I. INTRODUCTION

It is well known that electric fields constitute a good tool for confining relativistic massive charged particles in two or more spatial dimensions (see a detailed discussion by R. Hall et al in [1]), although, due to the strong Klein tunnel effect, they are not so useful in the case of two-dimensional massless particles [2, 3]. For this reason, in order to confine massless Dirac electrons in graphene, it is much better to use magnetic than electric fields [4–8]. However, due to its potential importance in practical applications, the electrical confinement of massless particles has also been investigated in a recent series of papers [9–17]. Note that since the origin of the transparency of potentials for zero-mass particles comes from transverse moments, electric fields could still be useful, for example in the study of quantum wires, where the momentum is parallel to the wall of the confining potential. Another case in which electric fields could produce confinement is when the quantum dot has a symmetry (for example, radial), with a well-defined angular momentum quantum number corresponding to a circular motion parallel to the quantum dot (there is a close connection between classical non-chaotic trajectory, superintegrability and transparent potentials [10]). However, it is also worth considering massive particles in two dimensions due to the introduction of new materials similar to graphene, such as silicene, germanene, stanene, and phosphorene, quite attractive from the point of view of topological insulators, where the charges acquire an effective mass from the spin-orbit interaction and from perpendicular electric fields [18].

An important issue closely related to the analysis of the confinement properties is the search for resonances. From a certain point of view, resonances are characterized by a kind of complex energy values $E = E_R + iE_I$, such that when the energy of an incident wave packet on the potential well (or barrier) is close to $E_R$ and if the imaginary part $E_I$ is very small, the outgoing wave packet produced may have a long stay in the region where the potential is significant, giving rise to a quasi-bound state [11]. The delay time can be calculated using Wigner’s formula for the phase change in the scattering states [19–21]. We will carry out this approximation on resonances for Dirac particles, with or without mass, moving in a plane to learn more about the phenomenon of confinement.

The objective of the present work is to draw some conclusions about the role of mass in the problem of confinement under electrostatic potentials. The most notable difference that we will find between the massive and massless cases is that increasing the depth of the well implies that the massive particles shift their bound states and their resonances towards negative energies, while in the case of the massless particles the well assumes the role of sink for positive energy resonances, but has little
influence on negative resonances, as we will see later.

The structure of the paper is as follows. Section 2 begins with the study of the bound states, resonance and dispersion of massive particles, showing the effect that increasing the depth of the well has on these states. Section 3 addresses the same issues for massless particles, while the final section is dedicated to discussing the main conclusions.

II. BOUND STATES AND RESONANCES FOR MASSIVE TWO-DIMENSIONAL DIRAC PARTICLES

We will start with the two-dimensional (2D) Dirac Hamiltonian to describe the interaction of particles of mass $m$ and charge $e$ with an external electrostatic potential $V(x)$:

$$H = v_F \sigma \cdot p + mc^2 \sigma_z + eV(x), \quad x = (x, y) \in \mathbb{R}^2. \quad (1)$$

Here, $\sigma = (\sigma_x, \sigma_y)$ and $\sigma_z$ are Pauli matrices, $p = (p_x, p_y) = -i\hbar(\partial_x, \partial_y)$ the momentum operators, and $v_F$ is the Fermi velocity of the material. To be more specific, we will consider a potential with radial symmetry $V(x) = V(r)$, so naturally from now on we will use polar coordinates $(r, \theta)$ to separate variables in the time-independent Dirac equation. In the case we are dealing with, the Hamiltonian commutes with the total angular momentum operator defined as

$$J_z = L_z + \Sigma, \quad \text{with} \quad L_z = -i\hbar \partial_\theta \quad \text{and} \quad \Sigma = \frac{1}{2} \hbar \sigma_z. \quad (2)$$

Thus, we can look for the eigenfunctions $\Phi(r, \theta)$ of $H$ that at the same time are eigenfunctions of $J_z$,

$$H\Phi(r, \theta) = E\Phi(r, \theta), \quad J_z\Phi(r, \theta) = \pm j\hbar\Phi(r, \theta). \quad (3)$$

It is quite easy to prove that the second of the equations that appear in (3) leads to eigenfunctions with the following spinor form

$$\Phi(r, \theta) = \begin{pmatrix} \phi_1(r)e^{i(j-\frac{1}{2})\theta} \\ i\phi_2(r)e^{i(j+\frac{1}{2})\theta} \end{pmatrix}, \quad j = \ell + \frac{1}{2}, \quad \ell = 0, \pm 1, \ldots \quad (4)$$

where $\ell = j-1/2$ and $\ell+1 = j+1/2$ are, respectively, the integer orbital angular momentum of the upper and lower scalar component of the spinor, and the imaginary unit in the second component is introduced for convenience.

After replacing (4) in the eigenvalue equation for $H$ in (3), we get a reduced equation in the variable $r$:

$$\begin{pmatrix} 0 & -iA^- \\ iA^+ & 0 \end{pmatrix} \begin{pmatrix} \phi_1(r) \\ i\phi_2(r) \end{pmatrix} = \begin{pmatrix} E - eV(r) - mc^2 & 0 \\ 0 & E - eV(r) + mc^2 \end{pmatrix} \begin{pmatrix} \phi_1(r) \\ i\phi_2(r) \end{pmatrix}. \quad (5)$$

where the operators $A^\pm$ are given by

$$A^- = \hbar v_F \left( \frac{\partial_r + (\ell + 1)}{r} \right), \quad A^+ = \hbar v_F \left( -\partial_r + \frac{\ell}{r} \right), \quad \ell \in \mathbb{Z}. \quad (6)$$

In this work we will choose the electric potential to be a typical two-dimensional radial well of the form

$$V(r) = \begin{cases} V_0, & r < R, \\ 0, & r > R. \end{cases} \quad (7)$$

We redefine variables in natural units for this problem as

$$\rho = \frac{r}{R}, \quad \varepsilon = \frac{ER}{h v_F}, \quad v = \frac{eV_0 R}{h v_F} < 0, \quad \mu = \frac{mc^2 R}{h v_F}, \quad (8)$$

where we will assume that the effective potential inside the dot $(\varepsilon)$ is constant and negative unless otherwise stated (for $v > 0$ we would have the problem of a potential barrier instead of a well), and the potential outside the dot is zero. We could have used another shape for the potential well, but we preferred the option mentioned above to be able to compare the results to be obtained with other relevant references available in the literature.

With the new notation we have introduced in (8), equations (5)-(6) become the following coupled differential system

$$\begin{cases} \phi_2,\alpha(\rho) + \frac{\ell + 1}{\rho} \phi_2,\alpha(\rho) = \varepsilon_\alpha^- \phi_1,\alpha(\rho), \\ -\phi_1,\alpha(\rho) + \frac{\ell}{\rho} \phi_1,\alpha(\rho) = \varepsilon_\alpha^+ \phi_2,\alpha(\rho), \end{cases} \quad (9)$$

where the subindex $\alpha$ can be either $\alpha = i$ in the inner region of the dot $(0 < \rho < 1)$, or $\alpha = o$ in the outer region of the dot $(\rho > 1)$, being

$$\varepsilon_\alpha^\pm = \varepsilon - v \pm \mu, \quad 0 \leq \rho < 1, \quad \varepsilon_\alpha^\pm = \varepsilon \pm \mu, \quad \rho > 1. \quad (10)$$

Next, we will solve this set of equations in these two regions. Note that both the potential and the parameters $\varepsilon_\alpha^\pm$ are constant, although different, in each of these two intervals. The connection between the solutions for each of the two regions is obtained by imposing the continuity of the components $\phi_1,\alpha$ and $\phi_2,\alpha$ at the point $\rho = 1$. The equation for $\phi_1,1,\alpha$, obtained from (9), is

$$\rho^2 \phi_1,1,\alpha'(\rho) + \rho \phi_1,1,\alpha'(\rho) + (\rho^2_\alpha \rho^2 - \ell^2) \phi_1,1,\alpha(\rho) = 0, \quad (11)$$

where the ‘momentum’ $\rho$ in each interval $\alpha = i, o$ is

$$p_1 = \sqrt{(\varepsilon - v)^2 - \mu^2}, \quad 0 \leq \rho < 1, \quad p_0 = \sqrt{\varepsilon^2 - \mu^2}, \quad \rho > 1, \quad (12)$$

where $p_1$ and $p_0$ are valid, respectively, in the inner and outer regions. This means that, as long as $p_1$ and $p_0$ are nonzero (later we will discuss what happens when one of them is zero), the general solution within each interval can be expressed either as a linear combination of Bessel functions of the first and second kind $(J_\ell, Y_\ell)$ [22], or as
a linear combination of Hankel functions of the first and second kind \((H_{\ell}^{(1)}, H_{\ell}^{(2)})\), in the form
\[
\phi_{1,\alpha}(\rho) = a_{\alpha} J_{\ell}(p_{\alpha}\rho) + b_{\alpha} Y_{\ell}(p_{\alpha}\rho) \quad (13)
\]
\[
\phi_{2,\alpha}(\rho) = \frac{p_{\alpha}}{\varepsilon_{\alpha}} [a_{\alpha} J_{\ell+1}(p_{\alpha}\rho) + b_{\alpha} Y_{\ell+1}(p_{\alpha}\rho)] \quad (14)
\]
The arbitrary constant coefficients \(a_{\alpha}, b_{\alpha}, \ldots\) are used for the inner region, while \(a_{\alpha}, b_{\alpha}, \ldots\) are used for the outer region. The second radial function \(\phi_{2,\alpha}\) of the spinor can be obtained from the previous expression for \(\phi_{1,\alpha}\) and the second equation of (9). Using well-known properties of the Bessel and Hankel functions [22], we get
\[
\phi_{2,\alpha}(\rho) = \frac{p_{\alpha}}{\varepsilon_{\alpha}} [a_{\alpha} J_{\ell+1}(p_{\alpha}\rho) + b_{\alpha} H_{\ell+1}^{(2)}(p_{\alpha}\rho), \quad \alpha = i, o.
\]

Now, we are going to use this equation in various situations of physical interest: when there are bound states, resonances or the so-called critical states. We will assume in the following that \(\ell \geq 0\), since for negative values, although it is not an equivalent situation, the results are similar.

### A. Bound states

To characterize the bound states in the problem we are analyzing, the right solutions are chosen by means of the appropriate boundary conditions at the origin \(\rho = 0\), at the junction point \(\rho = 1\), and at \(\rho \to \infty\).

- In the inner region, \(0 \leq \rho < 1\), the Bessel functions that are bounded at the origin are only those of the first kind (even in the case where \(p_{i}\) be an imaginary number), so, according to (4), in this interval the solutions must have the form
  \[
  \Phi_{I}(\rho, \theta) = a_{i} \left( \frac{J_{\ell}(p_{i}\rho) \ e^{i\ell\theta}}{p_{i} \ \varepsilon_{i}^{\ell} J_{\ell+1}(p_{i}\rho) \ e^{i(\ell+1)\theta}} \right), \quad \ell \in \mathbb{Z}. \quad (15)
  \]
The critical value \(p_{i} = 0 \implies \varepsilon - v = \pm\mu\), gives no additional solution.

- In the outer region, \(\rho > 1\), the appropriate solution to study the bound states is the Hankel function of the first kind, since its asymptotic behavior when \(\rho \to \infty\) is
  \[
  H_{\ell}^{(1)}(p_{o}\rho) \sim \sqrt{\frac{-2}{\pi p_{o}\rho}} e^{i(p_{o}\rho - \ell\pi/2 - \pi/4).} \quad (16)
  \]
Therefore, from (16) it is clear that bound states will appear if and only if \(p_{o} = \sqrt{\varepsilon^{2} - \mu^{2}}\) is an imaginary number, that is,
\[
\varepsilon^{2} < \mu^{2} \implies -\mu < \varepsilon < \mu, \quad (17)
\]
where \(\varepsilon\) and \(\mu\) are the energy and the mass in the units defined in (8). In other words, relativistic bound states in electric fields can only take place for a range of energy which is bounded from below/above by minus/plus the particle mass. The special cases \(\varepsilon = \pm\mu\) where \(p_{o} = 0\) are called critical points and will be studied separately in the next subsection. Thus, the wave function of the bound states in the outer region must take the form
\[
\Phi_{o}(\rho, \theta) = a_{o} \left( \frac{H_{\ell}^{(1)}(p_{o}\rho) \ e^{i\ell\theta}}{p_{o} \ H_{\ell+1}^{(1)}(p_{o}\rho) \ e^{i(\ell+1)\theta}} \right), \quad \ell \in \mathbb{Z}. \quad (18)
\]
Finally, the eigenvalues of the bound states are obtained by imposing the condition of continuity of the two spinor functions (15) and (18) at the point \(\rho = 1\), obtaining the following secular equation:
\[
\varepsilon_{i}^{+} J_{\ell}(p_{i}) \frac{p_{i} J_{\ell+1}(p_{i})}{p_{i} J_{\ell+1}(p_{i})} = \varepsilon_{o}^{+} H_{\ell}^{(1)}(p_{o}). \quad (19)
\]
The solutions corresponding to the discrete values of the energy \(\varepsilon\) that arise from the secular equation (19) will have their corresponding eigenfunctions \(\Phi(\rho, \theta)\) constructed from the matching of (15) and (18).

### B. Critical and supercritical states

The critical points are the eigenvalues corresponding to bound (or quasi–bound) states such that \(p_{o} = 0\) or \(\varepsilon^{2} = \mu^{2}\), i.e., they correspond to the maximum \(\varepsilon = \mu\) or minimum \(\varepsilon = -\mu\) possible eigenvalues of the energy (the latter case is often called supercritical). Critical eigenvalues can only be reached for some special values of the coefficient \(v\). The associated eigenstates are referred to as critical and supercritical states. In other words, we must look for the possible values of the potential depth \(v\), such that there are states with eigenvalue \(\varepsilon = \mu\) or \(\varepsilon = -\mu\) and with a bounded behavior as \(\rho \to 0\) and \(\rho \to \infty\), corresponding to a square integrable function (or at least bounded for quasi-bound states).

**Critical states: \(\varepsilon = \mu\)**

Inside the potential well those states are described by Bessel functions, as in (15), with the following values of the parameters that appear there: \(\varepsilon_{i}^{+} = 2\mu - v > 0\) and \(p_{i} = \sqrt{v^{2} - 2\mu v} > 0\).

In the outer region, taking into account that \(\varepsilon_{o}^{+} = 2\mu\) and \(\varepsilon_{o}^{-} = 0\), the components \(\phi_{1,\alpha}(\rho)\) and \(\phi_{2,\alpha}(\rho)\) satisfy this particular form of equations (9)
\[
\begin{align*}
\phi_{2,\alpha}'(\rho) + \ell + 1 \phi_{2,\alpha}(\rho) &= 0, \\
-\phi_{1,\alpha}'(\rho) + \phi_{1,\alpha}(\rho) &= 2\mu \phi_{2,\alpha}(\rho),
\end{align*}
\]
whose acceptable solutions are given by

$$\Phi_\ell(\rho, \theta) = a_\ell \left( i \rho^{-(\ell+1)} e^{i(\ell+1)\theta} \right), \ell = 0, 1, 2, \ldots$$  \hspace{1cm} (21)$$

Note that in the case $\ell = 0$ the critical state is not really a bound state, but just a quasi-bound state: the wave function is not square integrable, although it is bounded. For values of $\ell = 1, 2, \ldots$, the critical state wave functions, according to (21), vanish as $\rho \to \infty$. This kind of solutions of critical states will also be discussed in detail for the massless case of the next section.

The matching condition of the solutions of the critical wave functions (15) and (21) at $\rho = 1$ produce the following secular equations:

$$\ell \left( \mu + \sqrt{\mu^2 + \rho^2} \right) J_\ell(p_i) = \mu p_i J_{1+\ell}(p_i), \ell = 0, 1, 2 \ldots$$  \hspace{1cm} (22)$$

Remember that the solutions $p_i(\ell, \mu)$ of these transcendental equations allow us to determine the depth of the well $v$, which will depend on the parameters $\ell$ and $\mu$, as in the present case $v = \mu - \sqrt{\mu^2 + p_i^2}$.

**Supercritical states: $\varepsilon = -\mu$**

Again, the eigenfunctions in the inner interval $0 \leq \rho < 1$ are Bessel functions of the first kind (15), with the following values of the parameters that appear there: $\varepsilon^+_i = -v > 0$ and $p_i = \sqrt{v^2 + 2\mu v} > 0$, while outside, for $\rho > 1$, the components satisfy this particular form of equations (9)

$$\begin{align*}
\phi_{2,0}(\rho) + \frac{\ell + 1}{\rho} \phi_{2,0}(\rho) &= -2\mu \phi_{1,0}(\rho), \\
-\phi_{1,0}'(\rho) + \frac{\ell}{\rho} \phi_{1,0}(\rho) &= 0.
\end{align*}$$  \hspace{1cm} (23)$$

Then, the solutions bounded in the region $\rho > 1$ are

$$\Phi_\ell(\rho, \theta) = a_\ell \left( \rho^{-(\ell+1)} e^{i(\ell+1)\theta} \right), \ell = 0, 1, 2, \ldots$$  \hspace{1cm} (24)$$

The matching condition in the present situation leads to the following secular equations:

$$J_\ell(p_i) = 0, \ell = 0, 1, 2, \ldots$$  \hspace{1cm} (25)$$

which allow to determine the values of $p_i(\ell, \mu)$, and therefore those of $v$ for each $\ell$, taking into account that in the present case $v = -\mu - \sqrt{\mu^2 + p_i^2}$.

Some examples of bound energy levels together with critical and supercritical values are given in Figure 1. In these graphs we can see that for $\ell = 0$ bound states appear for any negative value of the well depth $v$, however for the case $\ell = 2$ bound states will only appear for negative values of $v$ lower than the value $v = v = -\mu - \sqrt{\mu^2 + p_{1,1}^2}$, where $p_{1,1}$ is the first strictly positive root of the transcendental equation (25), $v = -2.7558$ in the case shown in Figure 1. This fact is due to the centrifugal potential caused by the orbital momentum $\ell$. Note that from Figure 1 we can see that for this particular value of mass and for any negative value of $v$ there will be no more than two bound states for each $\ell$. Thus, the maximum number of bound states depends on the value of $\mu$. The depth $v$ of the dot modulates the energy of the bound states. If the potential depth $v$ is more negative than $-\mu$, the lowest bound states can pass into the continuous spectrum of antiparticles. This result is quite different from that of non-relativistic wells, which allow to have any number.
of bound states just by taking more negative depths \( v \) without any risk of leaking into the ‘negative sea’.

### C. Resonances and pure outgoing states

In a scattering process, a resonance in the real energy \( E_r \) occurs if the incoming state takes longer time to exit than it should, or if it suffers a delay time, within the significant range of the potential. One way to measure these resonances is by computing the phase shift \( \delta_r \) of the incoming and outgoing waves, as it will be done in the next subsection. The derivative of this phase shift with respect to the energy gives the so-called “Wigner time delay” \( \tau_r \) [19, 20], indeed

\[
\tau_r = 2 \frac{d\delta_r}{dE}. \tag{26}
\]

The presence of a maximum in this function (specially if it is a sharp one) is a clear resonance signal.

There is another approach also used to calculate the resonances of a potential well (or barrier) that consists of finding the complex energies \( E = E_R + iE_I \), where the stationary states satisfy purely outgoing boundary conditions. In fact, when these energies are real, in the interval \((−\mu, +\mu)\), they belong to the discrete spectrum and in other cases they can be interpreted as anti-bound states [20, 21, 23]. These features can best be appreciated in the momentum space, but we will limit ourselves here to the energy picture for simplicity. The real part \( E_R \) of the resonant energy is then identified with the energy of the incident wave, while the imaginary part \( E_I \) is related to the time delay of the resonance as follows: \( E_R \approx E_r \), while \( \tau_r \propto 1/E_I \). The consistence of these two criteria to detect resonances will be checked in the following two subsections.

1. Scattering states, shift phases, and resonances

In order to study the scattering states and their phase shifts, it is better to use in the outer region the basis \( \{ J_\ell, Y_\ell \} \). The asymptotic behavior of these functions for large values of \( \rho \) is the following [22]:

\[
J_\ell(p_\rho, \rho) \sim \sqrt{\frac{2}{\pi p_\rho \rho}} \cos(p_\rho \rho - \ell \pi/2 - \pi/4),
\]

\[
Y_\ell(p_\rho, \rho) \sim \sqrt{\frac{2}{\pi p_\rho \rho}} \sin(p_\rho \rho - \ell \pi/2 - \pi/4).
\]

Then, the spinors of the scattering states take the form (15) in the inner region \((0 \leq \rho < 1)\), and the following one in the outer region \((\rho > 1)\)

\[
\Phi_\alpha(p, \theta) = \left( \begin{array}{c} A J_\ell(p_\rho, \rho) + B Y_\ell(p_\rho, \rho) \bigg| e^{i\ell\theta} \\ i \frac{p_\rho}{\epsilon} \left( A J_{\ell+1}(p_\rho, \rho) + B Y_{\ell+1}(p_\rho, \rho) \right) e^{i(\ell+1)\theta} \end{array} \right).
\]

If for convenience we choose the form of the arbitrary constants \( A \) and \( B \) as \( A = a \cos \delta_r \), \( B = -a \sin \delta_r \), the asymptotic behavior of the spinor when \( \rho \to \infty \) is

\[
\Phi_\alpha(p, \theta) \sim \sqrt{\frac{2}{\pi p_\rho \rho}} \left( \begin{array}{c} \cos(p_\rho \rho - \ell \pi/2 - \pi/4 + \delta_r) \\ i \frac{p_\rho}{\epsilon} \sin(p_\rho \rho - \ell \pi/2 - \pi/4 + \delta_r) \end{array} \right). \tag{27}
\]

The quantity \( \delta_r \) is the phase shift due to the presence of the potential near the origin of coordinates, and it appears in the outer spinor wave function:

\[
\Phi_\alpha(p, \theta) = A \left( \begin{array}{c} J_\ell(p_\rho, \rho) - \tan \delta_r Y_\ell(p_\rho, \rho) \bigg| e^{i\ell\theta} \\ i \frac{p_\rho}{\epsilon} (J_{\ell+1}(p_\rho, \rho) - \tan \delta_r Y_{\ell+1}(p_\rho, \rho)) e^{i(\ell+1)\theta} \end{array} \right).
\]

The continuity condition of the spinor \( \Phi(p, \theta) \) at \( \rho = 1 \) leads to

\[
\epsilon \frac{p_\rho}{\epsilon} J_\ell(p_\rho, \rho) = \frac{\epsilon \frac{p_\rho}{\epsilon} J_{\ell+1}(p_\rho, \rho) - \tan \delta_r Y_{\ell+1}(p_\rho, \rho)}{\tan \delta_r Y_\ell(p_\rho, \rho)}, \tag{28}
\]

from where we obtain the explicit value of the phase \( \delta_r \):

\[
\tan \delta_r(\epsilon) = \frac{\epsilon \frac{p_\rho}{\epsilon} J_\ell(p_\rho, \rho) J_{\ell+1}(p_\rho, \rho) - \epsilon \frac{p_\rho}{\epsilon} J_{\ell+1}(p_\rho, \rho) J_\ell(p_\rho, \rho)}{\epsilon \frac{p_\rho}{\epsilon} J_{\ell+1}(p_\rho, \rho) Y_{\ell+1}(p_\rho, \rho) - \epsilon \frac{p_\rho}{\epsilon} J_\ell(p_\rho, \rho) Y_{\ell+1}(p_\rho, \rho)}. \tag{29}
\]

Once the potential depth \( v \) is fixed, the phase shift \( \delta_r \) will depend on the energy \( \varepsilon \), taking into account (10) and (12). Furthermore, for a potential well with depth \( v \) it is easy to show that at the high energy limit (note that the constraint (17) is no longer valid now) the phase shift is

\[
\lim_{\epsilon \to \infty} \tan \delta_r(\epsilon) = -\tan v. \tag{30}
\]

The values for which the derivative with respect to the energy of (29) is maximum will correspond to resonances, since this will mean that with this energy the time that a wave packet spends inside the well (26) will be longer than it should. It also turns out that as we increase the depth of the well we trap bound states that leave the continuum. In this process, when a value of \( v \) is reached that gives rise to a new bound state, the corresponding phase shift undergoes an abrupt change, increasing by \( \pi \). These features are shown in Figure 2.

2. Complex resonances and outgoing states

In the scattering process discussed in the previous subsection, we had an incoming wave and an outgoing wave, and we calculated the phase shift of these waves due to the potential near the origin. Next, we will look for energy values \( \varepsilon \) such that we have a pure outgoing wave. This situation may not have a physical realization, but it will provide us with useful information. In fact, this condition in general will be satisfied for complex energies: \( \varepsilon = \varepsilon_R + i\varepsilon_I \). Therefore, the states with pure outgoing boundary conditions must satisfy the following conditions:
(i) when \( \rho \to 0 \) the spinor \( \Phi_i(\rho, \theta) \) must be kept bounded,

(ii) when \( \rho \to \infty \) each component must behave as the first Hankel function (16), and the spinor \( \Phi_{\text{a}}(\rho, \theta) \) as in (18).

In other words, the purely outgoing wave conditions are the same as the bound state conditions (19), except that now what we want to find are the complex solutions \( \varepsilon = \varepsilon_R + i\varepsilon_I \) of this secular equation, and therefore the corresponding eigenfunctions may diverge when \( \rho \to \infty \).

Obviously, the resonances depend on the depth \( \nu \), in the same way that the energies of the bound states also depend on \( \nu \), as shown in Figure 1. As the results must necessarily be obtained numerically or graphically from (19), we will choose, as an example, a particle with mass \( \mu = 2 \) and we will follow its trajectory \( \varepsilon(v) = \varepsilon_R(v) + i\varepsilon_I(v) \) as a function of \( v \). We start from a resonance such that \( \varepsilon_R(v_0) > \mu = 2 \), \( \varepsilon_I(v_0) \neq 0 \), and little by little we decrease the depth of the potential well until we reach the value \( v_1 \) for which we have precisely the first bound state: \( \varepsilon(v_1) = \varepsilon_R(v_1) = \mu, \varepsilon_I(v_1) = 0 \). As we continue to decrease the value of \( v \), the bound energy is getting lower up to the minimum bound eigenvalue \( \varepsilon(v_2) = \varepsilon_R(v_2) = -\mu = -2 \). Below this value \( v_2 \), the bound energy will again be a complex resonance \( \varepsilon(v_3) = \varepsilon_R(v_3) + i\varepsilon_I(v_3) \), with \( \varepsilon_R(v_3) < -2 \) and \( \varepsilon_I(v_3) \neq 0 \). The whole process can be followed in Figure 3.

In Figure 4, the relation between Wigner time delay \( \tau_L \)
and complex energies is shown: the dots correspond to resonances in the energy representation and the curves represent the Wigner time delay. For $\ell > 0$, we see that the real part of the resonances perfectly matches the peak of the time delay, but for $\ell = 0$ this correspondence is not so good.

III. BOUND STATES AND RESONANCES FOR MASSLESS TWO-DIMENSIONAL DIRAC PARTICLES

In this section, we will analyze the bound states and resonances of the problem under study when particles are assumed to be massless, something that can be seen as a limit of the treatment shown in the previous section for massive particles. However, we will see that the behavior in massive and massless cases has important differences that we will highlight below.

A. Bound states

The potential well has the same shape as in the case of non-zero masses (7). The equations for the components $\phi_1, \phi_2$ also have the same form as their massive analogs (9), although now the mass disappears from the equations as $\mu = 0$. Consequently, the energy constants simplify:

$$\varepsilon_i^{\pm} = \varepsilon - v, \quad 0 \leq \rho < 1, \quad \varepsilon_o^{\pm} = \varepsilon, \quad \rho > 1,$$  \hspace{1cm} (29)

as well as

$$p_i = \varepsilon - v, \quad 0 \leq \rho < 1, \quad p_o = \varepsilon, \quad \rho > 1.$$ \hspace{1cm} (30)

By the same arguments on the asymptotic behavior of the wave functions of the massive case (17), in the present situation the bound states must have zero energy $\varepsilon = 0$, that is, they coincide with the critical and supercritical states. In order to find them we concentrate in the outer region $\rho > 1$, because in the inner region the solutions are the same as in the $\mu \neq 0$ case. In the outer region, since $\varepsilon_o^\pm = 0$, equations (9) become

$$\phi_2^\pm + \frac{\ell + 1}{\rho} \phi_2 = 0, \quad -\phi_1^\pm + \frac{\ell}{\rho} \phi_1 = 0.$$ \hspace{1cm} (31)

Notice that in these two equations there is the symmetry $\ell \to -(\ell + 1)$ and changing the components $\phi_1 \to \phi_2$, due to the fact that $\mu = 0$. The solutions to these equation are

$$\Phi_o(\rho, \theta) = \left( \begin{array}{c} c_1 \rho^{\ell} e^{i\ell \theta} \\ i c_2 \rho^{-(\ell+1)} e^{i(\ell+1)\theta} \end{array} \right), \quad \ell = 0, \pm 1, \pm 2, \ldots$$

where $c_1, c_2$ are arbitrary integration constants. Thus, if either $\ell > 0$ or $\ell < -1$, the physically acceptable bound states are described by

$$\Phi_o(\rho, \theta) = \left( \begin{array}{c} 0 \\ i c_2 \rho^{-(\ell+1)} e^{i(\ell+1)\theta} \end{array} \right), \quad \ell > 0,$$

$$\Phi_o(\rho, \theta) = \left( \begin{array}{c} c_1 \rho^{\ell} e^{i\ell \theta} \\ 0 \end{array} \right), \quad \ell < -1,$$

which go to zero as $\rho \to \infty$ and are square integrable. For the cases $\ell = 0, -1$ the corresponding solutions, although also vanish at infinity, are not square integrable, and thus they do not correspond to ‘true’ bound states.

The formula for the potential depth $v$ is the same as (25) taking the limit $\mu \to 0$,

$$J_{\ell}(p) = 0, \quad \ell \geq 0, \quad J_{1+\ell}(p) = 0, \quad \ell < 0, \quad p_i = |v|.$$ \hspace{1cm} (32)

Then, the roots of $J_\ell(p) = 0$, give us the values for $v$ corresponding the bound states. In Figure 5 two graphics represent the first of these values corresponding to the angular momenta $\ell = 0$ and $\ell = 2$. 

FIG. 4: The red dots represent complex resonances with coordinates ($\varepsilon_R, \varepsilon_I$) and the blue curves are the Wigner time delay as a function of the real part of the energy. At the top the results for $\mu = 2, \ell = 2, v = -2$, and at the bottom for $\mu = 2, \ell = 0, v = -5$. The maxima of the Wigner time delay are reached at the real part of the resonance energies $\varepsilon_R$ as shown in the top figure. However, in the figure below the coincidence is not so good (this may be due to the fact that for $\ell = 0$ there is no good critical bound state because it is not square integrable).
B. Scattering states and resonances

With respect to scattering states with positive energy $\varepsilon > 0$, the situation is completely similar to that of Subsection II C 1 for massive particles. The phase shift of scattering states are computed using (27), whose limit value in $\varepsilon \to \infty$ is (28). Some examples of phase shift of scattering states and their derivatives with respect to energy (interpreted as Wigner time delays) are shown in Figure 6. The resonances are the values $\varepsilon_r(v)$ for which the Wigner time delay reach a maximum. We have checked numerically that even for $\ell = 0$ the time delay and resonances exist. However, for $\ell > 0$ they become quite strong, especially for energies close to a bound state.

At the top of Figure 6 we have represented the phase shifts for a massless charge with angular momentum $\ell = 2$ for three values of $v$: $-4$, $-5.13562$, and $-6$. The special value $v = -5.13562$ corresponds to the capture of a bound state with $\varepsilon = 0$. For the value $v = -4$ (which is a little above $v = -5.13562$) the phase shifts suffer a strong change at a certain value $\varepsilon$ near $\varepsilon = 0$ (dotted curve). However, for capture value $v = -5.13562$ (solid curve) or for a slightly lower value ($v = -6$, dashed curve), the phase shifts are smoother and start with a $\pi$-jump at $\varepsilon = 0$. Also, in the limit $\varepsilon \to \infty$ it is seen that these phase shifts tend to the corresponding value $|v|$.

At the bottom of Figure 6 we have represented the derivatives of these three phase shifts, which are identified with Wigner time delays. The potential $v = -4$ (a little above the capture value $v = -5.13562$) has a very high maximum of the time delay $\tau$, reached at a certain value $\varepsilon_r$ which we identify with a strong resonance. However, for the capture potential $v = -5.13562$ or slightly lower values ($v = -6$), the maxima of time delays are much lower, corresponding to weak resonances.

As we have already mentioned, another way to define (complex) resonances is through complex eigenvalues of energy corresponding to eigenfunctions that satisfy purely outgoing boundary conditions. Thus, we must look for complex solutions $\varepsilon(v) = \varepsilon_R + i \varepsilon_I$ of equation (19), but now having in mind that the mass vanishes and therefore we also have (29)-(30). Hence, we get

$$\frac{J_\ell(\varepsilon - v)}{J_{\ell+1}(\varepsilon - v)} = \frac{H_\ell^{(1)}(\varepsilon)}{H_{\ell+1}^{(1)}(\varepsilon)},$$

Some solutions of this equation are shown in Figure 7 for several values of $v$.

As we have already shown the previous subsection, bound states appear only for some special well depth values. In Figure 7 such bound states takes place for $v = -5.13$ and for $v = -8.65$. The resonances shown in Figure 6 correspond to the values of the well depth $v = -4, -5.13, -6$, which are part of Figure 7.
value \( v = -4 \) the resonance closest to the origin, shown in Figure 7 by a blue circle, is represented by the maximum of the dotted curve in Figure 6 (right). This is a strong resonance. For the next value \( v = -5.13 \), that resonance becomes a bound state with zero energy and the next resonance is represented by the blue line in Figure 6. In this case the resonance is much weaker. Finally, for the value \( v = -6 \), this resonance is close to the origin and becomes stronger, as it is seen by the maximum of the dashed curve in Figure 6.

If both approaches to resonance phenomena correspond to the same physical concept, described by different properties, then we should have \( \varepsilon_r(v) \approx \varepsilon_R(v) \), where \( \varepsilon_r(v) \) is the real energy of the scattering state and \( \varepsilon_R(v) \) is the real part of a complex resonance. The imaginary part \( \varepsilon_I \) should be inversely proportional to the time delay. From Figure 8, we can see the close relation between complex resonances of Figure 7 (represented by red dots with coordinates \((\varepsilon_R, \varepsilon_I)\)) and Wigner time delay (represented by blue curves) for some examples with \( \ell = 2 \). We observe that the first coordinate \( \varepsilon_R \) of the red dots (representing complex resonances) are very close to the values of the maxima \( \varepsilon_r \) of Wigner time delays, specially for the value \( v = -4 \) which is near (from above) to the value \( v = -5.1356 \) corresponding to the capture of a bound state, as it is shown in Figure 7.

**IV. CONCLUSIONS**

The aim of this article is to search for specific properties of both massive and massless Dirac particles, in so-called Dirac materials with electric quantum dots, including graphene. We start with particles of mass \( \mu \neq 0 \), where the bound states are limited, due to a relativistic effect, to the range of energies \( -\mu < \varepsilon < \mu \). We also consider resonant states, a kind of eigenstates, generally with complex energy, that satisfy purely outgoing boundary conditions that include bound states as a particular case. These resonant states, corresponding to complex energies, can be related with scattering resonances and in fact we have observed throughout some examples that these resonant states coincide quite well with the scattering resonances defined by the Wigner time delay conditions. We pay attention to the behavior of complex resonances \( \varepsilon(v) = \varepsilon_R(v) + i\varepsilon_I(v) \) as functions of the depth \( v \) of the potential. We have seen that \( \varepsilon(v) \) is a continuous function with a reasonable evolution: as the well deepens (\( v \) more negative), \( \varepsilon_R(v) \) goes from positive to negative values. In a certain interval of \( v \) (depending
on each particular resonance) each resonance becomes a bound state with $-\mu \leq \varepsilon_R(v) \leq \mu$ and $\varepsilon_I(v) = 0$, i.e., the imaginary part vanishes. For lower, more negative, values of $v$, the bound states turn into complex resonances again. This ‘movement’ of the resonances in the complex plane as a function of $v$ was illustrated by a sequence of graphs in Figure 3. We could conclude that the number of resonances is conserved, as a function of $v$; all of them (at least those with small imaginary part, which are physically more important) flow in the same direction. In this motion, the complex resonances become bound states within an interval of the potential depth $v$.

In the case of massless particles we have a special situation. Since $\mu = 0$, the bound states take place at zero energy $\varepsilon = 0$, which is at the same time a critical and supercritical point. If we study the evolution of complex resonances $\varepsilon(v) = \varepsilon_R(v) + i\varepsilon_I(v)$ as we did for massive particles, we observe a peculiar behavior: resonances initially located ‘to the right’ ($\varepsilon_R(v) > 0$) move to the left ($E < 0$) as $v$ takes increasingly negative values. If we pay attention to one of these resonances we will see that there is a particular value of $v$ such that it becomes a bound state with $\varepsilon_R(v) = 0$ and $\varepsilon_I(v) = 0$. If we take deeper values than that particular $v$, then the bound state seemingly disappears (or collapses) without leaving any trace. This behavior is repeated for all complex resonances located to the right side of the lower complex plane. Thus, we could say that for massless charges the electric well acts as a sink for complex resonances. On the other hand, complex resonances located on the left side of the complex plane ($\varepsilon_R(v) < 0$) are kept ‘to the left’, with little position change as depth $v$ decreases.

It is true that this behavior of resonance for massless charges seems quite surprising and must be explained in physical terms with a more in-depth analysis. In our case, the origin of the difference between resonances with mass and without mass comes from the formulas (29) and (30), which are not the same as those obtained by simply taking the limit $\mu \rightarrow 0$. Some of the properties that have been exposed in this work are based on numerical calculations, so until we have a clear interpretation, they should be taken with some caution. We plan to answer these open questions in the near future.

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