

# Ghost condensation and Ostrogradskian instability on low derivative backgrounds

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The observation of stable low-derivative degrees of freedom, relevant for low energy observers, amid the likely unstable effects of higher-time derivative terms can be treated as a built-in constraint in the dynamics. This constraint effectively turns degenerate a class of higher-derivative sectors and does not trivialize their effects. We show this new class of self-interacting and ghost-free higher-derivative extensions for every low derivative scalar. They lead to suppressed fluctuations atop low derivative backgrounds instead of a ghost. In contrast to some models with first-order derivative interactions with applications for dark energy and inflation, these theories with *necessarily constrained* second-order derivative self-interactions do not modify the speed of propagation, neither the cone of influence for the field equations, which are prescribed by the low-derivative background (source) that is also essential to condense the ghost; hence, avoiding the superluminality issues of the former.

## I. INTRODUCTION

In spite of the phenomenological success of the standard model of particle physics (SM) and Einstein gravity in a wide range of energies, it is understood that they do not give the full picture. New effects are explored beyond the currently tested energy scales, for instance, by adding effective terms to the SM which must cause negligible corrections in at least some energy regimes. In striking contrast, *non-degenerate* higher-derivative terms do not induce small corrections but *radically* modify the physics through non-perturbative effects, as they bear a fundamental instability [1]. They enlarge the dimensionality of phase space including a ghost that catastrophically destabilizes the low-derivative degrees of freedom (*dofs*) upon interaction [1–6]. Hence, higher-derivative extensions are either neglected, or applied only in the regime of the effective theory, for instance, by the method of “perturbative constraints” which can at best hide the ghost at some order, although not eliminate it [1–4, 7–12].

Galileons [13], and *degenerate* theories [6, 14–17] are among some exceptions [18–21]. Concretely, the theorem of Ostrogradsky states that every *non-degenerate* higher-derivative theory entails an unbounded energy from below, which is ultimately seen in the propagation of an additional ghostly *dof* [1, 2, 5, 22]. Constrained degenerate theories circumvent the issue at the expense of introducing an additional *ad-hoc* low-derivative *dof* that must non-trivially realize the degeneracy, and eliminate the ghost. All in all, the higher-derivative *dof* may lack physical interpretation of its own. A less restrictive class of degenerate, quadratic Lagrangians was detailed in [6].

In contrast, we propose a class of degenerate and ghost-free higher-derivative extensions for every low-derivative

scalar that are necessarily self-interacting, where the stability follows in a similar mechanism as in ghost condensation [23]. As a distinctive feature, their stability critically hinges on the propagation of the corresponding low-derivative *dofs*, which are taken as a *source* for any effect requiring higher derivative terms for its description. In short, this class is stable by promoting the “incidental observation of low-derivative *dofs*” to be a genuine built-in *constraint* amid the would-be unstable higher-derivative dynamics. This constraint, first, takes care for the observation of the low-derivative *dofs* in the background. Second, it generates secondary constraints that turn *degenerate* and ghost-free only this compatible class. This setup keeps the dimensionality of phase space at most the same as for the corresponding low-derivative theory, without a ghost, second-order equations, and the new effects are suppressed corrections atop the low-derivative mode.

The layout of this work is the following: In section II we motivate the setup by considering fluctuations about solutions to the low-derivative theory. In section III we summarize the proposal in a two-step setup. In section IV we verify the stability and the low dimensionality of phase space. In section V we show with an example the new dynamics, and the general properties for the scalar ghost condensate. We stress on the speed of propagation, the (acoustic) cone of influence for the wave equation, and subluminality despite the built-in higher-derivative self-interactions. We give the conclusions in section VI.

## II. FLUCTUATIONS ABOUT LOW DERIVATIVE DYNAMICS

As a motivation for the main results in the sections below, we first consider an expansion about solutions to the low-derivative theory. This expansion is meaningless for an (unconstrained) unstable higher-derivative theory.

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Thus, consistency requirements and smallness of the fluctuations will reveal, first, a special class of higher derivative sectors, and second, a minimal compatible choice of necessary constraints, both *prescribed by* the background low-derivative dynamics.

Let us first consider a theory with Lagrangian depending on up to first time derivatives of the dynamical variable  $\phi$ , and possibly, spatial derivatives  $\mathcal{L}^{(1)}(\partial\phi, \phi)$ . We will assume it is non-degenerate. That is, with  $\dot{\phi}$  time derivative of  $\phi$ ,

$$\frac{\partial^2 \mathcal{L}^{(1)}}{\partial \dot{\phi}^2} \neq 0, \quad (1)$$

such that the Euler-Lagrange equation for  $\mathcal{L}^{(1)}$  depends linearly on  $\ddot{\phi}$ . For definiteness, we assume that  $\mathcal{L}^{(1)}$  propagates only one *dof* and the dimensionality of phase space is 2, since two initial conditions must be imposed in order to solve the second order equation of motion.

Now, higher derivative extensions to the low derivative theory  $\mathcal{L}^{(1)}$ ,

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}, \quad (2)$$

where  $\mathcal{L}^{(2)}(\partial^2\phi, \partial\phi, \phi)$  depends on second time derivatives of  $\phi$ , increase the number of *dofs* (introduces a ghost) in the case that  $\mathcal{L}^{(2)}$  is non degenerate: namely, the term

$$\frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{\phi}^2}, \quad (3)$$

does not vanish. With this non degenerate condition, the theory (2) is unstable and non unitary upon quantization [1–6, 14, 15, 17]. For definiteness, we will assume: (3) is not *identically zero* (instead, we will construct a constrained version of (2) where (3) vanishes only on-shell). Furthermore, for the moment, we consider  $\mathcal{L}^{(2)}$  strictly high on derivatives. That is,  $\mathcal{L}^{(2)}$  can be written proportional to  $\ddot{\phi}$  and these higher derivatives cannot be eliminated by integration by parts. Finally, we will canonically normalize  $\phi$  based on  $\mathcal{L}^{(1)}$ , in such a way that with  $\mathcal{L}^{(2)}$ , we introduce a new energy scale ( $\Lambda$ ). Unless stated otherwise in the next sections, to avoid distractions with non-essential field formalism, we will first analyze mechanics of the sole dynamical variable  $\phi(t)$  depending only on time.

Note that, if the higher derivative terms are not degenerate, they cannot induce small corrections to the low derivative dynamics of  $\mathcal{L}^{(1)}$ . First, the equation of motion is of 4th order and 4 initial conditions are required; hence, the dimension of phase space for dynamics of the theory (2) is larger than in the theory  $\mathcal{L}^{(1)}$ . Second, upon interaction, the low derivative *dof* propagated with  $\mathcal{L}^{(1)}$  dynamics is catastrophically destabilized. In other words, (2) leads to unstable dynamics regardless if there is any small parameter suppressing the higher derivative sector [1]. As noted long ago, the Ostrogradsky's instability is a non-perturbative effect [1–5]. Perturbative expansions for (2) can at best hide

the ghost [1–4, 7–12].

Below, we construct a class of *constrained* modifications of the *unstable theory* (2), which we denote as  $\mathcal{L}'(\phi)$ . They have the property that fluctuations about the background set by the low derivative dynamics are degenerate, stable (in the sense of Ostrogradsky) and can be arbitrarily small. In other words, the higher derivative terms in the constrained theory  $\mathcal{L}'(\phi)$ ,

1. induce only corrections to the low-derivative modes described by  $\mathcal{L}^{(1)}$  (no new modes are integrated-in). These corrections are suppressed by  $\Lambda$ , and,
2. they are naturally stabilized by the low-derivative modes described by  $\mathcal{L}^{(1)}$  (the fluctuations are degenerate on the  $\mathcal{L}^{(1)}$ -background).

Note that we are selecting  $\mathcal{L}^{(1)}$  as special for the dynamics. A primary motivation is the phenomenological success of low derivative theories at least at low energies. Hence, we refer to the low derivative dynamics derived from  $\mathcal{L}^{(1)}$  as the *low energy sector* and the corresponding solutions as the *low energy modes*.

By the properties of the corrected theory  $\mathcal{L}'$ , we consider a perturbative expansion with the distinctive feature that the leading, 0-th order approximation is the standard, low derivative dynamics. Namely, denoting the Euler-Lagrange equation derived from an action with Lagrangian  $F(\psi_1, \psi_2, \dots)$ , for the dynamical variable  $\psi_1$ , as,

$$\Theta(F; \psi_1) = 0, \quad (4)$$

we denote the solution for the *low energy sector* as  $\phi_0$  such that,

$$\Theta(\mathcal{L}^{(1)}; \phi_0) = 0. \quad (5)$$

That is,  $\phi_0$  represents the *low energy modes*. Now, let us decompose  $\phi$  in terms of a fluctuation ( $\pi$ ) about this 0-th order solution as,

$$\phi = \phi_0 + \epsilon\pi, \quad (6)$$

where  $\epsilon$  is an *arbitrarily small* dimensionless parameter, whose physical meaning will be argued below. Note that this expansion is meaningless for the unstable theory (2). However, for the corrected theory  $\mathcal{L}'$ , the expansion must be meaningful and small fluctuations about the low energy solution  $\epsilon\|\pi\| \ll \|\phi_0\|$  must not become large arbitrarily fast, as one would expect from an Ostrogradsky unstable theory (2).

Let us first expand about  $\phi_0$  in (2) up to order  $\mathcal{O}(\epsilon^2)$  and read out from the signatures of the instability the structure that  $\mathcal{L}'$  must have.  $\mathcal{L}$  becomes,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\pi, \quad (7)$$

where  $\mathcal{L}_0$  can be written as a total time derivative, as it only depends on the fixed background  $\phi_0$ , and  $\mathcal{L}_\pi$  is,

$$\mathcal{L}_\pi = \epsilon^2 \frac{1}{2} \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \dot{\phi}^2} \Big|_0 \ddot{\pi} + \tilde{\mathcal{L}}_\pi + \mathcal{O}(\epsilon^3), \quad (8)$$

where the term  $\tilde{\mathcal{L}}_\pi$  is, up to a total time derivative, a second order polynomial of  $\epsilon\pi$  and  $\epsilon\dot{\pi}$ , and the subindex  $|_0$  means evaluation at the background  $\phi_0$ . Notice again that the  $\mathcal{O}(\epsilon^2)$  approximation in (8) cannot be justified for the naive, unstable theory (2), however, it should be meaningful for the corrected theory. Let us take a first step to construct  $\mathcal{L}'$ : at order  $\mathcal{O}(\epsilon^2)$  the only term that signals the Ostrogradsky's instability is the first on the right hand side of equation (8) ( $\propto \ddot{\pi}^2$ ). It leads to a linearized equation of 4-th order for  $\pi$ . Clearly, the constrained dynamics  $\mathcal{L}'$  should not have such terms in the perturbative expansion about the low energy mode  $\phi_0$ . Let us see the options: on the one hand, if (2) contains no self-interactions in the higher derivative sector, then (3) should be zero in order to eliminate the signature of the instability in (8). This is the trivial degeneracy requirement such that (3) identically vanishes, which we excluded in first place. On the other hand, if there are nonlinear terms in the higher derivative sector, then (3) is a function of  $\phi$  and derivatives. Since this function must be such that, if it is evaluated on the 0-th order solution  $\phi_0$ , it vanishes, a clear choice is,

$$\frac{\partial^2 \mathcal{L}^{(2)}}{\partial \ddot{\phi}^2} \propto \Theta(\mathcal{L}^{(1)}; \phi), \quad (9)$$

such that the perturbative expansion for  $\mathcal{L}'$  reduces to,

$$\mathcal{L}_\pi = \tilde{\mathcal{L}}_\pi + \mathcal{O}(\epsilon^3), \quad (10)$$

and the fluctuations  $\pi$  at  $\mathcal{O}(\epsilon^2)$  solve a second order equation with (space-)time dependent coefficients fixed by  $\phi_0$ . Let us note that (9) is not identically zero, and yet, it vanishes for fluctuations about low energy modes at  $\mathcal{O}(\epsilon^2)$ . Below, we must implement this non-trivial degeneracy at order  $\mathcal{O}(\epsilon^3)$ .

Furthermore note that (9) is not a necessary condition; however, it encloses the interesting property that the stability of the higher derivative theory hinges on the propagation of the well-known low energy modes. Thus, one could expect that corrections due to  $\mathcal{L}^{(2)}$  could be masked in low energy scatterings in the ( $\mathcal{L}^{(2)}$ -stabilizing) observation of the low-derivative degrees of freedom.

Now, Ostrogradsky's instability is a non-perturbative effect which cannot be hidden at  $\mathcal{O}(\epsilon^3)$ . Therefore, as is well known,  $\mathcal{L}'$  must have constraints and be degenerate [2, 3, 6, 14–17]. Let us show the constraint in  $\mathcal{L}'$  that not only stabilizes at non-perturbative level, but also makes meaningful the above-defined expansion about the low energy mode: in the unstable theory (2) the origin of the issue is that the low energy mode  $\phi_0$  is not meaningful as a 0-th order solution in a perturbative expansion. In other words, fluctuations, “small” with respect to  $\phi_0$ , would become rapidly “large” due to the  $\mathcal{O}(\epsilon^3)$  terms. Therefore, for the corrected theory  $\mathcal{L}'$  it must be guaranteed in first place that the low energy mode  $\phi_0$  is a background on top of which fluctuations can be built despite the higher derivative terms.

Altogether, in  $\mathcal{L}'$  this can be treated as a genuine constraint on the dynamical variable  $\phi$ ,  $J(\phi)$ , imposed by an

auxiliar variable  $a$  (Lagrange multiplier),

$$\mathcal{L}'(\phi, a) = \mathcal{L}(\phi, \partial\phi, \partial^2\phi) + aJ(\phi), \quad (11)$$

where  $\mathcal{L}$  is given by (2) satisfying (9), and the obvious choice for  $J(\phi)$  that guarantees the low derivative background  $\phi_0$ , which however does not trivialize the dynamics because  $a$  must become dynamical, solving a 2nd-order differential equation, and bearing the higher-derivative effects, is,

$$J(\phi) := \Theta(\mathcal{L}^{(1)}; \phi). \quad (12)$$

Equivalently,  $J(\phi)$  sources the auxiliar  $a$ , and the physical interpretation is clear: the dynamics of the low energy mode ( $\phi_0$ ) sources any effect requiring higher derivative terms for its description. Following Dirac's programme we will see the non trivial character of this setup, as it generates more constraints than (12) itself. Hence, (12) is not identically zero and  $\mathcal{L}'$  dynamics can be richer than the corresponding low derivative sector (5). It will reduce the dimensionality of phase space, leaving no ghost, and relating  $a$  and  $\phi$  such that altogether there is at most one truly dynamical degree of freedom.

As a first taste of this analysis notice that the constraint (12) implies the required degeneracy only on-shell: namely, (9) does not vanish identically, but it does vanish when it is valued on solutions to the Euler-Lagrange equation for  $a$ ,

$$\Theta(\mathcal{L}'; a) = J(\phi) = 0, \quad (13)$$

which sets  $\phi = \phi_0$  (only on-shell) and however, *it does not trivialize  $\mathcal{L}'$*  because there is an additional dynamical variable ( $a$ ) that is taking care for the observation of these low energy modes amid the higher-derivative effects of  $\mathcal{L}^{(2)}$ . Indeed,  $a$  solves a differential equation on its own:

### A taste of the new dynamics

The differential equation for the *ghost condensate* ( $a$ ), with the notation (4), is deduced from,

$$\Theta(\mathcal{L}'; \phi) = \Theta(\mathcal{L}; \phi) + \Theta(aJ(\phi); \phi) = 0,$$

where we can use the other Euler-Lagrange equation (13),

$$\Theta\left(a\Theta(\mathcal{L}^{(1)}; \phi); \phi\right)\Big|_0 = -\Theta(\mathcal{L}^{(2)}; \phi)\Big|_0. \quad (14)$$

More explicit, by assumption (1) the term  $\Theta(\mathcal{L}^{(1)}; \phi)$  depends linearly on  $\dot{\phi}$  such that, for instance, in the mechanics of a point particle the left hand side of (14),

$$\left(\frac{d^2}{dt^2} \frac{\partial}{\partial \ddot{\phi}} - \frac{d}{dt} \frac{\partial}{\partial \dot{\phi}} + \frac{\partial}{\partial \phi}\right) \left(a\Theta(\mathcal{L}^{(1)}; \phi)\right)\Big|_0,$$

is a second-order differential operator acting on  $a$ , which may contain damping terms and non vanishing time dependent coefficients valued on the sourcing, low energy mode  $\phi_0$ . Finally, the right hand side of (14) is a forcing

term driving the non homogeneous equation for  $a$ , which is defined by the dynamics of the higher derivative sector  $\mathcal{L}^{(2)}$ . Notice that it is non-zero because it is valued on  $\phi_0$ , and also, it is necessarily suppressed by the scale  $\Lambda$ . Prominently, in the case of relativistic field theory, this setup matches the speed of propagation and (acoustic) cone of influence for  $a$ -wave equation (14) with the wave-equation for the low energy mode (13), even though there are derivative interactions of  $\phi_0$  forcing  $a$  on the right hand side of (14). That there are second order equations *does not mean* that the system is stable in the sense of Ostrogradsky [6]. This requires a critical verification of the energy and dimensionality of phase space [2, 5, 6]. (For more details of the ghost condensate dynamics, see section V).

### A taste of the stability, and the criticality

Let us review the key aspects that lead to “stable” dynamics for (11), while the initial theory (2) is unstable (Here we only refer to “stability” only in the sense of Ostrogradsky): It is easy to see that the conserved quantity derived from the time homogeneity of a second-order time derivative action with Lagrangian  $\mathcal{L}$  (which we associate with the energy for a standard low derivative theory), is given by,

$$E_\phi = \frac{\partial \mathcal{L}}{\partial \ddot{\phi}} \ddot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} - \dot{\phi} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{\phi}}. \quad (15)$$

The configuration space is determined by the coordinates  $\phi, \dot{\phi}, \ddot{\phi}$  and  $\ddot{\phi}$ . All terms on the right hand side of (15) with the exception of the last one, depend on  $\phi, \dot{\phi}$  and  $\ddot{\phi}$  in a non trivial way. However, expanding the total derivative in the last term, it is easy to see that it depends linearly on  $\ddot{\phi}$  as,

$$-\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi}^2} \ddot{\phi}. \quad (16)$$

If (3) does not vanish and there are no constraints, this linear dependance implies that the energy is not bounded from below [1–6, 14, 15, 17, 24]. On the other hand, for a higher derivative sector with the structure (9), first note that in the energy for the small fluctuations about the low energy mode ( $E_\pi$ ), for the Lagrangian  $\mathcal{L}_\pi$  (8) up to order  $\mathcal{O}(\epsilon^2)$  this term vanishes,

$$-\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi}^2} \Big|_0 \ddot{\pi} \ddot{\pi} \propto -\Theta(\mathcal{L}^{(1)}(\partial\phi_0, \phi_0)) \ddot{\pi} \ddot{\pi},$$

Second, without restricting to small fluctuations, let us use the full structure of the corrected theory  $\mathcal{L}'$ : because the energy is expressed in terms of solutions to the equations of motion, we can use again the Euler-Lagrange equation for  $a$  (13), which imposes the low energy mode that by definition vanishes (9) *only on-shell*, as well as the critical would-be linear term  $\sim \ddot{\phi}$ . As we will see below,

the linear terms in  $a$  in the energy are only temporary, because  $a$  is a Lagrange multiplier that must be solved. Similar considerations can be done for a field theory, as we detail in the next sections.

The vanishing of the linear dependance on  $\ddot{\phi}$  of the energy amounts to a would-be linear momentum in the Hamiltonian that is constrained to other canonical coordinates; thus, eliminating the linear dependance on the hypersurface of constraints. This constraint follows from (9) and the constraint (12) (associated with the use of (13) in the previous analysis). They imply the reduction of dimensionality of phase space from 4, in the unstable higher derivative theory (2), to at most 2 in a (consistent) constrained theory (11) (For more details see the Hamiltonian analysis in section IV). Therefore, up to further cancellations, the amount of initial data for the  $\mathcal{L}'$  dynamics is the same as for the low energy theory  $\mathcal{L}^{(1)}$  that is *specialy* chosen at the preparation of the scattering (which sets apart  $\mathcal{L}^{(1)}$  dynamics as a circumstantial *source* for any new dynamics to be probed in the high energy scattering,  $\mathcal{L}^{(2)}$ ).

Finally, let us note: from the energy  $E_\phi$  with the term (16), the condition (9) amounts to have designed the higher-derivative theory  $\mathcal{L}'$  such that the 0-th order solution for the perturbative expansion (low-derivative dynamics) is a critical point of the energy,

$$\frac{\partial E_\phi}{\partial \ddot{\phi}} = -\frac{\partial^2 \mathcal{L}'}{\partial \ddot{\phi}^2} \dot{\phi} \propto \Theta(\mathcal{L}^{(1)}; \phi) = 0.$$

### III. SUMMARY OF THE SETUP

The setup proposed in this letter can be summarized in two observations: first, the stability of a class of higher-derivative theories  $\mathcal{L}'$  ( $\mathcal{L}^{(1)}$ ) could be granted by the propagation of the *low derivative mode* that is described with *its own low-derivative sector*,  $\mathcal{L}^{(1)}$ . Second, this does not trivialize the higher derivative effects.

More concretely, there exists a class of constrained and non-trivially degenerate higher-derivative theories  $\mathcal{L}'(\partial^2\phi, \partial\phi, \phi, a)$  that

1. can be written as,

$$\mathcal{L}'(\phi, a) = \mathcal{L}(\phi, \partial\phi, \partial^2\phi) + a\Theta(\mathcal{L}^{(1)}; \phi), \quad (17)$$

2. and satisfies,

$$\frac{\partial^2 \mathcal{L}'}{\partial \ddot{\phi}^2} = c(\phi) \Theta(\mathcal{L}^{(1)}; \phi), \quad (18)$$

where  $\mathcal{L}^{(1)}(\phi, \partial\phi)$  is a non degenerate low-derivative sector in  $\mathcal{L}(\phi, \partial\phi, \partial^2\phi)$ ;  $\Theta(\mathcal{L}^{(1)}; \phi)$  was defined in (4) (by (1) it depends linearly on  $\ddot{\phi}$ ), and  $c(\phi)$  is a non-zero, non-singular function which may depend on up to second derivatives of  $\phi$ ,

such that  $\mathcal{L}'(\phi, a)$  propagates at most one degree of freedom associated with the single dynamical variable  $\phi$ , which is not a ghost. In particular, there must be at least one such  $\mathcal{L}^{(1)}$ -sector that remains after all terms proportional to  $\propto \ddot{\phi}$  and  $\propto a$  are removed from  $\mathcal{L}'$ . This was the particular choice in the previous section.

This setup can be generalized to the case of field theory. For instance if  $\phi(x)$  is a real scalar field, the generalization for a relativistic theory is straightforward:

$$\frac{\partial^2 \mathcal{L}'}{\partial \phi_{,\mu\nu} \partial \phi_{,\rho\sigma}} = c^{\mu\nu\rho\sigma}(\phi) \Theta(\mathcal{L}^{(1)}; \phi), \quad (19)$$

where  $\phi_{,\mu\nu} \equiv \partial_\mu \partial_\nu \phi$  and the assumptions on  $c$  ( $\equiv c^{0000}$ ) are extended to  $c^{\mu\nu\rho\sigma}(\phi, \partial\phi, \partial^2\phi)$ .

Let us stress that (18), or (19) is non-zero and only vanishes on-shell. Further details of  $\mathcal{L}^{(1)}$  and the remaining part of  $\mathcal{L}'$  were given in the discussion following (1) and (3). Note that the explicit term  $\mathcal{L}^{(1)}$  in the total Lagrangian  $\mathcal{L}'$  (17) contributes to the energy, even though it is redundant to derive the *classical* dynamics. Finally note that this construction does not avoid the possibility of instabilities due to unbounded potential or other origins different than Ostrogradsky's.

#### IV. OSTROGRADSKY'S STABILITY FOR THIS SETUP

Let us check explicitly that this mechanism completely avoids the Ostrogradsky's instability. We will work in a mechanical context in order to make the analysis clearer, but generalization to fields is straightforward.

The six canonical coordinates corresponding to  $a(t)$  and  $\phi(t)$  in (17) are [2, 22]:

$$\begin{aligned} x_1 &= \phi & p_1 &= \frac{\partial \mathcal{L}'}{\partial \dot{\phi}} - \frac{d}{dt} \frac{\partial \mathcal{L}'}{\partial \ddot{\phi}} \\ x_2 &= \dot{\phi} & p_2 &= \frac{\partial \mathcal{L}'}{\partial \dot{\phi}} \\ a &= a & p_a &= \frac{\partial \mathcal{L}'}{\partial \dot{a}} \end{aligned} \quad (20)$$

where the elementary non-zero Poisson brackets are  $\{a, p_a\} = 1$ ,  $\{x_i, p_j\} = \delta_{ij}$ ,  $i, j = 1, 2$ . The assumption of non-degeneracy (18) leads to a conjugate momentum that depends on the acceleration  $\ddot{\phi}$ ,

$$p_2 = p_2(\ddot{\phi}, x_2, x_1, a) \quad (21)$$

Thus,  $\ddot{\phi}$  can be inverted in terms of the canonical coordinates  $p_2, x_1, x_2, a$ . On the other hand, the  $p_1$  conjugate momentum depends linearly on  $\ddot{\phi}$ ,

$$p_1 = G(x_1, x_2, a, \dot{a}, \ddot{\phi}(p_2, x_1, x_2, a)) - \ddot{\phi} \frac{\partial^2 \mathcal{L}'}{\partial \dot{\phi}^2} \quad (22)$$

$$p_1 = G(x_1, x_2, a, \dot{a}, \ddot{\phi}(p_2, x_1, x_2, a)) - \ddot{\phi} c(\phi) \Theta(\mathcal{L}^{(1)}; \phi)$$

where we have used the definition of  $\mathcal{L}'$  (18) and the explicit form of  $G$  is irrelevant for our discussion. Since neither the Lagrangian nor the other canonical coordinates depend on  $\ddot{\phi}$ , the linear dependence of  $p_1$  on  $\ddot{\phi}$  will

remain linear upon the Legendre transform that gives the Hamiltonian,

$$\begin{aligned} \mathcal{H} &= p_1 x_2 + p_2 \ddot{\phi}(p_2, x_1, x_2, a) + \lambda \xi_1 \\ &\quad - \mathcal{L}'(p_2, x_1, x_2, a, \ddot{\phi}(p_2, x_1, x_2, a)) \end{aligned} \quad (23)$$

where  $\lambda$  is a Lagrange multiplier for the primary constraint  $\xi_1 = p_a$ ,

$$\xi_1 = p_a \approx 0 \quad (24)$$

due to the no dependence on  $\dot{a}$ . It is easy to verify that (23) generates correct (lagrangean) time evolution [2], thus, it is the right functional form for the energy.

However, as has been widely discussed in the literature [1–6, 14, 15, 17, 24], unless a constraint expresses  $x_2$  in terms of  $p_1$ , the term  $p_1 x_2$  in the Hamiltonian is the most basic signal of the Ostrogradsky's instability. It is linear in the conjugate momentum  $p_1$  and it renders the Hamiltonian unbounded from below. Let us see in two steps how the stability arises in this construction:

1- *Consistency and secondary constraints (Dirac's programme)*: The conservation in time of  $\xi_1$  implies the low energy dynamics as a secondary constraint  $\xi_2$ ,

$$\dot{\xi}_1 = \{\xi_1, \mathcal{H}\} = -\frac{\partial \mathcal{H}}{\partial a} \approx 0 \quad (25)$$

$$\dot{\xi}_1 = -p_2 \frac{\partial \ddot{\phi}}{\partial a} + \frac{\partial \ddot{\phi}}{\partial a} \frac{\partial \mathcal{L}'}{\partial \ddot{\phi}} + \Theta(\mathcal{L}^{(1)}; \phi) = \Theta(\mathcal{L}^{(1)}; \phi) \approx 0,$$

where the equality holds on the hypersurface of constraints. The low energy dynamics  $\xi_2 = \Theta(\mathcal{L}^{(1)}; \phi)$  depends on the canonical coordinates  $x_1, x_2$  and the acceleration  $\ddot{\phi}(p_2, x_1, x_2, a)$  by assumption (1), thus, in principle, on the hypersurface of constraints  $\xi_2$  only relates  $a$  to the canonical coordinates  $x_1, x_2, p_2$ ,

$$\xi_2 = \xi_2(\ddot{\phi}(p_2, x_1, x_2, a), x_1, x_2) \approx 0. \quad (26)$$

The consistency of the Hamiltonian requires again the time conservation of (26),  $\dot{\xi}_2 \approx 0$ , which does not imply any more constraints by the consistency procedure followed above (It expresses  $\lambda = \dot{a}$  in terms of canonical coordinates, because  $\xi_2$  depends on  $a$ , such that  $\{\xi_2, \xi_1\} \neq 0$ ).

2- *Built-in constraints from the structure of  $\mathcal{L}'$* : The distinctive feature of the higher derivative sector (18) at work together with the constrained structure of the theory (17) is that the secondary constraint (25) that serves to express the auxiliary variable  $a$ , and  $\dot{a}$  in terms of other canonical coordinates, necessarily implies a new, independent secondary constraint  $\xi_3$ , which is seen in the definition of conjugate momentum (22),

$$\xi_3 = p_1 - G = -\ddot{\phi} c(\phi) \Theta(\mathcal{L}^{(1)}; \phi) \approx 0, \quad (27)$$

where  $G$  depends on  $x_2$  and other canonical coordinates. This is an additional constraint because the

only term containing  $\ddot{\phi}$  which lead to the definition of the independent momentum  $p_1$  (required to express all time derivatives of  $\phi$  in terms of canonical coordinates) vanishes on the hypersurface of constraints as a by-product of  $\Theta(\mathcal{L}^{(1)}; \phi) \approx 0$ . In other words,  $p_1$  is no longer independent but related to  $x_2$  and other canonical coordinates on the hypersurface of constraints, which gets rid of the linear dependence of  $\mathcal{H}$  on  $p_1$  (23). In fact, (27) is a new constraint that must be added to the set  $\{\xi_1, \xi_2\}$  to solve suitably all together, and new constraints may appear. For the moment, let us stress that the desired expression of  $x_2$  in terms of  $p_1$  is inherent in  $\mathcal{L}'$  and *the Ostrogradsky's instability is not present* ( $\mathcal{H} = p_1 x_2(p_1, \dots) + \dots$ ).

Let us stress that the independent constraint  $\xi_3$  that relates the would-be linear momentum  $p_1$  and other canonical coordinates is entirely due to the intimate relation between the structure of the higher derivative sector (18) and the constraint that takes care of the propagation of the low energy modes (12). In other words, had the condition (18) or the constraint imposed by  $a$  (low energy mode) (12) been different, the independent constraint (27) would not arise and the Ostrogradsky's stability would still be present. A particular case of such a failure was proven in [6]: namely, consider the case that  $a$  imposes a secondary constraint (12) that is also a second-order equation for  $\phi$ , but different than  $\Theta(\mathcal{L}^{(1)}; \phi)$  which is used to define  $\mathcal{L}'$  (18). Then, similarly, there would be a system of two second-order equations of motion, but critically, no additional  $\xi_3$  constraint would be present and Ostrogradsky's instability would remain. The dimensionality of phase space would remain high, signaling the ghost. In short, the constraint (12) must be compatible with the condition for  $\mathcal{L}'$  (18) such that  $\xi_3 \approx 0$  emerges.

Let us see the reduction of dimensionality in this setup: the special prescription (18) together with the propagation of the low energy mode in (17) implies in principle a 6-dimensional phase space (20), which is reduced by the 3 independent constraints  $\xi_1, \xi_2, \xi_3$  to at most 3 dimensions. As we saw, it guarantees the elimination of Ostrogradsky's instability, and in accordance with [6], the dimensionality of phase space has been reduced from 4 in the unstable theory (2) to at most 3 for  $\mathcal{L}'$  (17). Finally, depending on the particular theory (17), new constraints may arise. In fact, a consistent non singular theory must lead to an even-dimensional phase space. Thus, a consistent theory  $\mathcal{L}'$  (17) must imply an at most 2-dimensional phase space, or equivalently, propagates at most 1 degree of freedom. This agrees with the low energy dynamics  $\mathcal{L}^{(1)}$  which leads to the same dimensionality of phase space and same amount of degrees of freedom, in contrast to the unstable, unconstrained theory (2) that implies a 4-dimensional phase space, or 2 degrees of freedom, one of them being a ghost.

That the dimensionality of phase space for  $\mathcal{L}'$  dynamics reduces to at most the same of the corresponding low

derivative theory  $\mathcal{L}^{(1)}$  clearly does not imply that the only propagated degree of freedom is the same as the low energy mode. Indeed, for  $\mathcal{L}'$  dynamics the trajectory in phase space lies on a 2-dimensional hypersurface in a 6-dimensional phase space.

## V. DYNAMICS OF THE GHOST CONDENSATE

Below we discuss the generalities of the ghost condensate dynamics that arise in this setup. We stress on the speed of propagation, the (acoustic) cone of influence for the wave equation, as well as the stability and subluminality properties that can be inherited from a healthy low-derivative sector, despite the built-in derivative self-interactions in the higher-derivative sector. We start with the simplest case in mechanics of a single particle to show how the low-dimensionality of phase space, which is intimately related to the elimination of the ghostly degree of freedom, is indeed tied to the boundedness from below of the energy. We only show a solution of the ghost condensate for this simple case.

### A. A first example, the energy, and the dimensionality of phase space

The Hamiltonian for the Lagrangian  $\mathcal{L}'$  (17), with the constraint structure in section IV properly solved, first, would give the correct time evolution for at most one truly dynamical variable (at most a 2-dimensional phase space for a consistent model), and second, it would be bounded from below for the one dynamical variable (no Ostrogradsky's instability). We showed these two aspects in the last section and how they are intimately linked, in accordance with [6]. However, it is difficult to write the Hamiltonian in its explicit form for a particular model because of the built-in self-interactions that are necessary for ghost condensation: integrating  $\mathcal{L}'$  from the condition (18), assuming that  $c$  is a polynomial function of  $\ddot{\phi}$ , and because  $\Theta(\mathcal{L}^{(1)}; \phi)$  is linear in  $\ddot{\phi}$ , there is a highest order for  $\ddot{\phi}$  in  $\mathcal{L}'$  that scales at least as,

$$\mathcal{L}^{(2)} \propto \ddot{\phi}^p,$$

with  $p \geq 3$ . Hence, although we will not solve all the constraints neither compute the explicit Hamiltonian, we will compute, both the lagrangean dynamics and the energy function, knowing from the Hamiltonian analysis that the low-dimensionality of phase space will be intimately tied to the boundedness from below of the energy function.

Let us consider a first example: take the harmonic oscillator as the low-derivative sector,

$$\mathcal{L}^{(1)} = \frac{1}{2} \dot{\phi}^2 - m^2 \phi^2, \quad (28)$$

and consider a simple higher-derivative sector keeping the

symmetry  $\phi \rightarrow -\phi$ ,

$$\mathcal{L}^{(2)} = \frac{1}{2\Lambda^5} \ddot{\phi}^2 \left( -\frac{1}{6} \ddot{\phi}^2 + m^4 \phi^2 \right), \quad (29)$$

where  $\phi^2$ ,  $\Lambda^{-1}$ ,  $m^{-1}$  have units of time, such that in this setup the complete theory takes the form (17),

$$\mathcal{L}'(\phi, a) = \mathcal{L}^{(2)} + \mathcal{L}^{(1)} + a\Theta(\mathcal{L}^{(1)}; \phi),$$

where, we have used the notation (4), and the condition (18) holds,

$$\frac{\partial^2 \mathcal{L}'}{\partial \ddot{\phi}^2} = \frac{\partial^2 \mathcal{L}^{(2)}}{\partial \ddot{\phi}^2} = \frac{(\ddot{\phi} - m^2 \phi)}{\Lambda^5} \Theta(\mathcal{L}^{(1)}; \phi).$$

The Euler-Lagrange equation for  $a$  and  $\phi$  are respectively: the low derivative sector for  $\phi(t)$ , whose solutions we have denoted as  $\phi_0$ ,

$$\Theta(\mathcal{L}'; a) = \Theta(\mathcal{L}^{(1)}; \phi) = \left( \frac{d^2}{dt^2} + m^2 \right) \phi_0 = 0,$$

and the differential equation for  $a(t)$ ,

$$\Theta(\mathcal{L}'; \phi) = \Theta(\mathcal{L}^{(1)}; \phi) + \Theta(\mathcal{L}^{(2)}; \phi) + \Theta(a J(\phi); \phi) = 0,$$

where the last term with  $J(\phi) = \Theta(\mathcal{L}^{(1)}; \phi)$  reads,

$$\Theta(a J(\phi); \phi) = \left( \frac{d^2}{dt^2} \frac{\partial}{\partial \ddot{\phi}} - \frac{d}{dt} \frac{\partial}{\partial \dot{\phi}} + \frac{\partial}{\partial \phi} \right) a(-\ddot{\phi} - m^2 \phi),$$

which is a linear differential operator acting on  $a(t)$ , left hand side of,

$$\left( \frac{d^2}{dt^2} + m^2 \right) a = \frac{m^5}{\Lambda^5} \left( 3m^3 \phi_0^3 - 4m\phi_0 \dot{\phi}_0^2 \right), \quad (30)$$

where we have used  $\phi = \phi_0$  and the right hand side is the contribution from the higher-derivative dynamics  $\Theta(\mathcal{L}^{(2)}; \phi)|_0$  also written in terms of the low derivative solutions, hence the subindex  $|_0$ . As anticipated, the explicit term  $\mathcal{L}^{(1)}$  in the total Lagrangian  $\mathcal{L}'$  is redundant to derive the *classical* equations, however, it does contribute with the standard form of energy, as we verify below.

Note that both equations are of second order. Even though it seems that 4 initial conditions are required to find particular solutions,  $a$  is an auxiliary variable that is linked to the only 2 initial conditions that fix the particular solution  $\phi_0$ . Indeed, we know from the constrained evolution, which is implied by the Hamiltonian, that phase space is at most 2-dimensional. Let us see: the energy function (15) has a contribution from the harmonic oscillator  $\mathcal{L}^{(1)}$ , which is the low energy sector,

$$\mathcal{E}^{(1)} = \frac{1}{2} \dot{\phi}_0^2 + \frac{m^2}{2} \phi_0^2,$$

and a new contribution from the constrained higher derivative sector,

$$\mathcal{E}^{(2)} = \frac{m^5}{\Lambda^5} \left( 2m\phi_0^2 \dot{\phi}_0^2 + \frac{1}{4} m^3 \phi_0^4 \right) + \dot{\phi}_0 \dot{a} + m^2 \phi_0 a,$$

where all terms but the last two of the last line (dependent on  $a$  and  $\dot{a}$ ) are positive contributions to the energy  $\mathcal{E} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)}$ . Let us recall that  $a(t)$  is an auxiliary Lagrange multiplier, and it must be solved: to do so, we showed in section IV that the constraint that eliminates the Ostrogradsky's instability (27) is also necessarily associated to the constraint (25) and derived conditions that allow to express  $a$  and  $\dot{a}$  in terms of other canonical coordinates. In other words, the particular solution for  $a(t)$  (30) is linked to the solution to the other canonical coordinate  $\phi = \phi_0$  by the boundedness from below of the energy (elimination of Ostrogradsky's instability (27)), such that the dimensionality of phase space is at most 2. Indeed, if the particular solution for  $\phi_0$  is,

$$\phi_0(t) = c_1 y_1(t) + c_2 y_2(t)$$

where  $c_1, c_2$  are fixed by initial conditions and  $y_1 = \cos(mt)$ ,  $y_2 = \sin(mt)$ , then,  $a(t)$  is written as the solution to the homogeneous equation of (30), namely, with the same basis of independent functions  $\{y_1, y_2\}$ , plus a solution to the non homogeneous equation (30),  $N(t)$ ,

$$a(t) = c_3 y_1(t) + c_4 y_2(t) + \frac{1}{\Lambda^5} N(t),$$

where,

$$\frac{1}{\Lambda^5} N(t) = -\frac{1}{m} y_1 \int dt y_2 \Theta(\mathcal{L}^{(2)}; \phi)|_0 + (y_1 \leftrightarrow y_2),$$

depends only on  $c_1$  and  $c_2$ , and  $\Theta(\mathcal{L}^{(2)}; \phi)|_0$  is given by the right hand side of (30). The key aspect is: we can find  $c_3$  and  $c_4$  in terms of  $c_1$  and  $c_2$  just by writing the energy, which must be free of linear instabilities (bounded from below), as the Hamiltonian analysis revealed in section IV. Let us see: the last two terms in  $\mathcal{E}^{(2)}$  are expressed with these solutions as,

$$\dot{\phi}_0 \dot{a} + m^2 \phi_0 a = m^2 (c_1 c_3 + c_2 c_4) (y_1^2 + y_2^2) + \frac{\dot{\phi}_0 \dot{N} + m^2 \phi_0 N}{\Lambda^5},$$

where the quadratic first term is clearly not positive definite for every particular solution of  $\phi_0(t)$  and  $a(t)$  *unless*  $c_3 = c_1$  and  $c_4 = c_2$ . That there is such a non trivial choice in the energy that links a low-dimensionality of phase space with the boundedness from below is a signature of the elimination of Ostrogradsky's instability, which would be evident in the full Hamiltonian. The last two terms dependent on  $N(t)$ , and  $\dot{N}(t)$  are non linear functions of  $y_1, y_2$  and integrals, and thus, are not critical in this analysis (however, they could show other types of instabilities depending on the particular model).

All in all, the ghost condensate takes the form of a  $\Lambda$ -suppressed correction ( $N(t)$ ) superposed to the low energy mode  $\phi_0$ ,

$$a = \phi_0 + \frac{1}{\Lambda^5} N, \quad (31)$$

such that  $a(t)$  is fixed once the *two* initial conditions for the preparation of the low energy mode ( $\phi_0$ ) are given. This is reminiscent of the Ansatz ( $\phi = \phi_0 + \epsilon\pi$ ) for the perturbative expansion about  $\phi_0$  in  $\mathcal{L}'$  dynamics, which motivated this setup in section II, where we identify the solution to all orders  $\epsilon\pi = \Lambda^{-5}N$ . Let us stress that this form of the solution for the ghost condensate is not restricted to the particular higher-derivative sector (29). In fact, the form of the ghost condensate  $a(t)$  as a superposition of a  $\Lambda$ -suppressed correction atop  $\phi_0$  (31), holds for any higher derivative sector  $\mathcal{L}^{(2)}$  compatible to the harmonic oscillator (28). Namely, satisfying (18). This follows in such a simple way because the equations for  $a(t)$  and  $\phi(t)$  share the same *linear* differential operator,

$$\left(\frac{d^2}{dt^2} + m^2\right).$$

More intricate relations arise with nonlinearities in the low derivative sector  $\mathcal{L}^{(1)}$ .

### B. Scalar ghost condensate, the wavefront velocity and the causal structure

Ghost condensates in this setup inherit many of the features of propagation of the low energy modes. This holds because the differential operator for the ghost condensate is derived from the dynamics of the low energy mode ( $\Theta(\mathcal{L}^{(1)}; \phi)$ ), and most prominently, because the effective metric coincides for both field equations. We can easily see this: consider the complete theory with the form (17),

$$\mathcal{L}'(\phi, a) = \mathcal{L}^{(2)} + \mathcal{L}^{(1)} + a\Theta(\mathcal{L}^{(1)}; \phi),$$

where the low derivative sector is the most general self-interacting real scalar field  $\phi(t, \vec{x})$  with Lorentz invariant Lagrangian  $\mathcal{L}^{(1)}$  that depends only on powers of  $\partial_\mu\phi\partial^\mu\phi$  and  $\phi$ . The equation of motion for  $\phi$  is derived from,

$$\Theta(\mathcal{L}'; a) = \Theta(\mathcal{L}^{(1)}; \phi) = 0,$$

and defining  $2\mathcal{X} = \partial_\mu\phi\partial^\mu\phi$ , with  $g^{\mu\nu}$  as flat space-time metric, it can be written as,

$$-G^{\mu\nu}\partial_\mu\partial_\nu\phi - 2\mathcal{X}\frac{\partial^2\mathcal{L}^{(1)}}{\partial\phi\partial\mathcal{X}} + \frac{\partial\mathcal{L}^{(1)}}{\partial\phi} = 0, \quad (32)$$

where  $G^{\mu\nu}$  depends on  $\phi$  and its first derivatives,

$$G^{\mu\nu} = \frac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}}g^{\mu\nu} + \frac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2}\partial^\mu\phi\partial^\nu\phi. \quad (33)$$

$G^{\mu\nu}$  defines the characteristic curves of the field equation and the propagating character of solutions to (32). Namely, whether the equation is hyperbolic, parabolic or elliptic. In the case it is hyperbolic, there are indeed propagating solutions. In other words, the characteristic curves are real and they serve to identify the wavefront and its velocity [25, 26]. In short,  $G^{\mu\nu}$  fixes the speed of

sound for the wave equation (32) and the acoustic cone of influence [26–28]. Hence, it usually receives the name of effective, or emergent metric.

On the other hand, the equation for the scalar ghost condensate ( $a(x)$ ) is derived from  $\Theta(\mathcal{L}'; \phi) = 0$ . Denoting with  $\phi_0$  the solutions to (32), the equation for  $a(x)$  takes the form,

$$\Theta\left(a\Theta(\mathcal{L}^{(1)}; \phi); \phi\right)\Big|_0 = -\Theta(\mathcal{L}^{(2)}; \phi)\Big|_0. \quad (34)$$

The left hand side is a second order differential operator acting on  $a(t, \vec{x})$ ,

$$\left(\partial_\mu\partial_\nu\frac{\partial}{\partial(\partial_\mu\partial_\nu\phi)} - \partial_\mu\frac{\partial}{\partial(\partial_\mu\phi)} + \frac{\partial}{\partial\phi}\right)a\Theta(\mathcal{L}^{(1)}; \phi)\Big|_0;$$

as  $\Theta(\mathcal{L}^{(1)}; \phi)$  (left hand side of (32)) depends linearly on  $\partial_\mu\partial_\nu\phi$ , it takes the form,

$$(G^{\mu\nu}\partial_\mu\partial_\nu + v^\mu\partial_\mu + M^2)a, \quad (35)$$

where we encounter the same effective metric  $G^{\mu\nu}(\phi_0)$ . Thus we can identify the same characteristic curves for the field equation of the ghost condensate as for the respective low derivative theory (32). In other words, the higher derivative effects in this setup do not modify the speed of propagation, neither the (acoustic) cone of influence of the low derivative theory, because necessarily, the principal part of the differential operator ( $G^{\mu\nu}\partial_\mu\partial_\nu$ ) is kept invariant by the built-in constraint (12). Although this built-in property is what keeps the second-order equations of motion in this setup, thus helping to eliminate the ghost, these features do not arise from the critical definition of the higher derivative sector (18) that is necessary to keep the low dimensionality of phase space and fully eliminate the ghost, as we verified in the previous example. Therefore, although the coincidence of the causal structures follow from the necessary constraint (12), this property can be interpreted only in part as a by-product of the setup. All in all, if the effective metric  $G^{\mu\nu}$  implied by the low-derivative sector  $\mathcal{L}^{(1)}$  of the corresponding higher-derivative theory  $\mathcal{L}'$  satisfies the hyperbolicity, stability and subluminality conditions that were recognized long ago by Aharonov, Komar and Susskind [29] (Appendix A),

$$\frac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}} > 0, \quad \frac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2} \geq 0, \quad \frac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}} + 2\mathcal{X}\frac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2} > 0,$$

then, the scalar ghost condensate inherits these properties. The higher-derivative sector  $\mathcal{L}^{(2)}$  with the structure (19), constrained by (12) is limited to force the condensate  $a(x)$  as in the right hand side of (34), but not to define the propagating character of solutions neither the dimensionality of phase space.

On the other hand, the mass term for the ghost condensate is,

$$M^2(\phi_0) = -\Theta\left(\Theta(\mathcal{L}^{(1)}; \phi); \phi\right)\Big|_0,$$

and the damping term is,

$$v^\mu(\phi_0) = \left( 2\partial_\nu G^{\mu\nu} + \frac{\partial\Theta(\mathcal{L}^{(1)}; \phi)}{\partial(\partial_\mu\phi)} \right) \Big|_0,$$

such that  $v^\mu$  vanishes if the low derivative sector  $\mathcal{L}^{(1)}$  contains no derivative self-interactions. Consider an analogous example to (28) and (29), where the low derivative sector  $\mathcal{L}^{(1)}$  is the massive real scalar field ( $\phi$ ) and,

$$\mathcal{L}^{(2)} = \frac{1}{2\Lambda^8}(\Box\phi)^2 \left( -\frac{(\Box\phi)^2}{6} + m^4\phi^2 \right),$$

satisfies (19).  $\phi$ ,  $\Lambda$  and  $m$  have dimension of mass. With  $\mathcal{L}'$  in the standard form (17), the equation for the scalar ghost condensate  $a(x)$  is,

$$\Box a + m^2 a = \frac{m^5}{\Lambda^8}(3m^3\phi_0^3 - 4m\phi_0\partial_\mu\phi_0\partial^\mu\phi_0),$$

where  $\phi_0$  are solutions to the Klein-Gordon equation. Let us stress that the metric is flat for the  $a(x)$  field equation, and the speed of light is not endangered by the derivative self-interactions of the low energy mode ( $\phi_0$ ) on the right hand side. This contrasts to the typical, unconstrained low-derivative self-interactions that can be obtained with  $\mathcal{L}^{(1)}$ , whose non-perturbative effects can have disastrous consequences such as superluminality [26–32]. This holds in general: namely, in the case that  $\mathcal{L}^{(1)}$  contains no derivative self-interactions, the right hand side of (34) still contains derivative interactions of the low-energy mode forcing the ghost condensate, which are induced by the high-derivative sector  $\mathcal{L}^{(2)}$ , and however, do

not enclose contributions to the metric that could potentially spoil causality, or generate other undesirable effects (See related discussions in [26–32]).

## VI. CONCLUSIONS

We showed a new class of healthy degenerate higher-derivative extensions for every low-derivative scalar. As a distinctive feature, their stability critically hinges on the propagation of the corresponding low-derivative modes that are relevant at low energies, which were interpreted as a *source* for any effect requiring higher derivative terms. The analysis was summarized in a two-step setup, which, for a low-derivative theory, first, defines the compatible class of higher-derivative terms, and second, the minimal choice of necessary constraints that turns degenerate the latter class, and altogether eliminates the ghost. This setup leads to second-order equations, and keeps the dimensionality of phase space up to the same as for the corresponding low-derivative theory. These *necessarily constrained high-order* derivative self-interactions do not modify the principal part of the differential equation (effective metric), neither the cone of influence and speed of propagation; hence, avoiding the superluminality issues that may arise for *unconstrained first-order* derivative interactions [32]. Instead of integrating-in a ghost, they generate suppressed fluctuations atop the low-derivative mode.

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