A142}$$

where ω is a system of factors of G, i.e.,

$$G \times G \xrightarrow{\omega} U(1) \tag{A143}$$

such that

$$\omega(g_1,g_2)\,\omega(g_1g_2,g_3) = \omega(g_2,g_3)\,\omega(g_1,g_2g_3)\,, \qquad \forall g_1,g_2,g_3 \in G\,. \tag{A144}$$

and

$$\omega(e,e) = \omega(e,g) = \omega(g,e) = 1\,, \qquad e = \text{identity element of}\, G\,,\;\; \forall g \in G\,. \tag{A145}$$

The *factors* or *factor system* ω is a 2-cocycle. The set of 2-cocycles is denoted by $\mathbf{Z}^2(G,U(1))$ [34]. We recall that

$$U(g_1g_2) = \omega(g_1,g_2)\,U(g_1)\,U(g_2)\,. \tag{A146}$$

Two factor systems ω_1 and ω_2 are said equivalent if there is a$\lambda : G \to U(1)$ such that

$$\omega_1(g_1,g_2) = \lambda(g_1)\,\lambda(g_2)\,\lambda(g_1,g_2)^{-1}\,\omega_2(g_1,g_2) \tag{A147}$$

A factor system ω is said trivial or equivalent to 1 (or a 2-coboundary) if

$$\omega_1(g_1,g_2) = \lambda(g_1)\,\lambda(g_2)\,\lambda(g_1,g_2)^{-1}\,. \tag{A148}$$

The 2-cocycles verifying (A148) or 2-coboundaries belong to $\mathbf{B}^2(G, U(1))$. The set of classes de equivalence of 2-cocycles determines the second cohomology group of G: $\mathbf{H}^2(G, U(1)) = \mathbf{Z}^2(G, U(1))/\mathbf{B}^2(G, U(1))$.

Let us consider a nonconnected Lie group, a subgroup $H \subset G$ of index 1 or 2 in G and a realization of G on the group of linear and antilinear operators in a Hilbert space such that $O(g)$ be linear or antilinear if $g \in H$ or $g \in G - H$. Hence the action on a function $f(x)$ would be

$$(U(g)f)(x) = \eta(g, x)\, f^g(g^{-1}x)\,, \tag{A149}$$

such that $f^g(x) = f(x)$ or $f^g(x) = f(x)^*$ if $g \in H$ or $g \in G - H$, respectively. We have the following relation

$$\eta(g', gx)\, \eta(g, x)^{g'} = \omega(g', g)\, \eta(g'gx)\,. \tag{A150}$$

The factor system verifies

$$\omega(g_1, g_2)\, \omega(g_1g_2, g_3) = \omega(g_2, g_3)^{g_1}\, \omega(g_1, g_2g_3)\,. \tag{A151}$$

Let G be a nonconnected Lie group which is a semidirect product $G = G_o \odot V$, where G_o is the connected component of the identity and $V = \pi_o(G)$ is the group of the connected components, with the action

$$g \in G \xrightarrow{\alpha \in V} g^\alpha \in G\,. \tag{A152}$$

By H we denote a closed subgroup of G of index 1 or 2. The action of G on $U(1)$ is denoted by ${}^*H$ such that

$$\beta \in U(1) \quad \xrightarrow[H \subset G]{g \in G} \quad \beta^g = \begin{cases} \beta & \text{if } g \in H \\ \beta^* & \text{if } g \in G - H \end{cases} \tag{A153}$$

and their restrictions to G_o and V give the actions of G_o and V on $U(1)$ (denoted by ${}^*H\big|_{G_o}$ and ${}^*H\big|_V$ respectively). In this case ${}^*H\big|_{G_o}$ is trivial. Then for each $[\omega] \in \mathbf{H}^2_{{}^*H}(G, U(1))$ we can find a factor system ω which is an element of $\mathbf{Z}^2_{{}^*H}(G, U(1))$ given by

$$\omega^G(g_1, \alpha_1; g_2, \alpha_2) = \omega^{G_o}(g_1, g_2^{\alpha_1})\, \omega^V(\alpha_1, \alpha_2)\, \Lambda(g_2, \alpha_1)\,, \tag{A154}$$

where $\omega^{G_o} \in \mathbf{Z}^2_{{}^*H\big|_{G_o}}(G_o, U(1))$, $\omega^V \in \mathbf{Z}^2_{{}^*H\big|_V}(V, U(1))$ and $\Lambda : G_o \times V \to U(1)$ verifying

$$\omega^{G_o}(g_1^\alpha, g_2^\alpha) = \omega^{G_o}(g_1, g_2)^{{}^*H\big|_V(\alpha)}\, \Lambda(g_1g_2, \alpha)(\Lambda(g_1, \alpha)\, \Lambda(g_2, \alpha))^{-1}\,, \tag{A155}$$

$$\Lambda(g, \alpha_1\alpha_2) = \Lambda(g^{\alpha_2}, \alpha_1)\, (\Lambda(g, \alpha_2))^{{}^*H\big|_V(\alpha_1)}\,. \tag{A156}$$

For more details see [13] and references therein.

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