

Variational principle and boundary terms in gravity *a la* Palatini

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A general $f(\mathcal{R})$ gravitational theory is considered within the Palatini formalism. By applying the variational principle and the usual conditions on the boundary, we show explicitly that a surface term remains such that as in other metric compatible theories, an additional surface term has to be added in the action, which plays a fundamental role when calculating the entropy of the black hole as shown along the paper.

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INTRODUCTION

Gravitational theories are described by the spacetime metric that measures lengths and the corresponding covariant derivatives that provides the way vectors are parallelly transported. Such construction departs from a Lagrangian that is generally given by scalars from the Riemann and Ricci tensors that depend on derivatives of the connection and on the metric. As far as one assumes a metric compatible connection, the so-called Levi-Civita connection, one only cares about variations over the spacetime metric, which leads to the gravitational field equations but also to non-null surface contributions. This is the origin of the so-called Gibbons-Hawking-York boundary term that compensates the surface term in General Relativity. Similar boundary terms have to be added in scalar-tensor theories and metric $f(R)$ gravities [1–3]. Nevertheless, in the so-called Palatini formalism, one considers the spacetime metric and the connection as independent fields in principle, and then by applying the variational principle, the corresponding field equations for the metric and the connection are obtained which show that actually the connection is metric compatible with a conformal metric to the spacetime one [4, 5].

Nevertheless, in the Palatini formalism, surface terms are removed by considering Dirichlet boundary conditions on the variations of the connection, such that no boundary terms remain. In this letter, we reformulate the variational principle when applied in the Palatini formalism by assuming that the connection is not the fundamental field but the metric compatible tensor is. By doing so, the corresponding boundary terms are obtained, and we propose a surface term similar to the Gibbons-Hawking-York to be added to the gravitational action. As in GR, such new term which depends conformally on the geometry of the boundary, will play a fundamental role in several frameworks as the hamiltonian approach of the theory. Here we obtain the entropy of the Schwarzschild black hole in these theories through the Euclidean semiclassical approach, and shown that the entropy is induced by the new surface term. The ex-

pression of the entropy coincides with the one calculated previously in the Palatini formalism by using Noether charges [6].

PALATINI FORMALISM

The general gravitational action is given by:

$$S = S_G + S_m = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_m, \quad (1)$$

where the metric of the spacetime is given by $g_{\mu\nu}$ and we are assuming that the matter action S_m just depends on the metric and the matter fields, preserving the Equivalence Principle, while the constant $\kappa^2 = 8\pi G$. The Ricci scalar \mathcal{R} is defined as the contraction of the Ricci tensor with the spacetime metric:

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu}(\Gamma). \quad (2)$$

In the Palatini formalism the connection is assumed to be in principle an independent field, which might not be metric compatible with $g_{\mu\nu}$. The Ricci tensor is expressed in terms of the independent connection Γ as follows:

$$\mathcal{R}_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\sigma\lambda}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\sigma\nu}^\lambda \Gamma_{\mu\lambda}^\sigma. \quad (3)$$

Then, variations of the gravitational action (1) consist on variations over the metric and over the connection:

$$\begin{aligned} \delta S_G &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[f_{\mathcal{R}} \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(\mathcal{R}) \right] \delta g^{\mu\nu} \\ &+ \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f_{\mathcal{R}} g^{\mu\nu} \delta \mathcal{R}_{\mu\nu}, \end{aligned} \quad (4)$$

where $f_{\mathcal{R}} = \frac{df}{d\mathcal{R}}$. Hence, variations with respect to the metric $g_{\mu\nu}$ leads to the following set of equations:

$$f_{\mathcal{R}} \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f = \kappa^2 T_{\mu\nu}, \quad (5)$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor. The issue arises when dealing with the variations of the

gravitational action with respect to the connection. The variation of the Ricci tensor $\delta\mathcal{R}_{\mu\nu}$ is given by:

$$\delta\mathcal{R}_{\mu\nu} = \nabla_\sigma \delta\Gamma_{\nu\mu}^\sigma - \nabla_\nu \delta\Gamma_{\sigma\mu}^\sigma, \quad (6)$$

such that the second term in (4) yields:

$$\begin{aligned} & \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f_{\mathcal{R}} g^{\mu\nu} \delta\mathcal{R}_{\mu\nu} = \\ & = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f_{\mathcal{R}} g^{\mu\nu} [\nabla_\sigma \delta\Gamma_{\nu\mu}^\sigma - \nabla_\nu \delta\Gamma_{\sigma\mu}^\sigma], \quad (7) \end{aligned}$$

which after integrating by parts can be decomposed as follows:

$$\begin{aligned} & \int d^4x \nabla_\sigma [\sqrt{-g} f_{\mathcal{R}} (g^{\mu\nu} \delta\Gamma_{\nu\mu}^\sigma - g^{\mu\sigma} \delta\Gamma_{\lambda\mu}^\lambda)] \\ & - \delta\Gamma_{\nu\mu}^\sigma [\nabla_\sigma (\sqrt{-g} g^{\mu\nu} f_{\mathcal{R}}) - \delta_\sigma^\nu \nabla_\lambda (\sqrt{-g} g^{\mu\lambda} f_{\mathcal{R}})]. \quad (8) \end{aligned}$$

Here we have omitted the antisymmetric part of the connection, as the Ricci scalar $R(\Gamma)$ only depends on the symmetric part of the connection due to the projective invariance of the scalar curvature, such that Γ is considered symmetric under the low indexes [4]. The first term in (8) is a boundary term and will be analysed later, while the second term after some calculations leads to the well known result:

$$\nabla_\lambda (\sqrt{-g} f_{\mathcal{R}} g^{\mu\nu}) = 0, \quad (9)$$

which basically states that the connection is compatible with the conformal metric:

$$q_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 = f_{\mathcal{R}}. \quad (10)$$

And General Relativity is recovered as far as $f_{\mathcal{R}} = 1$. The equation (9) becomes:

$$\nabla_\lambda (\sqrt{-q} q^{\mu\nu}) = 0. \quad (11)$$

Moreover, by taking the trace of the equation (5), one obtains:

$$f_{\mathcal{R}} \mathcal{R} - 2f = \kappa^2 T. \quad (12)$$

This is an algebraic equation for the scalar curvature \mathcal{R} , such that can be solved as a function of the trace of the energy-momentum tensor $\mathcal{R} = \mathcal{R}(T)$. From the metric compatibility equation (11), one might consider to apply the conformal transformation (10) on the Ricci tensor:

$$\begin{aligned} \mathcal{R}_{\mu\nu}(q) &= R_{\mu\nu}(g) + \frac{4}{\Omega^2} \nabla_\mu \Omega \nabla_\nu \Omega - \frac{2}{\Omega} \nabla_\mu \nabla_\nu \Omega \\ & - g_{\mu\nu} \frac{g^{\rho\sigma}}{\Omega^2} \nabla_\rho \Omega \nabla_\sigma \Omega - g_{\mu\nu} \frac{\square \Omega}{\Omega}. \quad (13) \end{aligned}$$

Finally, the field equation (5) can be expressed as:

$$\begin{aligned} R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) &= \frac{\kappa^2}{f_{\mathcal{R}}} T_{\mu\nu} - g_{\mu\nu} \frac{\mathcal{R} f_{\mathcal{R}} - f}{2f_{\mathcal{R}}} \\ & - \frac{3}{2f_{\mathcal{R}}^2} \left[\nabla_\mu f_{\mathcal{R}} \nabla_\nu f_{\mathcal{R}} - \frac{1}{2} g_{\nu\mu} \nabla_\lambda f_{\mathcal{R}} \nabla^\lambda f_{\mathcal{R}} \right] \\ & + \frac{1}{f_{\mathcal{R}}} [\nabla_\mu \nabla_\nu f_{\mathcal{R}} - g_{\mu\nu} \square f_{\mathcal{R}}]. \quad (14) \end{aligned}$$

This is the usual approach when dealing with non linear terms in the action within the Palatini formalism, as the equations are now expressed in terms of the spacetime metric in the right hand side while the left hand side depends solely on the energy-momentum tensor and its trace by the relation (12). In general this approach allows to study different spacetimes in a simple way.

BOUNDARY TERMS IN THE PALATINI FORMALISM

Turning back to the surface term in (8), this is usually considered to be null in the literature just by applying standard Dirichlet conditions on the boundary, i.e. by assuming that variations of the fields are null on the boundary, in this case the variations of the connection $\delta\Gamma$. However, variations of the derivatives of the fields are uncomfortable to be killed, what leads in General Relativity to a non-null surface term that is compensated by the so-called Gibbons-Hawking-York boundary term [2]. The misunderstanding when applying the variational principle to non linear actions in the Palatini formalism has to do with assuming the -in principle- arbitrary connection Γ as the fundamental field, while this is not, as shown by the metricity condition (11), but it is the compatible metric $q_{\mu\nu}$ the fundamental field that plays the game. Hence, by taking this assumption, the boundary term in (8) leads to:

$$\frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \nabla_\sigma V^\sigma = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \partial_\sigma V^\sigma \quad (15)$$

where

$$V^\sigma = \sqrt{-g} f_{\mathcal{R}} (g^{\mu\nu} \delta\Gamma_{\nu\mu}^\sigma - g^{\mu\sigma} \delta\Gamma_{\lambda\mu}^\lambda). \quad (16)$$

Then, by using the Gauss-Stokes theorem, it yields:

$$\int_{\mathcal{M}} d^4x \partial_\sigma V^\sigma = \int_{\delta\mathcal{M}} d^3x \sqrt{|\gamma|} \epsilon n_\sigma V^\sigma, \quad (17)$$

where γ is the induced metric on the boundary $\delta\mathcal{M}$, n_σ is the unit normal vector to $\delta\mathcal{M}$ and $\epsilon = \pm 1$ depending whether the boundary is spacelike or timelike. By lowering the index in V , this can be expressed in terms of the variations of the metric $q_{\mu\nu}$ as follows:

$$\begin{aligned} V_\sigma &= g_{\sigma\lambda} V^\lambda = \Omega^{-2} q_{\sigma\lambda} V^\lambda = \\ & = g^{\alpha\beta} [\partial_\beta (\delta q_{\sigma\alpha}) - \partial_\sigma (\delta q_{\alpha\beta})]. \quad (18) \end{aligned}$$

And the boundary term (17) yields:

$$\int_{\delta\mathcal{M}} d^3x \sqrt{|\gamma|} \epsilon n^\sigma g^{\alpha\beta} [\partial_\beta (\delta q_{\sigma\alpha}) - \partial_\sigma (\delta q_{\alpha\beta})]. \quad (19)$$

The spacetime metric can be expressed in terms of the induced metric on the boundary as:

$$g^{\alpha\beta} = \gamma^{\alpha\beta} + \epsilon n^\alpha n^\beta. \quad (20)$$

And (17) leads to:

$$\int_{\delta\mathcal{M}} d^3x \sqrt{|\gamma|} \epsilon n^\sigma (\gamma^{\alpha\beta} + \epsilon n^\alpha n^\beta) [\partial_\beta (\delta q_{\sigma\alpha}) - \partial_\sigma (\delta q_{\alpha\beta})] \quad (21)$$

The projections along the normal direction cancel each other, while the first term is the tangential derivative on the boundary of the variation $\delta q_{\sigma\alpha}$, which becomes null just by imposing standard Dirichlet boundary conditions:

$$\delta q_{\sigma\alpha}|_{\delta\mathcal{M}} = 0 . \quad (22)$$

Finally, the boundary term (15) leads to:

$$-\frac{1}{2\kappa^2} \int_{\delta\mathcal{M}} d^3x \sqrt{|\gamma|} \epsilon \gamma^{\alpha\beta} n^\sigma \partial_\sigma (\delta q_{\alpha\beta}) . \quad (23)$$

This is the boundary term that remains in the Palatini approach. It may be expressed in a more convenient way by using the induced conformal metric on the boundary:

$$\tilde{\gamma}_{\alpha\beta} = q_{\alpha\beta} - \epsilon \tilde{n}_\alpha \tilde{n}_\beta . \quad (24)$$

where the normal vector \tilde{n}_μ and $h_{\alpha\beta}$ are related to the induced metric γ and the normal vector n_μ by conformal transformation (10):

$$\begin{aligned} \tilde{\gamma}_{\alpha\beta} &= \Omega^2 \gamma_{\alpha\beta} , \\ \tilde{n}_\alpha &= \Omega n_\alpha . \end{aligned} \quad (25)$$

Then, the boundary term (23) yields:

$$-\frac{1}{2\kappa^2} \int_{\delta\mathcal{M}} d^3x \sqrt{|\tilde{\gamma}|} \epsilon \tilde{\gamma}^{\alpha\beta} \tilde{n}^\sigma \partial_\sigma (\delta q_{\alpha\beta}) . \quad (26)$$

This is nothing but the variation of the Gibbons-Hawking-York boundary term of the conformal metric, such that the appropriate term to be added to the gravitational action is given by:

$$S_B = \frac{1}{\kappa^2} \int_{\delta\mathcal{M}} d^3x \sqrt{|\tilde{\gamma}|} \epsilon \mathcal{K} , \quad (27)$$

where the conformal extrinsic curvature is given by:

$$\mathcal{K} = \nabla_\alpha \tilde{n}^\alpha , \quad (28)$$

being the covariant derivative defined as compatible to the metric $q_{\mu\nu}$. This is related to the extrinsic curvature as defined by the covariant derivative of the normal vector to the boundary n_μ by:

$$\mathcal{K} = \Omega^{-1} K + 3\Omega^{-2} n^\mu \partial_\mu \Omega , \text{ with } K = \nabla_\alpha^0 n^\alpha , \quad (29)$$

where ∇_α^0 is the covariant derivative compatible with the spacetime metric $g_{\mu\nu}$. Hence, the corresponding surface term has been obtained within the Palatini formalism. In the next section, we consider a direct application through the Euclidean semiclassical formalism.

EUCLIDEAN APPROACH AND THERMODYNAMICS OF SCHWARZSCHILD BLACK HOLES

Let us consider the Schwarzschild black hole, which can be described by standard coordinates as

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 , \quad (30)$$

where $d\Omega^2$ is the line element of a two sphere. This spacetime metric is a solution of the gravitational action (1) as far as $f(\mathcal{R} = 0) = 0$ and the algebraic equation (12) has the root $\mathcal{R} = 0$. The Euclidean approach consists of approximating the partition function in gravity, which is described by the path integral [7]

$$Z[\beta] = \int d[g] d[\Gamma] d[\psi] e^{iS} . \quad (31)$$

Here S is the gravitational action that includes the corresponding matter fields ψ . By applying the saddle point approximation, the main contribution to the path integral is given by the classical action of a Euclidean solution:

$$Z[\beta] = e^{\beta F} \approx e^{-S_E} , \quad (32)$$

where F is the free energy and $\beta = T^{-1}$ with T the temperature of the system, while the Euclidean action after applying a Wick rotation $t \rightarrow i\tau$ yields:

$$\begin{aligned} S_E &= S_{EG} + S_{EB} = \\ &= \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \sqrt{-g} f(\mathcal{R}) - \frac{1}{\kappa^2} \int_{\delta\mathcal{M}} d^3x \sqrt{|\tilde{\gamma}|} \mathcal{K} \end{aligned} \quad (33)$$

In the case of the Schwarzschild spacetime metric (30), the only non zero contribution to the Euclidean integral (33) is the surface term, as $\mathcal{R} = \mathcal{R}_0$ is a constant and $f(\mathcal{R}_0) = 0$. The hypersurface of integration is given by $r = R$ and in order to make the Euclidean integral convergent at infinity, one subtracts the corresponding contribution of the asymptotic flat spacetime [8]:

$$\begin{aligned} S_E - S_E^0 &= S_{EB} - S_{EB}^0 = \\ &= \frac{1}{\kappa^2} \int_{\delta\mathcal{M}} d^3x \left[\sqrt{|\tilde{\gamma}|} \mathcal{K} - \sqrt{|\tilde{\gamma}^0|} \mathcal{K}^0 \right] . \end{aligned} \quad (34)$$

Here $\tilde{\gamma}$ and $\tilde{\gamma}^0$ are the conformal transformation of the induced metrics on the hypersurface $r = R$ for the Schwarzschild spacetime and the Minkowski spacetime respectively:

$$\begin{aligned} \gamma_{\mu\nu} dx^\mu dx^\nu &= \left(1 - \frac{2GM}{R}\right) d\tau^2 + R^2 d\Omega^2 , \\ \gamma_{\mu\nu}^0 dx^\mu dx^\nu &= d\tau_0^2 + R^2 d\Omega^2 . \end{aligned} \quad (35)$$

As usual the Euclidean time τ is integrated over the period $\beta = T^{-1} = 2\pi/\kappa = 8\pi GM$, with κ being the surface

gravity and T the temperature. To make the lengths equal for the Euclidean time in the Schwarzschild and Minkowski spacetimes, one imposes $\beta^0 = \beta(1 - \frac{2GM}{R})^{1/2}$. In addition, the corresponding extrinsic curvatures on the hypersurface $r = R$ are given by:

$$\begin{aligned} K^0 &= -\frac{2}{R}, \\ K &= -\frac{2}{R} - \frac{MG}{R^2(1 - \frac{2GM}{R})}. \end{aligned} \quad (36)$$

Hence, by using (29) the Euclidean integral (34) leads to:

$$S_E - S_E^0 = f_{\mathcal{R}} \frac{\beta MG}{2} \left(1 - \frac{2GM}{R}\right)^{1/2}, \quad (37)$$

which in the limit $R \rightarrow \infty$ yields:

$$S_E - S_E^0 = f_{\mathcal{R}} \frac{\beta MG}{2} = f_{\mathcal{R}} \frac{\beta^2}{16\pi G}. \quad (38)$$

From the free energy in (32), one has $\beta F = S_E - S_E^0 = f_{\mathcal{R}} \frac{\beta^2}{16\pi G}$ and by using the well known thermodynamical relation for the entropy:

$$\mathcal{S} = \left(\beta \frac{\partial}{\partial \beta} - 1\right) \beta F, \quad (39)$$

the entropy \mathcal{S} of the Schwarzschild black hole in Palatini $f(\mathcal{R})$ theories is given by:

$$\mathcal{S} = f_{\mathcal{R}} \frac{\beta^2}{16\pi G} = \frac{f_{\mathcal{R}}}{4G} A. \quad (40)$$

Here we have used the area of the horizon $A = 4\pi r_s^2$ with $r_s = 2GM$. Hence, this is the expression for the entropy in the Palatini formalism which has its origin in the surface term obtained in the previous section. This coincides with the expression in the metric formalism for $f(R)$ gravitational theories [9]. Note also that this result was previously obtained through Noether charges in [6] and extended in [10], such that both approaches coincide.

CONCLUSIONS

In this letter, we have shown that the surface terms corresponding to a theory of gravity formulated in the Palatini formalism can not be removed but provide a contribution similar to the well known Gibbons-Hawking-York term of General Relativity and other higher other theories including scalar tensor theories [2, 3]. Nevertheless, the surface term here depends

upon the conformal geometry of the boundary which is sensible to the gravitational action itself through the derivative of the action $f_{\mathcal{R}}$.

As in other metric compatible theories, the boundary term is necessary for multiple analysis, as the hamiltonian formulation of the theory or the calculation of the black hole entropy in Schwarzschild spacetime. The latter has been explicitly shown here, where the expression of the entropy matches the one obtained before in the literature by using Noether charges [6]. Next efforts should lie on the formulation of the hamiltonian approach of the theory and the corresponding ADM energy.

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