Extremal cosmological black holes in Horndeski gravity and the anti-evaporation regime

Ismael Ayuso*
Departamento de Física and Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências, Universidade de Lisboa, Edifício C8, Campo Grande, 1769-016 Lisboa, Portugal

Diego Sáez-Chillón Gómez†
Department of Theoretical Physics, Atomic and Optics, Campus Miguel Delibes, University of Valladolid UVA, Paseo Belén, 7, 47011 - Valladolid, Spain

Extremal cosmological black holes are analysed in the framework of the most general second order scalar-tensor theory, the so-called Horndeski gravity. Such extremal black holes are a particular case of Schwarzschild-De Sitter black holes that arises when the black hole horizon and the cosmological one coincide. Such metric is induced by a particular value of the effective cosmological constant and is known as Nariai spacetime. The existence of this type of solutions is studied when considering the Horndeski Lagrangian. Also the so-called anti-evaporation regime, an instability on the radius of the horizon induced by external fields, is studied. Contrary to other frameworks, the radius of the horizon remains stable for some cases of the Horndeski Lagrangian when considering perturbations at linear order.

I. INTRODUCTION

General Relativity (GR) has shown its power of prediction over more than one hundred years, and despite some important issues, it is still considered as the best description of gravity. Nevertheless, there are some fundamental questions to be answered in the future in the context of theoretical physics. From an UV completation of gravity to cosmological late-time acceleration, among other also relevant problems, the scientific community is doing a great effort to afford them. In the context of cosmology, the main issue lies on the unknown dark energy (also on dark matter), which have been widely contrasted by observational data and many theoretical models have been proposed to explain its main consequence, the late-time acceleration of the universe expansion (for a review see [1]). Some of such dark energy models are focused on modifications of GR, which may provide a natural solution to the problem which might be connected to the corrections expected from some UV completations of GR, as string theory [2]. In this sense, the simplest way of modifying GR is by introducing an scalar field, which incorporates an additional scalar mode while keeping the well known predictions by GR unbroken through screening mechanism that can be implemented by an appropriate potential -chameleon mechanism- as by the kinetic term -Vainshtein mechanism. In addition, scalar-tensor theories are well known and well understood, from Brans-Dicke theory to Horndeski gravity there are a wide range of scalar field models that have been widely analysed and used not only to provide a natural explanation for dark energy but also to get a better understanding of GR itself [3]. Generalisations of standard scalar-tensor theories have been widely studied lately, mainly in the context of cosmology, as the so-called K-essence which presents a non-canonical kinetic term and provides a natural explanation for dark energy [4], or the so-called Galileons, that incorporates a galilean-like symmetry and which can also reproduces in a simple way the late-time acceleration [5]. This type of models have in common that may avoid the so-called Ostrogradsky instability which arises in higher order theories, while is absent in second order theories as the ones cited above. This class of scalar-tensor models are encompassed in the so-called Horndeski gravity [6], which represents the most general theory with second order field equations (for a review see [7]). Horndeski gravity is shown to be a generalisation of Galileon in its covariant form [8], which is also connected to k-essence fields [9]. Nevertheless, there have been some healthy extensions of Horndeski gravity also implying second order derivatives for the field equations [10]. In general, Horndeski gravity is well understood in many contexts, inflationary models have been widely analysed as well as the growth of cosmological perturbations [11], also consequently dark energy models can be easily implemented in Horndeski gravity [12], whose predictions and restrictions are analysed [13–16]. Also in light of the era of gravitational waves [17], Horndeski gravity is shown to carry just an additional -scalar- mode [18], but the theory is well constrained by the speed of propagation of the graviton [19, 20], which implies sever restrictions on the full Lagrangian [21].

*Electronic address: iayuso@fc.ul.pt
†Electronic address: diego.saez@uva.es

Also static spherically symmetric solutions, as black holes, have been widely studied in the literature within theories beyond GR [22], as may provide a way to regularise such type of solutions [23], a better understanding of Birkhoff's theorem [24], or directly new ways for testing General Relativity [25]. In Horndeski gravity, there have been plenty of works where such type of solutions are studied, mainly when dealing with compact objects as black holes [26], but also when assuming the constraints imposed on the full Horndeski Lagrangian by the speed of propagation of gravitational waves [27], and the stability of such type of spacetimes [28]. Here we are interested in analysing a particular class of static spherically symmetric solutions, the Schwarzschild-de Sitter spacetime, and particularly its extremal case, the so-called Nariai spacetime [30]. Schwarzschild-(Anti) De Sitter spacetime arise naturally in GR as a solution when considering a (negative) cosmological constant. In particular, Schwarzschild-Anti-De Sitter black holes have been of great interest as they show a thermodynamical equilibrium when analysing Hawking radiation [29]. In the case of a positive cosmological constant, the Schwarzschild-de Sitter spacetime shows in general two horizons, one corresponding to the black hole event horizon and the other one to a cosmological horizon. The extreme case arises when both horizons coincide at the same hypersurface, leading to an interesting structure for the spacetime and the trajectories of geodesics [31] as well as for its spectrum [32]. In addition, the stability of such extreme spacetime has been studied [33], and when some corrections in terms of scalar fields are included, an interesting phenomena occurs, the radius of the horizon becomes instable and grows, what has been called black hole anti-evaporation [34]. Such behaviour has also been observed in the presence of conformal scalars [35], $F(R)$ gravity [36] and Gauss-Bonnet gravities [37].

The aim of the present paper is to analyse Nariai spacetime in Horndeski gravity and the emergence of the anti-evaporation regime. To do so, we study the existence of Schwarzschild-de Sitter solutions in its extremal version for the Lagrangians that compose Horndeski gravity, which also show some implications on an extended version of Birkhoff's theorem for scalar-tensor theories. Finally, we analyse the stability of such solution for shorter version of the full Horndeski Lagrangian, motivated by keeping the less free functions as possible and which coincides with the viable terms restricted by the speed of GW's.

The paper is organised as follows: In section II a brief introduction to Horndeski gravity and Nariai metric is provided. Section III is devoted to the analysis of the viable Lagrangians that contain Nariai metric as a solution. In section IV, the anti-evaporation regime is analysed. Finally, section V gathers the conclusions.

II. NARIAI SPACETIME IN HORNDESKI GRAVITY

Let us start by writing the general action that we are dealing with along this manuscript. This is the Hilbert-Einstein action plus the so-called Horndeski Lagrangian:

$$S_G = \int dx^4 \sqrt{-g} \left[\frac{R}{16\pi G} + \mathcal{L}_{\rm Hr} + \mathcal{L}_m \right] \ , \tag{1}$$

where $\mathcal{L}_m$ is the matter Lagrangian which encompasses all the matter species of the system under study while the Horndeski Lagrangian $\mathcal{L}_{\rm Hr}$ is given by:

$$\begin{aligned} \mathcal{L}_{\rm Hr} \ = \ & G_2(\phi,X) - G_3(\phi,X)\Box\phi + G_4(\phi,X)R + G_{4X}(\phi,X)\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right] + G_5(\phi,X)\phi_{;\mu\nu}G^{\mu\nu} \\ & - \frac{G_{5X}(\phi,X)}{6}\left[(\Box\phi)^3 - 3\Box\phi\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\nu\lambda}\phi^{;\mu}_{;\lambda}\right] \ . \end{aligned} \tag{2}$$

Here ϕ is an scalar field, $G_{\mu\nu}$ is the Einstein tensor, $_{;\mu} = \nabla_\mu$ is the covariant derivative, $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is the kinetic term, $G_i(\phi,X)$ are arbitrary functions of ϕ and X, and $_X$ is the derivative with respect to X. As it is well known, the Lagrangian (2) represents the most general scalar-tensor Lagrangian that leads to second order field equations despite it depends on second derivatives of the field ϕ at the level of the action as well as on non-minimally couplings terms to the Ricci scalar. As shown in Ref. [9], this is just the generalization of the so-called covariant Galileon field, whose covariant version loses the Galilean shift symmetry that provides its name [8]. Hence, by varying the action (1) with respect to the metric $g_{\mu\nu}$ and with respect to the scalar field ϕ, the corresponding field equations can be obtained and we can analyse how some particular spacetimes behave within this class of theories.

Along this paper, we are interested in studying the Nariai spacetime, which is the extremal case of the Schwarzschild-de Sitter black hole, as is shown below. The general Schwarzschild-de Sitter metric can be expressed in spherical coordinates as follows:

$$ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2 d\Omega_2^2 \ . \tag{3}$$

where $d\Omega_2^2$ is the metric of a 2D sphere and,

$$A(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 \ . \tag{4}$$

Here $\Lambda > 0$ and $M > 0$. If $0 < M^2 < \frac{1}{9\Lambda}$, the function $A(R)$ has two positive roots r_{BH} and r_c, which corresponds to the black hole event horizon and to the cosmological horizon respectively. The global structure of this spacetime has been widely analysed in the literature [29]. The crucial point here is that whenever $M \to \frac{1}{3\sqrt{\Lambda}}$, the size of the black hole event horizon r_{BH} increases and approaches the cosmological horizon r_c at $r = 3M$, such that the function (4) tends to:

$$A(r) = -\frac{(r-3M)^2(r+6M)}{27M^2 r} \ . \tag{5}$$

This is the extremal case of the Schwarzschild-de Sitter black hole, which is known as the Nariai spacetime [30]. As shown in (5), it leads to a degenerate horizon that corresponds to the black hole one and to the cosmological one simultaneously. The causal structure of this particular case is well understood and the geodesics in such spacetime are well described in Ref. [31]. Note that $A(r) \leq 0$, such that the radial coordinate becomes timelike and the time coordinate spacelike everywhere. Our aim here is to analyse the metric (3) for the extremal case in the framework of the Horndeski Lagrangian, and analyse the stability of such solution. For that purpose, let us express the metric (3) with some more appropriate coordinates, but firstly we express the extremal case as a limit in terms of a parameter $0 < \epsilon << 1$, [33]:

$$9M^2\Lambda = 1 - 3\epsilon \ . \tag{6}$$

As $\epsilon \to 0$, both horizons approach each other. Then, we can choose the following coordinates [34]:

$$t = \frac{1}{\epsilon\sqrt{\Lambda}}\psi \ , \qquad r = \frac{1}{\sqrt{\Lambda}}\left(1 - \epsilon\cos\chi - \frac{1}{6}\epsilon^2\right) \ . \tag{7}$$

In these new coordinates, the metric (3) becomes:

$$ds^2 = -\frac{1}{\Lambda}\left(1 + \frac{2}{3}\epsilon\cos\chi\right)\sin^2\chi d\psi^2 + \frac{1}{\Lambda}\left(1 - \frac{2}{3}\epsilon\cos\chi\right)d\chi^2 + \frac{1}{\Lambda}\left(1 - 2\epsilon\cos\chi\right)d\Omega_2^2 \ . \tag{8}$$

Here the black hole horizon is given by $\chi = 0$ whereas the cosmological one corresponds to $\chi = \pi$. The spatial topology is clearly $S_1 \times S_2$. By setting $\epsilon \to$, the extremal case is obtained and the metric yields (8):

$$ds^2 = \frac{1}{\Lambda}\left(-\sin^2\chi d\psi^2 + d\chi^2\right) + \frac{1}{\Lambda}d\Omega_2^2 \ . \tag{9}$$

Finally, we can implement another change of coordinates that simplifies the expression (9), which is described by the following coordinates:

$$x = \mathrm{Log}\left(\tan\frac{\chi}{2}\right) \ , \qquad \tau = \frac{\psi}{4} \ . \tag{10}$$

And the metric (9) for the Nariai spacetime becomes:

$$ds^2 = \frac{1}{\Lambda\cosh^2 x}\left(-d\tau^2 + dx^2\right) + \frac{1}{\Lambda}d\Omega_2^2 \ . \tag{11}$$

The new coordinates are defined in the domain $(-\infty, \infty)$, as can be easily shown by (10).

III. RECONSTRUCTING THE GRAVITATIONAL ACTION IN HORNDESKI GRAVITY

In this section, we analyse the particular Lagrangians within Horndeski gravity that reproduces Nariai solution. To do so, we use the metric as expressed in the coordinates given in (11). As shown, Nariai spacetime can be a solution for each of the Horndeski Lagrangians as far as some constraints are assumed on the $\mathcal{L}_i$ functions.

A. Case with $\mathcal{L}_2$

As a first approximation to Horndeski gravity in Nariai space-time, we will start studying the simplest case in which only $\mathcal{L}_2$ for $\mathcal{L}_{Hr}$ is considered,

$$\mathcal{L}_2 = G_2(\phi, X) \ , \tag{12}$$

which essentially is the usual term for K-essence theory. The first step will be to solve, at the background level, the equations of motion given by the Einstein's tensor plus an effective energy-tensor coming from metric variations of the matter Lagrangian plus the Lagrangian defined in (12):

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left[g_{\mu\nu}G_2(\phi, X) + \frac{\partial G_2(\phi, X)}{\partial X}\partial_\mu\phi\partial_\nu\phi + T^{(m)}_{\mu\nu} \ ,\right] \ , \tag{13}$$

where $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor of the matter Lagrangian, and which, for the case of our interest, we are going to consider zero to focus on vacuum, i.e. $T^{(m)}_{\mu\nu} = 0$. Therefore, this tensor equation leads to the following system of equations:

$$tt- \qquad \frac{1}{8\pi G}\frac{1}{\cosh^2 x} = -\frac{G_2(\phi, X)}{\Lambda\cosh^2 x} + \frac{\partial G_2(\phi, X)}{\partial X}\dot{\phi}^2 \ , \tag{14}$$

$$xt- \qquad 0 = -\frac{\partial G_2(\phi, X)}{\partial X}\partial_t\phi\partial_x\phi \ , \tag{15}$$

$$xx- \qquad \frac{1}{8\pi G}\frac{-1}{\cosh^2 x} = \frac{G_2(\phi, X)}{\Lambda\cosh^2 x} + \frac{\partial G_2(\phi, X)}{\partial X}\phi'^2 \ , \tag{16}$$

$$\theta\theta- \qquad \frac{-1}{8\pi G} = \frac{G_2(\phi, X)}{\Lambda} + \frac{\partial G_2(\phi, X)}{\partial X}\partial_\theta\phi\partial_\theta\phi \ , \tag{17}$$

$$\Phi\Phi- \qquad \frac{-\sin^2\theta}{8\pi G} = \sin^2\theta\frac{G_2(\phi, X)}{\Lambda} + \frac{\partial G_2(\phi, X)}{\partial X}\partial_\Phi\phi\partial_\Phi\phi \ , \tag{18}$$

where dot means time derivatives and $'$ derivatives with respect to x. The two main issues that we intend to solve are the form of $G_2(\phi, X)$ and $\phi(t, x, \theta, \Phi)$. By combining (17) with (18), it yields:

$$\partial_\Phi\phi\partial_\Phi\phi = \sin^2\theta\partial_\theta\phi\partial_\theta\phi \quad \rightarrow \quad \partial_\Phi\phi = \pm\sin\theta\partial_\theta\phi \ , \tag{19}$$

whose solution is:

$$\phi = g(t, x)\left[\Phi \pm \ln\left(\cot\frac{\theta}{2}\right)\right] + f(t, x) \ . \tag{20}$$

However, from (14) and (16) it is possible to deduce that $g(t, x)$ should vanish in order to keep the same dependence parameters on the left and right hand side of the equations, and therefore $\phi = \phi(t, x)$, which implies that $X = \Lambda\cosh^2(x)(\dot{\phi}^2 - \phi'^2)/2$, this is the formal way for showing that the scalar field has to be spherically symmetric as the metric is. In addition, for solving $G_2(\phi, X)$, we can use the trace equation of (13) where the scalar curvature for the Nariai metric is $R = 4\Lambda$ and therefore:

$$-\frac{\Lambda}{4\pi G} = 2G_2(\phi, X) - \frac{\partial G_2(\phi, X)}{\partial X}X \ , \tag{21}$$

whose solution is:

$$G_2(\phi, X) = -\frac{\Lambda}{8\pi G} + f(\phi)X^2 \ . \tag{22}$$

However, by the equation (15), the following condition is obtained:

$$2Xf(\phi)\dot{\phi}\phi' = 0 \ . \tag{23}$$

It is straightforward to show that by combining (23) with $xx-$ and $tt-$ equations, $\phi' = \dot{\phi} = 0$, such that $\phi = constant$. Hence, the solution of the background leads to the following constraint on the action:

$$G_2(\phi_0, 0) = -\frac{\Lambda}{8\pi G} \tag{24}$$

This solution mimics the one from General Relativity with a cosmological constant, but in this case induced by a constant scalar field ϕ. Such result satisfies the Birkhoff's theorem for scalar-tensor theories [24].

B. Case $\mathcal{L}_3$

For the case $\mathcal{L}_3$, the general gravitational action is given by

$$S_G = \int dx^4 \sqrt{-g} \left[\frac{R}{16\pi G} - G_3(\phi, X)\Box\phi \right] \ . \tag{25}$$

By varying the action (25) with respect to the metric $g_{\mu\nu}$, the corresponding field equations are obtained:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \ &= \ 8\pi G \left[G_{3\phi} \left(g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi - 2\nabla_\mu \phi \nabla_\nu \phi \right) \right. \\ &\left. + G_{3X} \left(-\nabla_\mu \phi \nabla_\nu \phi \Box \phi - g_{\mu\nu} \nabla_\alpha \phi \nabla_\beta \phi \nabla^\alpha \phi \nabla^\beta \phi + 2\nabla^\alpha \phi \nabla_{(\mu} \phi \nabla_{\nu)} \nabla^\alpha \phi \right) \right] \ . \end{aligned} \tag{26}$$

Here the subscript $_{()}$ refers to an anticonmutator among the indexes, while $_\phi$ and $_X$ are derivatives with respect to the scalar field ϕ and its kinetic term X respectively. The equation for the scalar field is obtained by varying the action (25) with respect to the scalar field:

$$\begin{aligned} &2G_{3\phi}\Box\phi + G_{3\phi\phi}(\nabla\phi)^2 + G_{3X\phi}\left[(\nabla\phi)^2\Box\phi + 2\nabla_\mu\phi\nabla^\mu X \right] + G_{3X}\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi - R_{\mu\nu}\nabla^\mu\nabla^\nu\phi \right] \\ &+ G_{3XX}\left[\nabla_\mu\phi\nabla^\mu X + (\nabla X)^2 \right] = 0 \ , \end{aligned} \tag{27}$$

where recall that $X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$ is the kinetic term of the scalar field. As in the previous Lagrangian, which can be also followed by the extended version of Birkhoff's theorem for scalar-tensor theories, a non-constant stationary scalar field, $\phi = \phi(x)$, has to be assumed. In order to show that the Nariai metric, expressed in the coordinates as in (11), may be a solution for the gravitational action (25), we use the $tt-$ and $xx-$ equations, which can be easily obtained from the field equations (26) and yield:

$$\begin{aligned} tt- \qquad \frac{1}{\cosh^2 x} &= 8\pi G \phi'^2 \left[-G_{3\phi} + G_{3X}\Lambda^2 \left(\sinh x \cosh x \phi' + \cosh^2 x \phi'' \right) \right] \ , \\ xx- \quad -\frac{1}{\cosh^2 x} &= 8\pi G \phi'^2 \left[-G_{3\phi} + G_{3X}\Lambda^2 \cosh x \sinh x \phi' \right] \ . \end{aligned} \tag{28}$$

The $\theta\theta-$ and $\varphi\varphi-$ equations are just redundant, since reproduce the $tt-$ equation up to proportional terms. In general, the system of equations (28) do not provide a solution $\phi(x)$ for an arbitrary $G_3(\phi, X)$, such that in general Nariai spacetime is not a solution for the gravitational Lagrangian (25). Nevertheless, equations (28) can be used as constraint equations to reconstruct the appropriate action that reproduces the Nariai spacetime (11). As the corresponding partial derivatives $G_{3\phi}$ and G_{3X} are at the end functions of the coordinate x, we can express both of them in terms of the scalar field and its derivatives through the equations (28), which lead to:

$$\begin{aligned} G_{3\phi}(x) \ &= \ \frac{1}{8\pi G} \frac{2\tanh x \ \phi' + \phi''}{\phi'^2 \phi'' \cosh^2 x} \ , \\ G_{3X}(x) \ &= \ \frac{1}{4\pi G\Lambda} \frac{1}{\phi'^2 \phi'' \cosh^4 x} \ . \end{aligned} \tag{29}$$

Hence, for a particular solution for the scalar filed $\phi(x)$, the corresponding Lagrangian (25) can be reconstructed as far as the expressions (29) are well defined for such particular scalar field $\phi(x)$. In addition, we may specify one of the dependence of function $G_3(\phi, X)$, so that the system of equations (28) can be solved for the scalar field. For instance, we can assume $G_{3\phi}(x) = kG_{3X}(x)$ with k a proportional dimensional constant, such that both are the same function with respect to x, then the system of equations (28) can be integrated, which yields:

$$\phi(x) = \phi_0 + \phi_0' \tanh x + \frac{2k}{\Lambda} \left[x \tanh x - \mathrm{Log}(\cosh x) \right] \ , \tag{30}$$

where ϕ_0 and ϕ_0' are integration constants that refer to the value of the scalar field and its derivative at $x = 0$. Then, for this particular choice on the partial derivatives of the function G_3, the corresponding solution for the scalar field can be found and the Nariai spacetime is reproduced under the gravitational action (25), what implies that not only an effective cosmological constant reproduces such a spacetime, as was found for $\mathcal{L}_2$, a result that can be easily extended to any case of Schwarzschild-AdS spacetime.

Moreover, the corresponding G_3 function can be reconstructed by assuming a preliminar solution for the scalar field trough the expressions (29). For instance we may assume the following dynamics for the scalar field:

$$\phi(x) = \phi_0 \mathrm{e}^{\mu x} \ . \tag{31}$$

Then, by following the equations (28) and keeping in mind that $G_{3\phi}$ and G_{3X} are at the end functions of the coordinate x, the following particular solutions are found:

$$\begin{aligned} G_{3\phi}(x) &= \frac{1}{8\pi G}\frac{\mathrm{sech}^2 x(\mu + 2\tanh x)\mathrm{e}^{-2\mu x}}{\mu^3\phi_0^2} \ , \\ G_{3X}(x) &= \frac{1}{4\pi G}\frac{\mathrm{sech}^4 x \mathrm{e}^{-3\mu x}}{\mu^4\phi_0^3\Lambda} \ . \end{aligned} \tag{32}$$

The full reconstruction of the gravitational action (25) would imply to express (32) in terms of ϕ and the kinetic term $X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$, and then integrate, which also imply to make an assumption on the dependence of $G_3(\phi, X)$. Nevertheless, such exercise does not provide any new feature. The main conclusions can be obtained by analysing these two examples. As shown in the field equations, and by the expressions of $G_{3\phi}(x)$ and $G_{3X}(x)$, a constant scalar field $\phi(x) = \phi_0$ is not a solution for the equations (28), at least whenever the Lagrangian (25) is considered as the solely action for gravity. In addition, the freedom of the function $G_3(\phi, X)$ leads to an infinite number of solutions for the scalar field, as far as its partial derivatives (29) are well defined.

C. Case $\mathcal{L}_4$

Let us now analyse the solutions when the Lagrangian $\mathcal{L}_4$ in (2) is considered as the solely gravitational action:

$$S_G = \int dx^4\sqrt{-g}\left[\frac{R}{16\pi G} + G_4(\phi, X)R + G_{4X}\left((\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right)\right] \ . \tag{33}$$

As usual, by varying the action (25) with respect to the metric $g_{\mu\nu}$, the corresponding field equations are obtained:

$$\left(\frac{1}{16\pi G} + G_4\right)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \nabla_\mu\nabla_\nu G_4 + g_{\mu\nu}\Box G_4 - \frac{1}{2}g_{\mu\nu}G_{4X}\left((\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right) + \ldots \text{(second order terms)} \ . \tag{34}$$

We can proceed as in the previous Lagrangian. However, the degree of freedom on the function $G_4(\phi, X)$ will lead to a set of infinite solutions for the scalar field, as shown above for $\mathcal{L}_3$, what does not provide any new insights on Nariai spacetime in Horndeski gravity, but just some similar features as in the previous case, i.e. for a given solution $\phi(x)$, one can in general reconstruct the appropriate action through $G_4(\phi, X)$, while the other way around, that is, given an arbitrary $G_4(\phi, X)$ function, the field equations (34) does not have any solution for the scalar field in general, except for some special cases of the $G_4(\phi, X)$ function, as also shown for $G_3(\phi, X)$ above. Nevertheless, let us explore the case where $G_4(\phi, X) = G_4(\phi)$, as this case may play a relevant role while analysing the stability of the solution below. In such a case, the field equations (34) read:

$$\begin{aligned} tt- &\quad \left(\frac{1}{16\pi G} + G_4\right)\mathrm{sech}^2\, x - \phi'^2 G_{4\phi\phi} - (\tanh x\phi' + \phi'')G_{4\phi} = 0 \ , \\ xx- &\quad -\left(\frac{1}{16\pi G} + G_4\right)\mathrm{sech}^2\, x - \tanh x\phi' G_{4\phi} = 0 \ , \\ \theta\theta- &\quad -\left(\frac{1}{16\pi G} + G_4\right)\mathrm{sech}^2\, x - \phi'^2 G_{4\phi\phi} + \phi'' G_{4\phi} = 0 \ . \end{aligned} \tag{35}$$

By combining the $xx-$ and $-\theta\theta$ equations, it yields:

$$\tanh x\ G_{4\phi}\phi' = 0 \ \rightarrow \quad \phi = \text{constant} \ . \tag{36}$$

Hence, the only solution leads to a constant scalar field, which is not a solution for any choice of G_4 of equations (35). Then, for the particular case (33) with $G_4 = G_4(\phi)$, Nariai spacetime and consequently Schwarzschild-(A)dS can not be reproduced for such Lagrangian. This is a natural consequence as Schwarzschild-(A)dS spacetime requires the presence of an effective cosmological constant, which does not emerge in this particular case. However, such issue can be easily sorted out by adding an scalar potential in the action,

$$S_G = \int dx^4 \sqrt{-g} \left[\frac{R}{16\pi G} + G_4(\phi, X) R - V(\phi) \right] \ . \tag{37}$$

The equations do not differ much from the ones above, but just up to a potential term,

$$\begin{aligned}
tt- \quad & \left(\frac{1}{16\pi G} + G_4\right) \text{sech}^2\, x - \phi'^2 G_{4\phi\phi} - (\tanh x \phi' + \phi'') G_{4\phi} - \frac{\text{sech}^2\, x}{2\Lambda} V(\phi) = 0 \ , \\
xx- \quad & -\left(\frac{1}{16\pi G} + G_4\right) \text{sech}^2\, x - \tanh x \phi' G_{4\phi} + \frac{\text{sech}^2\, x}{2\Lambda} V(\phi) = 0 \ , \\
\theta\theta- \quad & -\left(\frac{1}{16\pi G} + G_4\right) \text{sech}^2\, x - \phi'^2 G_{4\phi\phi} + \phi'' G_{4\phi} + \frac{\text{sech}^2\, x}{2\Lambda} V(\phi) = 0 \ .
\end{aligned} \tag{38}$$

As in the previous case, by combining the $xx-$ and $-\theta\theta$ equations, the constraint equation (36) is obtained, what leads to a constant scalar field $\phi(x) = \phi_0$, and by replacing in the equations (38), it leads to:

$$-G_4(\phi_0) + \frac{V(\phi_0)}{2\Lambda} = \frac{1}{16\pi G} \ . \tag{39}$$

Hence, Nariai spacetime is a solution for the gravitational action (37) as far as the algebraic equation (39) has a at least real solution.

Therefore, it is clear that Schwarzschild-(A)dS spacetime, and specifically Nariai spacetime is a solution for each of the Horndeski Lagrangians whereas some constraints are imposed on the Lagrangians $\mathcal{L}_i$. It is straightforward to show that Nariai metric is also a solution of the full Horndeski Lagrangian as the degrees of freedom added by each $\mathcal{L}_i$ provides a way of reconstruct the corresponding gravitational action, what will imply an infinite number of choices on the G_i functions and a degenerate solution for the scalar field, as has been shown for some of the Lagrangians above, and which will also affect the full gravitational action due to the freedom to choose the corresponding Lagrangians. In the next section, we analyse the stability of this extremal black holes for those cases that the Nariai metric imposes real constraints on the Lagrangians.

IV. ANTI-EVAPORATION REGIME IN HORNDESKI GRAVITY

In this section, we analyse the stability of Nariai spacetime when perturbations around the background solution are introduced. To do so, we focus on the first four terms of the Horndeski Lagrangian:

$$S_G = \int dx^4 \sqrt{-g} \left[\frac{R}{16\pi G} + G_2(\phi, X) - G_3(\phi, X) \Box\phi + G_4(\phi) R \right] \ . \tag{40}$$

Note that (40) is the most general Horndeski Lagrangian that keeps the speed of gravitational waves as the speed of light, which is given by [19]:

$$c_{GW} = \frac{G_4 - X(\ddot{\phi} G_{5X} + G_{5\phi})}{G_4 - 2X G_{4X} - X(H\dot{\phi} G_{5X} - G_{5\phi})} \ , \tag{41}$$

where H is the Hubble parameter. Hence, by assuming $G_4(\phi, X) = G_4(\phi)$ and $G_5 = 0$, the speed of propagation for GW's is kept as the speed of light $c_{GW} = 1$, satisfying the constraints obtained from the GW170817 detection [17]. As shown in the previous section, for a given solution $\phi(x)$ and the Nariai metric (11) one can reconstruct the corresponding Horndeski Lagrangian that reproduces such solution. Nevertheless, here we are assuming for simplicity while analysing the perturbations, a constant scalar field for the background $\phi(x,t) = \phi_0$, such that following the results from the above section, Nariai spacetime is a solution for the gravitational action (40) as far as the following constraint is satisfied:

$$\frac{G_{20}}{2\Lambda} + G_{40} = -\frac{1}{16\pi G} \ . \tag{42}$$

A useful way to define perturbations around the Nariai metric is:

$$ds^2 = e^{2\rho(x,t)}\left(-dt^2 + dx^2\right) + e^{-2\varphi(x,t)} d\Omega_2^2, \tag{43}$$

whose $\rho(x,t)$ and $\varphi(x,t)$ at the background level are: $\rho = -\ln\sqrt{\Lambda}\cosh x$ and $\varphi = \ln\sqrt{\Lambda}$. The perturbations on the metric and the scalar field (with spherical symmetry) can be expressed as follows:

$$\begin{aligned} \phi &\rightarrow \phi_0 + \delta\phi(t,x) \\ \rho &\rightarrow -\ln\sqrt{\Lambda}\cosh x + \delta\rho \\ \varphi &\rightarrow \ln\sqrt{\Lambda} + \delta\varphi \end{aligned} \tag{44}$$

While the field equations up to linear order lead to:

$$\left(\frac{1}{16\pi G} + G_4\right)\delta G_{\mu\nu} + G_{\mu\nu}G_{4\phi}\delta\phi - G_{4\phi}\nabla_\mu\nabla_\nu\delta\phi + g_{\mu\nu}G_{4\phi}\Box\delta\phi - \frac{1}{2}\left(G_{2\phi}g_{\mu\nu}\delta\phi + G_2\delta g_{\mu\nu}\right) = 0\ . \tag{45}$$

Note that the functions G_i and their derivatives are evaluated at $\phi = \phi_0$ and expanded up to first order in perturbations as follows:

$$\begin{aligned} G_2(\phi, X) &\rightarrow G_2(\phi_0, 0) + \left.\frac{\partial G_2(\phi,0)}{\partial\phi}\right|_{\phi_0}\delta\phi \\ G_4(\phi) &\rightarrow G_4(\phi_0) + \left.\frac{\partial G_4(\phi_0)}{\partial\phi}\right|_{\phi_0}\delta\phi \end{aligned} \tag{46}$$

In order to study the perturbations. The next step will be the introduction of these perturbations into the field equations, i.e. into the Einstein's tensor and the energy-momentum tensor, i.e. into $T_{\mu\nu}^{(k)}$:

$$\begin{aligned} -2G_{20}\text{sech}^2 x\delta\varphi + (G_{2\phi} + 2\Lambda G_{4\phi})\text{sech}^2 x\delta\phi - 2G_{20}(\tanh x\delta\varphi' + \delta\varphi'') + 2G_{4\phi}\Lambda(\tanh x\delta\phi' + \delta\phi'') &= 0\ , \\ -2G_{20}\text{sech}^2 x\delta\varphi + (G_{2\phi} + 2\Lambda G_{4\phi})\text{sech}^2 x\delta\phi + 2G_{20}\left(\tanh x\delta\varphi' + \delta\ddot{\varphi}\right) + 2G_{4\phi}\Lambda\left(\tanh x\delta\phi' + \delta\ddot{\phi}\right) &= 0\ , \\ G_{2\phi}\left(\tanh x\delta\dot{\varphi} + \delta\dot{\varphi}'\right) + G_{4\phi}\Lambda\left(\tanh x\delta\dot{\phi} + \delta\dot{\phi}'\right) &= 0 \end{aligned} \tag{47}$$

where $'$ denotes derivative with respect to the variable x and $\dot{}$ derivatives with respect to t. The $(tx)-$ equation can be rewritten as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[G_{2\phi}\left(\tanh x\delta\varphi + \delta\varphi'\right) + G_{4\phi}\Lambda\left(\tanh x\delta\phi + \delta\phi'\right)\right] = 0\ , \\ &\rightarrow\quad g(x,t)\tanh x + g'(x,t) = h(x)\ , \end{aligned} \tag{48}$$

where $h(x)$ is an integration function to be determined, while $g(x,t) = G20\delta\varphi + G_{40}\Lambda\delta\phi$, which integrating the equation (48) yields:

$$g(x,t) = G_{20}\delta\varphi + G_{40}\Lambda\delta\phi = f(t)\ \text{sech}x + \text{sech}x\int\cosh x\ h(x)dx\ . \tag{49}$$

Then, by combining the $tt-$ and $xx-$ equations, the functions $f(t)$ and $h(x)$ are determined,

$$\begin{aligned} &f(t) = C_1\text{e}^t + C_2 e^{-t}\ , \quad h(x) = C_3\tanh x + C_4\ , \\ \rightarrow\quad &g(x,t) = \left(C_1\text{e}^t + C_2 e^{-t}\right)\text{sech}x + C_3 + C_4\tanh x\ . \end{aligned} \tag{50}$$

Here, C_i's are integration constants. Then, the expression for the metric perturbation $\delta\varphi$ can be easily obtained:

$$\delta\varphi = \frac{C_1\text{e}^t + C_2 e^{-t}}{G_{20}}\text{sech}x + C_3\frac{G_{2\phi} + 2\Lambda G_{4\phi}}{G_{20}(G_{2\phi} + 4\Lambda G_{4\phi})} + C_4\tanh x \tag{51}$$

We can now calculate how horizon changes when considering the above perturbations on the metric. The horizon is a null hypersurface that can be defined as follows:

$$g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi = 0\ , \tag{52}$$

By introducing (44) and (51) in (52), the following relation is obtained:

$$C_1^2 \mathrm{e}^{4t} + C_2^2 - \left(C_4^2 + 2C_1 C_2 \cosh 2x\right) \mathrm{e}^{2t} + 2C_1 C_4 \mathrm{e}^{3t} \sinh x + 2C_2 C_4 \mathrm{e}^{t} \sinh x = 0 \tag{53}$$

Hence, the perturbation (51) on the metric at the horizon leads to:

$$\delta\varphi_h = \frac{1}{G_{20}} \left[C_3 \frac{G_{2\phi} + 2\Lambda G_{4\phi}}{G_{2\phi} + 4\Lambda G_{4\phi}} + \sqrt{4C_1 C_2 + C_4^2} \right] \tag{54}$$

Hence, the perturbation at the horizon remains constant. By the Nariai metric (43), one can identify the radius of the horizon when is perturbed as:

$$r_h = \frac{\mathrm{e}^{-\delta\varphi_h}}{\sqrt{\Lambda}} = \frac{\mathrm{e}^{-\frac{1}{G_{20}}\left[C_3 \frac{G_{2\phi}+2\Lambda G_{4\phi}}{G_{2\phi}+4\Lambda G_{4\phi}} + \sqrt{4C_1 C_2 + C_4^2}\right]}}{\sqrt{\Lambda}} \ . \tag{55}$$

Note that this expression is time independent, such that no anti-evaporation effect arises when considering the restricted Horndeski Lagrangian (40) in Nariai spacetime. The only effect is a shift of the horizon, which may increases or decreases depending on the values of the integration constants (initial conditions) and on the functions G_i and their derivatives evaluated at ϕ_0 (Horndeski Lagrangian). In addition, if we set the integration constants to zero $C_i = 0$, the radius turns out $r_h = 1/\sqrt{\Lambda}$, i.e. the radius for the horizon in the Nariai spacetime. Moreover, by calculating the perturbation on the scalar field $\delta\phi$ through (49), it yields:

$$\delta\phi(x,t) = \frac{2C_3}{G_{2\phi} + 4\Lambda G_{4\phi}} \ . \tag{56}$$

Hence, the scalar field perturbation does not propagate but just introduces a perturbation on the effective cosmological constant, what explains the absence of the anti-evaporation regime and the shift of the horizon radius when considering perturbations on Nariai spacetime in the framework of Horndeski gravity.

V. CONSLUSIONS

In the present paper we have analysed several aspects of Schwarzschild-de Sitter black holes, and particularly its extremal case when both horizons, the cosmological and the black hole ones coincide at the same hypersurface of the spacetime, the so-called Nariai metric. Focusing on the framework of Horndeski gravity, we have shown that the existence of such type solutions when Horndeski Lagrangians are considered can be easily achieved by the induction of an effective cosmological constant, which naturally arises when considering a constant scalar field for some of the Horndeski terms. In addition, we have found that not only a constant scalar field owns Nariai spacetime as a solution of the gravitational field equations but also non-constant scalar field can reproduce Schwarzschild-de Sitter extremal black holes when considering the appropriate functions on the gravitational Lagrangian. This result still satisfies the generalised Birkhoff's theorem for scalar-tensor theories [24], since the scalar field despite non-constant, is time-independent, which keeps the Nariai spacetime stationary.

By considering perturbations on the background scalar field, which is assumed constant, the induced perturbations on the metric turns out time dependent, which modifies the stationary regime of the metric, inducing an exponential expansion, a natural solution when considering an effective cosmological constant. Nevertheless, the linear regime just induces a slight modification on the horizon radius, keeping it constant. Contrary to other frameworks where perturbations on the Nariai spacetime have been considered [34–37] which reproduces the so-called anti-evaporation regime, where the radius of the horizon may grow with time, this effect seems to be absent for the type of Horndeski Lagrangian analysed here. One obviously expects to find a non-constant scalar field perturbation by going beyond of the linear regime, which will consequently induces the anti-evaporation regime. In addition, a non-constant scalar field for the background is also expected to produces such phenomena, as perturbations on its propagation will naturally induce effects on the horizon radius, making Nariai metric unstable.

Acknowledgments

DS-CG is funded by the University of Valladolid (Spain). This article is based upon work from CANTATA COST (European Cooperation in Science and Technology) action CA15117, EU Framework Programme Horizon 2020.

---

[1] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. **342**, 155 (2012) doi:10.1007/s10509-012-1181-8 [arXiv:1205.3421 [gr-qc]]; D. Huterer and D. L. Shafer, Rept. Prog. Phys. **81**, no. 1, 016901 (2018) doi:10.1088/1361-6633/aa997e [arXiv:1709.01091 [astro-ph.CO]]; J. Frieman, M. Turner and D. Huterer, Ann. Rev. Astron. Astrophys. **46**, 385 (2008) doi:10.1146/annurev.astro.46.060407.145243 [arXiv:0803.0982 [astro-ph]]; L. Heisenberg, Phys. Rept. **796**, 1-113 (2019) doi:10.1016/j.physrep.2018.11.006 [arXiv:1807.01725 [gr-qc]]; A. Joyce, B. Jain, J. Khoury and M. Trodden, Phys. Rept. **568**, 1-98 (2015) doi:10.1016/j.physrep.2014.12.002 [arXiv:1407.0059 [astro-ph.CO]].

[2] S. Nojiri, S. D. Odintsov and M. Sami, Phys. Rev. D **74**, 046004 (2006) doi:10.1103/PhysRevD.74.046004 [arXiv:hep-th/0605039 [hep-th]].

[3] E. Elizalde, S. Nojiri, S. D. Odintsov, D. Saez-Gomez and V. Faraoni, Phys. Rev. D **77**, 106005 (2008) doi:10.1103/PhysRevD.77.106005 [arXiv:0803.1311 [hep-th]].

[4] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000) doi:10.1103/PhysRevLett.85.4438 [astro-ph/0004134]; C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. D **63**, 103510 (2001) doi:10.1103/PhysRevD.63.103510 [arXiv:astro-ph/0006373 [astro-ph]].

[5] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D **79**, 064036 (2009) doi:10.1103/PhysRevD.79.064036 [arXiv:0811.2197 [hep-th]].

[6] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363-384 (1974) doi:10.1007/BF01807638

[7] T. Kobayashi, Rept. Prog. Phys. **82**, no.8, 086901 (2019) doi:10.1088/1361-6633/ab2429 [arXiv:1901.07183 [gr-qc]]; C. Deffayet and D. A. Steer, Class. Quant. Grav. **30**, 214006 (2013) doi:10.1088/0264-9381/30/21/214006 [arXiv:1307.2450 [hep-th]].

[8] C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D **79**, 084003 (2009) doi:10.1103/PhysRevD.79.084003 [arXiv:0901.1314 [hep-th]].

[9] C. Deffayet, X. Gao, D. A. Steer and G. Zahariade, Phys. Rev. D **84**, 064039 (2011) doi:10.1103/PhysRevD.84.064039 [arXiv:1103.3260 [hep-th]].

[10] M. Zumalacrregui and J. Garca-Bellido, Phys. Rev. D **89**, 064046 (2014) doi:10.1103/PhysRevD.89.064046 [arXiv:1308.4685 [gr-qc]]; J. Ben Achour, D. Langlois and K. Noui, Phys. Rev. D **93**, no.12, 124005 (2016) doi:10.1103/PhysRevD.93.124005 [arXiv:1602.08398 [gr-qc]]; J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, Phys. Rev. Lett. **114**, no.21, 211101 (2015) doi:10.1103/PhysRevLett.114.211101 [arXiv:1404.6495 [hep-th]];

[11] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Prog. Theor. Phys. **126**, 511-529 (2011) doi:10.1143/PTP.126.511 [arXiv:1105.5723 [hep-th]]; A. De Felice and S. Tsujikawa, Phys. Rev. D **84**, 083504 (2011) doi:10.1103/PhysRevD.84.083504 [arXiv:1107.3917 [gr-qc]]. X. Gao and D. A. Steer, JCAP **12**, 019 (2011) doi:10.1088/1475-7516/2011/12/019 [arXiv:1107.2642 [astro-ph.CO]]; A. De Felice, T. Kobayashi and S. Tsujikawa, Phys. Lett. B **706**, 123 (2011) doi:10.1016/j.physletb.2011.11.028 [arXiv:1108.4242 [gr-qc]]; X. Gao, T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. **107**, 211301 (2011) doi:10.1103/PhysRevLett.107.211301 [arXiv:1108.3513 [astro-ph.CO]].

[12] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, Phys. Rev. Lett. **108**, 051101 (2012) doi:10.1103/PhysRevLett.108.051101 [arXiv:1106.2000 [hep-th]]; E. J. Copeland, A. Padilla and P. M. Saffin, JCAP **1212**, 026 (2012) doi:10.1088/1475-7516/2012/12/026 [arXiv:1208.3373 [hep-th]].

[13] T. Kobayashi, Phys. Rev. D **94**, no.4, 043511 (2016) doi:10.1103/PhysRevD.94.043511 [arXiv:1606.05831 [hep-th]].

[14] L. Amendola, M. Kunz, M. Motta, I. D. Saltas and I. Sawicki, Phys. Rev. D **87**, no. 2, 023501 (2013) doi:10.1103/PhysRevD.87.023501 [arXiv:1210.0439 [astro-ph.CO]].

[15] M. Zumalacrregui, E. Bellini, I. Sawicki, J. Lesgourgues and P. G. Ferreira, JCAP **1708**, 019 (2017) doi:10.1088/1475-7516/2017/08/019 [arXiv:1605.06102 [astro-ph.CO]].

[16] J. M. Ezquiaga and M. Zumalacrregui, Phys. Rev. Lett. **119**, no. 25, 251304 (2017) doi:10.1103/PhysRevLett.119.251304 [arXiv:1710.05901 [astro-ph.CO]].

[17] B. Abbott *et al.* [LIGO Scientific and Virgo], Phys. Rev. Lett. **119**, no.16, 161101 (2017) doi:10.1103/PhysRevLett.119.161101 [arXiv:1710.05832 [gr-qc]]; B. Abbott *et al.* [LIGO Scientific, Virgo, Fermi-GBM and INTEGRAL], Astrophys. J. **848**, no.2, L13 (2017) doi:10.3847/2041-8213/aa920c [arXiv:1710.05834 [astro-ph.HE]].

[18] C. Deffayet, G. Esposito-Farese and D. A. Steer, Phys. Rev. D **92**, 084013 (2015) doi:10.1103/PhysRevD.92.084013 [arXiv:1506.01974 [gr-qc]].

[19] D. Bettoni, J. M. Ezquiaga, K. Hinterbichler and M. Zumalacrregui, Phys. Rev. D **95**, no.8, 084029 (2017) doi:10.1103/PhysRevD.95.084029 [arXiv:1608.01982 [gr-qc]].

[20] N. Bartolo, P. Karmakar, S. Matarrese and M. Scomparin, JCAP **05**, 048 (2018) doi:10.1088/1475-7516/2018/05/048 [arXiv:1712.04002 [gr-qc]].

[21] R. Kase and S. Tsujikawa, Int. J. Mod. Phys. D **28**, no.05, 1942005 (2019) doi:10.1142/S0218271819420057 [arXiv:1809.08735 [gr-qc]].

[22] A. de la Cruz-Dombriz, A. Dobado and A. Maroto, Phys. Rev. D **80**, 124011 (2009) doi:10.1103/PhysRevD.80.124011

[arXiv:0907.3872 [gr-qc]]; G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **84**, 124059 (2011) doi:10.1103/PhysRevD.84.124059 [arXiv:1110.0850 [gr-qc]]; G. J. Olmo, D. Rubiera-Garcia and H. Sanchis-Alepuz, Eur. Phys. J. C **74** (2014), 2804 doi:10.1140/epjc/s10052-014-2804-8 [arXiv:1311.0815 [hep-th]]. A. de la Cruz-Dombriz and D. Saez-Gomez, Entropy **14**, 1717-1770 (2012) doi:10.3390/e14091717 [arXiv:1207.2663 [gr-qc]]; T. P. Sotiriou and S. Zhou, Phys. Rev. Lett. **112**, 251102 (2014) doi:10.1103/PhysRevLett.112.251102 [arXiv:1312.3622 [gr-qc]]; T. Clifton, Class. Quant. Grav. **23**, 7445 (2006) doi:10.1088/0264-9381/23/24/015 [arXiv:gr-qc/0607096 [gr-qc]]; N. Yunes and C. F. Sopuerta, Phys. Rev. D **77**, 064007 (2008) doi:10.1103/PhysRevD.77.064007 [arXiv:0712.1028 [gr-qc]]; V. Cardoso and L. Gualtieri, Phys. Rev. D **80**, 064008 (2009) doi:10.1103/PhysRevD.81.089903 [arXiv:0907.5008 [gr-qc]].

[23] G. J. Olmo, D. Rubiera-Garcia and A. Sanchez-Puente, Phys. Rev. D **92** (2015) no.4, 044047 doi:10.1103/PhysRevD.92.044047 [arXiv:1508.03272 [hep-th]]. G. J. Olmo, D. Rubiera-Garcia and A. Sanchez-Puente, Eur. Phys. J. C **76** (2016) no.3, 143 doi:10.1140/epjc/s10052-016-3999-7 [arXiv:1504.07015 [hep-th]]; C. Bejarano, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D **95** (2017) no.6, 064043 doi:10.1103/PhysRevD.95.064043 [arXiv:1702.01292 [hep-th]]; S. Nojiri and S. Odintsov, Phys. Rev. D **96**, no.10, 104008 (2017) doi:10.1103/PhysRevD.96.104008 [arXiv:1708.05226 [hep-th]].

[24] V. Faraoni, Phys. Rev. D **81**, 044002 (2010) doi:10.1103/PhysRevD.81.044002 [arXiv:1001.2287 [gr-qc]]; K. Schleich and D. M. Witt, J. Math. Phys. **51**, 112502 (2010) doi:10.1063/1.3503447 [arXiv:0908.4110 [gr-qc]]; S. Capozziello and D. Saez-Gomez, Annalen Phys. **524**, 279-285 (2012) doi:10.1002/andp.201100244 [arXiv:1107.0948 [gr-qc]].

[25] J. Moffat, Eur. Phys. J. C **75**, no.3, 130 (2015) doi:10.1140/epjc/s10052-015-3352-6 [arXiv:1502.01677 [gr-qc]]; A. Davis, R. Gregory, R. Jha and J. Muir, JCAP **08**, 033 (2014) doi:10.1088/1475-7516/2014/08/033 [arXiv:1402.4737 [astro-ph.CO]]; M. Guo, N. A. Obers and H. Yan, Phys. Rev. D **98**, no.8, 084063 (2018) doi:10.1103/PhysRevD.98.084063 [arXiv:1806.05249 [gr-qc]].

[26] E. Babichev, C. Charmousis and A. Lehbel, JCAP **04**, 027 (2017) doi:10.1088/1475-7516/2017/04/027 [arXiv:1702.01938 [gr-qc]]; E. Babichev, C. Charmousis, A. Lehbel and T. Moskalets, JCAP **09**, 011 (2016) doi:10.1088/1475-7516/2016/09/011 [arXiv:1605.07438 [gr-qc]]; E. Babichev, C. Charmousis and A. Lehbel, Class. Quant. Grav. **33**, no.15, 154002 (2016) doi:10.1088/0264-9381/33/15/154002 [arXiv:1604.06402 [gr-qc]]; O. J. Tattersall and P. G. Ferreira, Phys. Rev. D **97**, no.10, 104047 (2018) doi:10.1103/PhysRevD.97.104047 [arXiv:1804.08950 [gr-qc]].

[27] O. J. Tattersall, P. G. Ferreira and M. Lagos, Phys. Rev. D **97**, no.8, 084005 (2018) doi:10.1103/PhysRevD.97.084005 [arXiv:1802.08606 [gr-qc]].

[28] E. Babichev, C. Charmousis, G. Esposito-Farse and A. Lehbel, Phys. Rev. Lett. **120**, no.24, 241101 (2018) doi:10.1103/PhysRevLett.120.241101 [arXiv:1712.04398 [gr-qc]]. A. Ganguly, R. Gannouji, M. Gonzalez-Espinoza and C. Pizarro-Moya, Class. Quant. Grav. **35**, no.14, 145008 (2018) doi:10.1088/1361-6382/aac8a0 [arXiv:1710.07669 [gr-qc]]. J. Sakstein, E. Babichev, K. Koyama, D. Langlois and R. Saito, Phys. Rev. D **95**, no.6, 064013 (2017) doi:10.1103/PhysRevD.95.064013 [arXiv:1612.04263 [gr-qc]]. H. Ogawa, T. Kobayashi and T. Suyama, Phys. Rev. D **93**, no.6, 064078 (2016) doi:10.1103/PhysRevD.93.064078 [arXiv:1510.07400 [gr-qc]].

[29] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977); Kayll Lake and R. C. Roeder, Phys. Rev. D **15**, 3513 (1977). S. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983) doi:10.1007/BF01208266

[30] H. Nariai, Sci. Rep. Tohoku Univ. Ser. I **35** 62 (1951). [General Relativity and Gravitation **31**, 963971 (1999) https://doi.org/10.1023/A:1026602724948]

[31] J. Podolsky, Gen. Rel. Grav. **31**, 1703 (1999) doi:10.1023/A:1026762116655 [gr-qc/9910029].

[32] A. Maassen van den Brink, Phys. Rev. D **68**, 047501 (2003) doi:10.1103/PhysRevD.68.047501 [gr-qc/0304092].

[33] P. Ginsparg and M.J. Perry, Nucl. Phys. B **222**, 245 (1983).

[34] R. Bousso and S. W. Hawking, Phys. Rev. D **57**, 2436 (1998) doi:10.1103/PhysRevD.57.2436 [hep-th/9709224].

[35] S. Nojiri and S. D. Odintsov, Int. J. Mod. Phys. A **14**, 1293 (1999) doi:10.1142/S0217751X9900066X [hep-th/9802160].

[36] S. Nojiri and S. D. Odintsov, Class. Quant. Grav. **30**, 125003 (2013) doi:10.1088/0264-9381/30/12/125003 [arXiv:1301.2775 [hep-th]].

[37] L. Sebastiani, D. Momeni, R. Myrzakulov and S. D. Odintsov, Phys. Rev. D **88**, no. 10, 104022 (2013) doi:10.1103/PhysRevD.88.104022 [arXiv:1305.4231 [gr-qc]].