

Gamow vectors formalism applied to the Loschmidt echo

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Abstract

Gamow vectors have been developed in order to give a mathematical description for quantum decay phenomena. Mainly, they have been applied to radioactive phenomena, scattering and to some decoherence models. They play a crucial role in the description of quantum irreversible processes, and in the formulation of time asymmetry in quantum mechanics as well. In this paper, we use this formalism to describe a well-known phenomenon of irreversibility called *Loschmidt echo*. The standard approach considers that the irreversibility of this phenomenon is the result of an additional term in the backward Hamiltonian. Here, we use the non-hermitian formalism, where the time evolution is non-unitary. Additionally, we compare the characteristic decay times of this phenomenon with the decoherence ones. We conclude that the Loschmidt echo and the decoherence, are two aspects of the same phenomenon, and that there is a mathematical relationship between their corresponding characteristic times.

1 Introduction

Gamow vectors have been introduced in the context of quantum unstable states, also called quantum resonances. Initially, they were used in nuclear physics to describe radioactive decay [1]. As is well known, experiments have shown that quantum resonances decay exponentially – at least for most observable times. Nevertheless, there are deviations of the exponential decay law [2] for very short (Zeno effect [3]) as well as for very long times (Khalfin effect [4]), which are difficult to be experimentally observed [5, 6]. Noise effects may also contribute to slight deviations of the exponential law for intermediate times [2]. In any case, this exponential law gives a good approximation, so that resonance states can be very well approximated by vector states (or wave functions) decaying exponentially with time. These are the Gamow vectors or Gamow functions. This means that we have either to generalize time evolution so as to allow for non-unitary time evolutions, or to extend the Hilbert space to a larger space containing the Gamow vectors.

The exponential decay of Gamow vectors is accomplished if they are defined as eigenvectors (with complex eigenvalues) of a total Hamiltonian $H = H_0 + V$, where V is a potential

responsible for the decay [7]. Since Hamiltonians are self-adjoint, this is not possible on the context of quantum mechanics on Hilbert space [8, 9, 10]. One consistent solution is to extend the Hilbert space to a rigged Hilbert space, also called *Gelfand triplet* [11]. It is built by adding two more spaces to the Hilbert space \mathcal{H} , so that they form a triplet, $\Phi \subset \mathcal{H} \subset \Phi^\times$, where Φ is a locally convex space, and Φ^\times is its dual [12, 13, 14, 15, 16, 17, 18]. Then, Gamow vectors are well defined in the larger space Φ^\times . On Φ^\times , time evolution can be defined as a non-unitary extension of the unitary time evolution on the Hilbert space \mathcal{H} [19, 38].

There are several applications where the Gamow vectors (or Gamow states) play an essential role. The Brussels school used them to develop an irreversible version of quantum mechanics proposed by Prigogine [21, 22, 23, 24, 25, 26, 27]. One of its obvious derivations is a formalism for time asymmetry in quantum mechanics [28, 29, 30], along with the investigation on a microphysical arrow of time [32, 33, 34, 35]. Other applications of Gamow vectors lie on the fields of quantum decoherence for closed systems [36, 37, 38], or even in non-linear optics [39, 40, 41]. All these applications have in common the fact that, in the temporal evolution of certain magnitudes, a decreasing exponential appears that cannot be described by a unitary time evolution.

Recently, we have developed a formalism that relies on the Gamow vectors to describe the quantum-to-classical transition from the point of view of the Heisenberg picture [42, 43, 44, 45]. Its formal properties were studied from the point of view of dynamical logics [44, 45], using an algebraic perspective [44].

The exponential decay also appears in the study of a completely different phenomenon, namely, the *Loschmidt echo*. A. Peres proposed it in 1984 as a measure of sensitivity and reversibility of quantum evolutions. The idea consists in considering an initial state $|\psi_0\rangle$ and evolve it during a time interval t under the action of a Hamiltonian H_1 , reaching the state $|\psi_t\rangle$. Then, apply a second Hamiltonian H_2 in such a way that it reverses the action of the first Hamiltonian during a time interval of the same duration, aiming to recover the initial state. H_1 is the Hamiltonian governing the forward evolution and H_2 is the Hamiltonian governing the backward evolution. Due to imperfections in the time-reversal or due to an intrinsic quantum irreversibility, the final state does not always coincide with the initial one. The Loschmidt echo measures this discrepancy and it is defined as

$$M(t) = |\langle \psi_0 | e^{-iH_2 t/\hbar} e^{-iH_1 t/\hbar} | \psi_0 \rangle|^2. \quad (1)$$

The time reversal operation has been widely studied from a theoretical point of view. However, the empirical study of the time reversal problem involves the search of physical operations for the implementation of the time inversion, which is a non trivial task. The development of new experimental techniques has allowed to perform precise experiments in this area. In some systems, it has been observed that the initial and the final states can differ to some degree [46]. To explain this discrepancy, two factors must be considered: the unitarity of the time evolution and the noise during the process.

When the evolution is unitary, the discrepancy can be attributed to an imperfection in the implementation of the time reversal operation. In this case, the Hamiltonian which is used to simulate the temporal reversion is not exactly the same as the theoretical one. This can be

attributed to the noise introduced by the environment, and it has been studied in a previous work [47].

In this paper, we propose an alternative description in order to explain the discrepancy between the initial and final states. We assume that the time inversion is perfect, but the time evolution is not unitary. To obtain a non-unitary evolution, we appeal to Gamow vectors, which are adequate for bringing into play exponential decays. Using this formalism, we show that the time evolution generated by Hamiltonians with complex eigenvalues is irreversible. Moreover, this approach allows us to account for the Loschmidt echo phenomenon.

2 Gamow vectors formalism

In this section, we introduce a model that will be useful to understand our description of the Loschmidt echo. This model is a variant of the Friedrichs one [52] with multiple resonances, and it is general enough to describe a wide range of realistic situations [53].

2.1 The Friedrichs model

As is well known, *scattering resonances* and *quasi-stationary states* are used to describe a scattering process in which a particle stays in the neighborhood of a center of forces –usually given by a potential– a time much longer than it would have remained if the center of forces were not present [8, 29]. Scattering resonances can be modeled using Hamiltonian pairs, $\{H_0, H\}$, where H_0 is the unperturbed Hamiltonian and $H = H_0 + V$, where V is the potential responsible for the creation of the quasi-stationary states. The simplest model for resonances is the Friedrichs model, in which H_0 is given by

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle \omega| d\omega. \quad (2)$$

Here, H_0 has a pure absolutely and non-degenerate continuous spectrum, which is $\mathbb{R}^+ \equiv [0, \infty)$ and an eigenvalue, $\omega_0 > 0$. Kets $|\omega\rangle$ are the eigenkets of H_0 with eigenvalues $\omega \in \mathbb{R}^+$, $H_0|\omega\rangle = \omega|\omega\rangle$. As in the case of plane waves, $|\omega\rangle$ are not normalizable states, which acquire meaning in suitable extensions of the Hilbert space called rigged Hilbert spaces or Gelfand triplets, for which there exists an extensive literature as mentioned in the Introduction. The eigenvector $|1\rangle$, $H|1\rangle = \omega_0|1\rangle$ belongs to Hilbert space and is normalized so that $\langle 1|1\rangle = 1$. Notice that the eigenvalue ω_0 is imbedded in the continuous spectrum of H_0 .

The potential intertwines discrete and continuous spectrum of H_0 and it usually has the form

$$V = \int_0^\infty f(\omega) [|\omega\rangle\langle 1| + |1\rangle\langle \omega|] d\omega, \quad (3)$$

where $f(\omega)$ is a real function, usually square integrable, called *form factor*. Since bound states and scattering states are orthogonal to each other, we have that $\langle \omega|1\rangle = \langle 1|\omega\rangle = 0$, for all $\omega \in \mathbb{R}^+$. In addition, $\langle \omega|\omega'\rangle = \delta(\omega - \omega')$ [53].

The total Hamiltonian has the form $H = H_0 + \lambda V$, where λ is a real coupling constant. The Hamiltonian H has a pure simple absolutely continuous spectrum, so that

$$H = \int_0^\infty \omega |\omega^\pm\rangle \langle \omega^\pm| d\omega, \quad (4)$$

where $H|\omega^\pm\rangle = \omega |\omega^\pm\rangle$ are eigenkets for $\omega \in \mathbb{R}^+$, the continuous spectrum of H . The signs \pm correspond to *in* and *out* states of scattering theory. More details can be found in [53].

This is the simplest possible Friedrichs model. The most straightforward generalization, is obtained by adding more bound states for H_0 , so that

$$H_0 = \sum_{k=1}^N \omega_k |k\rangle \langle k| + \int_0^\infty \omega |\omega\rangle \langle \omega| d\omega, \quad \omega_k > 0. \quad (5)$$

Other generalizations can be found in [53], although they do not play any role in the present discussion.

Thus, while H_0 has bound states, at least one, H possesses none. What has happened with the bound states of H_0 ? To see it, we need a mathematical definition for resonances. A precise definition based in the analytic properties of the resolvent $(H - zI)^{-1}$ of H , where z is a complex number, $z \in \mathbb{C}/\mathbb{R}^+$ and I the identity operator, was given in [54], page 55. In the case of the simplest Friedrichs model, in which H_0 has only one bound state, it is sufficient to consider the analytic properties of the *reduced resolvent function* given by

$$\eta(z) := \langle 1|(H - zI)^{-1}|1\rangle. \quad (6)$$

Under mild conditions on the form factor $f(\omega)$ [55], the function $\eta(z)$ admits an analytic continuation with a pair of poles at the points $z_0 = E_R - i\Gamma/2$ and $z_0^* = E_R + i\Gamma/2$, with $\Gamma > 0$. In addition, and following the formulation of resonances in rigged Hilbert spaces [19, 38], there are two non-normalizable vectors $|f^D\rangle$ and $|f^G\rangle$, such that

$$H|f^D\rangle = z_0 |f^D\rangle, \quad H|f^G\rangle = z_0^* |f^G\rangle. \quad (7)$$

For any $t > 0$ or $t < 0$, respectively, one has that

$$e^{-itH} |f^D\rangle = e^{-iE_R t} e^{-\Gamma t/2} |f^D\rangle, \quad e^{-itH} |f^G\rangle = e^{-iE_R t} e^{\Gamma t/2} |f^G\rangle, \quad (8)$$

which shows that $|f^D\rangle$ and $|f^G\rangle$ decay exponentially as $t \mapsto +\infty$ and $t \mapsto -\infty$, respectively. Thus, the superscripts D and G means decay and grow, respectively, for times going from $-\infty$ to ∞ .

Usually, $|f^D\rangle$ and $|f^G\rangle$ receive the names of *decaying* and *growing* Gamow vector, respectively. Under mild assumptions, z_0 is analytic in the coupling constant λ and has the property that $\lim_{\lambda \rightarrow 0} z_0 = \omega_0$ [55].

In correspondence with these two spectral decompositions for H given in (4), we have these other two:

$$H = z_0 |f^D\rangle\langle f^G| + \int_{\gamma^-} z |z^- \rangle\langle z^+| dz, \quad H = z_0^* |f^G\rangle\langle f^D| + \int_{\gamma^+} z |z^+ \rangle\langle z^-| dz. \quad (9)$$

The meaning of the first spectral decomposition (9) is the following: let us assume that the functions $\varphi(\omega) = \langle \omega | \varphi \rangle$ and $\phi(\omega) = \langle \omega | \phi \rangle$ are analytically continuable to the upper and lower half of the complex plane, respectively. This, in particular, implies that $\varphi^*(\omega) = \langle \varphi | \omega \rangle$ admits analytic continuation to the lower half plane. The value of these functions at the point z_0 in the lower half plane is given by $\varphi^*(z_0) = \langle \varphi | f^D \rangle$ and $\phi(z_0) = \langle f^G | \phi \rangle$. The values of $\varphi^*(\omega)$ and $\phi(\omega)$ at the complex number z with negative imaginary part are $\varphi^*(\omega) = \langle \varphi | z^- \rangle$ and $\phi(z) = \langle z^+ | \phi \rangle$, respectively. Thus, what really has proper meaning is the value of $\langle \varphi | H | \phi \rangle$. Finally, γ^- is a contour in the lower half plane starting at the origin and having the positive half of the real line as asymptotic. The meaning of the second spectral decomposition in (9) is similar in the upper half plane.

A Friedrichs model with free Hamiltonian as in (5) and potential given by

$$V := \sum_{k=1}^N f_k(\omega) [| \omega \rangle \langle k | + | k \rangle \langle \omega |] d\omega, \quad (10)$$

where the $f_k(\omega)$ are N form factor functions (the total Hamiltonian is $H = H_0 + \lambda V$), is expected to give N resonance poles (although for a given value of λ , two of these poles may coincide at the same point, giving a double resonance pole). Thus, for a given value of λ , we have resonance poles at the points z_1, z_2, \dots, z_N and their complex conjugates, with respective decaying and growing Gamow vectors, $|f_1^D\rangle, |f_2^D\rangle, \dots, |f_N^D\rangle$ and $|f_1^G\rangle, |f_2^G\rangle, \dots, |f_N^G\rangle$. Then, the spectral decompositions (9) generalize to

$$H = \sum_{k=1}^N z_k |f_k^D\rangle\langle f_k^G| + \int_{\gamma^-} z |z^- \rangle\langle z^+| dz, \quad H = \sum_{k=1}^N z_k^* |f_k^G\rangle\langle f_k^D| + \int_{\gamma^+} z |z^+ \rangle\langle z^-| dz. \quad (11)$$

The integral terms in (9) and (11) are responsible for the deviations of the exponential decay for very short (Zeno effect [3]) or very large values of time [4]. These effects are observable, although not easy to be observed [5, 6]. This means that, for the vast majority of experiments, they are not detected. Due to this reason, it makes sense to drop the integral term from (9) and (11), at least within a reasonable range of observations. Also, note that the spectral decompositions for H either in (9) or in (11) are time reversal of each other, so that they are completely equivalent [34]. Therefore, one may take as an effective Hamiltonian, valid for the majority of observations, the following:

$$H = \sum_{k=1}^N z_k |f_k^D\rangle\langle f_k^G|. \quad (12)$$

As in the case $N = 1$, we have that $H|f_k^D\rangle = z_k |f_k^D\rangle$, for $k = 1, 2, \dots, N$. Along (2), this suggests that

$$\langle f_s^G | f_k^D \rangle = \delta_{s,k}, \quad s, k = 1, 2, \dots, N. \quad (13)$$

Analogously, $H|f_k^G\rangle = z_k^* |f_k^G\rangle$ for all k , so that if we take as effective Hamiltonian the second sum in (11), we have that $\langle f_s^D | f_k^D \rangle = \delta_{s,k}$, for all values of s and k . Observe that this is also given by the pseudometrics proposed for the Gamow states in [43] (see also [56]). Another property derived from this pseudometrics is that $\langle f_k^D | f_s^D \rangle = 0 = \langle f_k^G | f_s^G \rangle$.

Resonances may be considered as independent of each other, so that the Gamow vectors $|f_1^D\rangle, |f_2^D\rangle, \dots, |f_N^D\rangle$ and $|f_1^G\rangle, |f_2^G\rangle, \dots, |f_N^G\rangle$ may be assumed to be linearly independent. Then, we may assume that linear combinations of resonance states may belong either to the N dimensional linear space, \mathcal{H}_N^D , spanned $|f_1^D\rangle, |f_2^D\rangle, \dots, |f_N^D\rangle$, or the space \mathcal{H}_N^G spanned by $|f_1^G\rangle, |f_2^G\rangle, \dots, |f_N^G\rangle$ or even to the $2N$ dimensional space, \mathcal{H}_{2N} , spanned by $|f_1^D\rangle, |f_2^D\rangle, \dots, |f_N^D\rangle$ and $|f_1^G\rangle, |f_2^G\rangle, \dots, |f_N^G\rangle$. Due to (13), the spaces \mathcal{H}_N^D and \mathcal{H}_N^G may be considered as duals of each other.

These ideas, along with (8), are the basis for the following comment. Assume that $|\psi\rangle$ and $|\varphi\rangle$ are arbitrary vectors in \mathcal{H}_N^D , so that

$$|\psi\rangle := \sum_{k=1}^N a_k |f_k^D\rangle, \quad |\varphi\rangle := \sum_{k=1}^N b_k |f_k^D\rangle. \quad (14)$$

Let us evolve $|\varphi\rangle$ in time, with the evolution governed by the total Hamiltonian H . The value of this vector after a time t has elapsed, is given by

$$|\varphi(t)\rangle = \sum_{k=1}^N b_k |f_k^D(t)\rangle, \quad (15)$$

where $|f_k^D(t)\rangle = e^{-itH} |f_k^D\rangle$ as in (8). Because of the duality between the spaces \mathcal{H}_N^D and \mathcal{H}_N^G , the bra corresponding to the ket $|\psi\rangle$ is

$$\langle\psi| = \sum_{k=1}^N a_k^* \langle f_k^G|, \quad (16)$$

where the star denotes complex conjugation. The probability amplitude of the state $|\varphi(t)\rangle$ be in the state $|\psi\rangle$ at time t is given by

$$A(t) = \langle\psi|\varphi(t)\rangle = \sum_{k,s=1}^N a_s^* b_k \langle f_s^G | f_k^D(t)\rangle = \sum_{k,s=1}^N a_s^* b_k A_{s,k}(t), \quad (17)$$

where,

$$A_{s,k}(t) = \langle f_s^G | e^{-itH} |f_k^D\rangle = e^{-iz_k t} \langle f_s^G | f_k^D\rangle = e^{-iz_k t} \delta_{s,k} = e^{-iE_k t} e^{-t\Gamma_k/2} \delta_{s,k}, \quad (18)$$

so that

$$A(t) = \sum_{k=1}^N a_k^* b_k e^{-iE_k t} e^{-t\Gamma_k/2}, \quad (19)$$

where E_k and $-\Gamma_k/2$ are the real and imaginary parts of the resonance pole z_k , respectively.

Instead, we may have considered the vectors $|\psi\rangle$ and $|\varphi\rangle$ to be in the space \mathcal{H}_{2N} , and follow a similar reasoning. Formally, the results would have been different, since to (19), we had to add a similar term. If this were the case,

$$|\psi\rangle := \sum_{k=1}^N a_k |f_k^D\rangle + \sum_{k=1}^N c_k |f_k^G\rangle, \quad |\varphi\rangle := \sum_{k=1}^N b_k |f_k^D\rangle + \sum_{k=1}^N d_k |f_k^G\rangle. \quad (20)$$

This construction, although useful for other purposes, may be incompatible with the formalism of Time Asymmetric Quantum Mechanics (TAQM) [28, 29, 30, 31], since in this formalism, time evolution for $|f_k^D\rangle$ is defined *only* for $t \geq 0$, while time evolution for $|f_k^G\rangle$ is *only* valid for $t \leq 0$. Nevertheless, a construction such that all growing and decaying Gamow vectors evolve rigorously for all values of time is possible [57], although this construction does not take into account the time asymmetry that is produced by decaying processes¹ Thus, we consider the discussion on \mathcal{H}_N^D as given above, as the most appropriate for our purposes.

2.2 Creation and annihilation operators

In this section we study the Friedrichs model in terms of the creation and annihilation operators. Its construction has been proposed, for instance, in [27]. The idea is the following: Let us assume that a and a^\dagger are the annihilation and creation of $|1\rangle$ in (2), respectively, and that b_ω and b_ω^\dagger play the same role in relation to $|\omega\rangle$. Then, the Friedrichs free Hamiltonian H_0 written in terms of these operators is a contribution of two terms $H_0 = H_A + H_R$, where [27],

$$H_A := \omega_0 a^\dagger a, \quad H_R := \int_0^\infty d\omega \omega b_\omega^\dagger b_\omega, \quad (21)$$

and the interaction is now written as

$$H_I = \lambda \int_0^\infty d\omega f(\omega) (a^\dagger b_\omega + a b_\omega^\dagger), \quad (22)$$

where λ is the coupling constant and $f(\omega)$ the form factor. Again, the total Hamiltonian is $H = H_0 + H_I = H_A + H_R + H_I$. The function

$$\eta(z) := \omega_0 - z - \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{\omega - z}, \quad (23)$$

is analytic with a branch cut on the positive semi-axis $[0, \infty)$. We denote its limits on this semi-axis from above to below and from below to above as $\eta^+(\omega) := \eta(\omega + i0)$ and $\eta^-(\omega) := \eta(\omega - i0)$, respectively. Then, a ‘‘diagonalization’’ of the total Hamiltonian H is given by [27]

¹It is also true that an algebraic formulation using operators on \mathcal{H}_{2N} is still compatible with TAQM, [43].

$$H = z_0 (A^\dagger)_{\text{in}} A_{\text{out}} + \int_0^\infty d\omega \omega \frac{\eta^+(\omega)}{\eta^-(\omega)} (\tilde{B}_\omega^\dagger)_{\text{in}} (\tilde{B}_\omega)_{\text{out}}. \quad (24)$$

Here, $(A^\dagger)_{\text{in}}$ and A_{out} are the creation and annihilation of the Gamow state from a vacuum $|0\rangle$, so that

$$|f^D\rangle = (A^\dagger)_{\text{in}} |0\rangle, \quad |0\rangle = A_{\text{out}} |f^D\rangle, \quad A_{\text{out}} |0\rangle = 0, \quad [A_{\text{out}}, A_{\text{in}}^\dagger] = 1, \quad (25)$$

while $(\tilde{B}_\omega^\dagger)_{\text{in}}$ and $(\tilde{B}_\omega)_{\text{out}}$ are creation and annihilation operators of states in the continuum, which were defined in [27]. The latter do not have here any interest for us, as we are solely interested in the resonance behavior. The subscripts “in” and “out” come from scattering theory and do not play any role in our discussion here, so that we shall omit them. Thus, the effective Hamiltonian, which concerns only to the presence of resonances is given by $H = z_0 A^\dagger A$ in our notation.

Now, rename $|f_1^D\rangle := |f^D\rangle$ and define $|f_n^D\rangle := (A^\dagger)^n |f_1^D\rangle$. If we denote by $N := A^\dagger A$, it readily comes that

$$N |f_n^D\rangle = n |f_n^D\rangle, \quad H |f_n^D\rangle = n z_0 |f_n^D\rangle, \quad (26)$$

which formally yields

$$e^{-itH} |f_n^D\rangle = e^{-itnz_0} |f_n^D\rangle. \quad (27)$$

In the next section, we are going to apply the formalism of Gamow vectors to the Loschmidt echo.

3 Loschmidt echo and Gamow vectors

We consider a quantum system in a given initial state $|\alpha(0)\rangle$. Let us suppose that this initial state is evolved during a time interval τ , and then, that it is evolved backwards during the same interval τ . We denote the final state by $|\alpha(2\tau)\rangle$. At this point, we can compare the initial state $|\alpha(0)\rangle$, with the final state $|\alpha(2\tau)\rangle$.

When the time-reversal procedure is perfect, and the time-evolution is unitary, both states are exactly the same. However, when the time-reversal procedure is imperfect, there will exist some degree of discrepancy between them. One way to quantify this discrepancy, is by using the Loschmidt echo. If the time-evolution of the quantum system is given by the operator $U(t)$, we define the decay rate by

$$M(\tau) := |\langle \alpha(0) | U^\dagger(\tau) U(\tau) | \alpha(0) \rangle|^2. \quad (28)$$

It is easy to see that $M(\tau) = 1$ when there is no discrepancy, and $M(\tau) = 0$ when the discrepancy is maximal.

$M(\tau)$ is maximal when the time evolution is unitary and the time reversal is perfect. In the usual approach, it is assumed that the time evolution is unitary and the time reversal imperfect (i.e., the backward Hamiltonian is not exactly opposite to the forward Hamiltonian).

In this work, we adopt an alternative approach. We will consider a perfect time reversal, but a non-unitary time evolution.

We cannot choose a formally unitary evolution $U(t)$ on the space of resonance states for various reasons, as has been discussed in [44]. In particular, if one has $U^\dagger(t)U(t) = I$, then $M(t)$ would be a constant (independent of time). On the space \mathcal{H}_{N+1}^D spanned by the resonance states $\{|f_n^D\rangle\}$, $n = 0, 1, 2, \dots, N$, the decaying Gamow vectors, we may define an evolution operator as

$$U(t) := \sum_{n=0}^N e^{-\frac{it}{\hbar} z_n} |f_n^D\rangle \langle f_n^G|. \quad (29)$$

Notice that, due to (29) and the discussion in the subsequent paragraph, the formal adjoint of $U(t)$ on the $N + 1$ dimensional space spanned by the Gamow vectors $\{|f_n^D\rangle\}$, cannot be

$$U^+(t) = \sum_{n=0}^N e^{\frac{it}{\hbar} z_n^*} |f_n^G\rangle \langle f_n^D|, \quad (30)$$

since the relations $\langle f_n^D | f_m^D \rangle = 0$, would imply that $U^+(t)$ is the zero operator on this space. Instead, we propose the following form for the adjoint of $U(t)$:

$$U^\dagger(t) = \sum_{n=0}^N e^{\frac{it}{\hbar} z_n^*} |f_n^D\rangle \langle f_n^G|. \quad (31)$$

Notice that, for all Gamow states $\{|f_n^D\rangle\}$, we have

$$U(t)|f_n^D\rangle = e^{-\frac{it}{\hbar} z_n} |f_n^D\rangle \quad \text{and} \quad U^\dagger(t)|f_n^D\rangle = e^{\frac{it}{\hbar} z_n^*} |f_n^D\rangle, \quad (32)$$

$U^\dagger(t)U(t)$ is well defined, and is different from the unity. Applying this time evolution in equation (28), we obtain

$$\begin{aligned} \langle \alpha(0) | U^\dagger(\tau) U(\tau) | \alpha(0) \rangle &= \langle \alpha(0) | \sum_{n=0}^N e^{\frac{i\tau}{\hbar} z_n^*} |f_n^D\rangle \langle f_n^G| \sum_{m=0}^N e^{-\frac{i\tau}{\hbar} z_m} |f_m^D\rangle \langle f_m^G| \alpha(0) \rangle = \\ &= \sum_{n=0}^N e^{\frac{i\tau}{\hbar} (z_n^* - z_n)} \langle \alpha(0) | f_n^D \rangle \langle f_n^G | \alpha(0) \rangle = \\ &= \sum_{n=0}^N e^{-\tau \gamma_n / \hbar} \langle \alpha(0) | f_n^D \rangle \langle f_n^G | \alpha(0) \rangle. \end{aligned} \quad (33)$$

Let us suppose that the initial state is a superposition of the Gamow vectors $|f_n^D\rangle$

$$|\alpha(0)\rangle = \sum_{n=0}^N a_n |f_n^D\rangle, \quad (34)$$

then

$$\langle \alpha(0) | = \sum_{n=0}^N a_n^* \langle f_n^G |. \quad (35)$$

Now, using the relations (13), obtain

$$\langle \alpha(0) | U^\dagger(\tau) U(\tau) | \alpha(0) \rangle = \sum_{n=0}^N e^{-\tau \gamma_n / \hbar} |a_n|^2. \quad (36)$$

This result shows that the Loschmidt echo decays with time, and its decay depends on the initial conditions a_n and the characteristic times γ_n^{-1} .

The decay of the Loschmidt echo has different regimes, such as parabolic, exponential, gaussian among others [58, 59, 60, 63, 64, 65]. The most simple case of equation (36) is when there is only one characteristic time, or when the initial conditions are such that only one mode is activated. In these cases, it is obtained a pure exponential decay. In this way, Gamow vectors can be used to model experimental situations. In the general case, the combination of exponential functions with different characteristic times, lead to more diverse time evolutions.

4 Decoherence and Loschmidt echo

So far we have seen how the Gamow vectors can be used to describe the Loschmidt echo. We saw that the type of decay of the quantity $M(\tau)$ is determined by the complex part of the Hamiltonian eigenstates and the initial conditions. In what follows we will show that there is a close relationship between the Loschmidt echo and the quantum decoherence phenomenon. This connection is because the decoherence time is also related with the imaginary part of the Hamiltonian eigenstates and the initial conditions.

To study the decoherence process in the Friedrich model, it is necessary to introduce the quasi coherent states, since they form the so-called privileged basis. Quasi-coherent states have been discussed in [48, 49, 50].

4.1 Initial conditions

For systems having an infinite number of resonances, it is possible in principle to define coherent resonance states. This states should have the following form for each complex number $z \in \mathbb{C}$:

$$|z^D\rangle = e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} |f_k^D\rangle. \quad (37)$$

These coherent states should be eigenvectors with eigenvalue z of certain annihilation operator B^- , satisfying $B^- |f_k^D\rangle = k |f_{k-1}^D\rangle$, for all $k = 1, 2, \dots$, where $|f_0^D\rangle = 0$, so that $B^- |z^D\rangle = z |z^D\rangle$. This can be implemented for some exactly solvable models for resonances, like the high barrier Pöschl-Teller one dimensional model [38, 66].

Unfortunately, up to our knowledge, nobody has constructed a solvable Friedrichs model with an infinite number of resonance poles. Let us assume that we have $N + 1$ resonances with

decaying Gamow vectors $|f_0^D\rangle, \dots, |f_N^D\rangle$. By definition a *quasi coherent state* at time $t = 0$, is a vector of the form:

$$|\alpha_1(0)\rangle := \left(\sum_{k=0}^N \frac{|\alpha_1(0)|^{2k}}{k!} \right)^{-1/2} \sum_{n=0}^N \frac{(\alpha_1(0))^n}{\sqrt{n!}} |f_n^D\rangle, \quad (38)$$

where $\alpha_1(0)$ is a given complex number. Note that for N going to infinity, (38) acquires the form (37), hence the name of quasi-coherent state. For any other complex number $\alpha_2(0)$, in an analogous way, we have

$$|\alpha_2(0)\rangle := \left(\sum_{k=0}^N \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1/2} \sum_{n=0}^N \frac{(\alpha_2(0))^n}{\sqrt{n!}} |f_n^D\rangle. \quad (39)$$

Notice that, in a rough – although efficient – approximation, we may consider the pair of vectors $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$ as a *Moving Preferred Basis* in the sense of [51]. The approximation is rough, because we have taken $t = 0$, so that it is not properly a moving basis. It is certainly efficient in the range of extremely short decoherence times t_D , as compared with the relaxation time t_R , $t_D \ll t_R$. In this sense, the choice $t_D = 0$ is a workable approximation. Let us consider an arbitrary normalized linear combination of these two vectors; $|\Phi(0)\rangle := a|\alpha_1(0)\rangle + b|\alpha_2(0)\rangle$. Its corresponding density matrix is given by

$$\begin{aligned} \rho_0 &= |\Phi(0)\rangle\langle\Phi(0)| \\ &= |a|^2 |\alpha_1(0)\rangle\langle\alpha_1(0)| + ab^* |\alpha_1(0)\rangle\langle\alpha_2(0)| + a^*b |\alpha_2(0)\rangle\langle\alpha_1(0)| + |b|^2 |\alpha_2(0)\rangle\langle\alpha_2(0)|. \end{aligned} \quad (40)$$

Note that, if $i = 1, 2$,

$$\langle\alpha_i(0)| = \left(\sum_{k=0}^N \frac{|\alpha_i(0)|^{2k}}{k!} \right)^{-1/2} \sum_{n=0}^N \frac{(\alpha_i^*(0))^n}{\sqrt{n!}} \langle f_n^G|. \quad (41)$$

After a time $t > 0$, the state becomes

$$\begin{aligned} \rho(t) &= |\Phi(t)\rangle\langle\Phi(t)| \\ &= |a|^2 |\alpha_1(t)\rangle\langle\alpha_1(t)| + ab^* |\alpha_1(t)\rangle\langle\alpha_2(t)| + a^*b |\alpha_2(t)\rangle\langle\alpha_1(t)| + |b|^2 |\alpha_2(t)\rangle\langle\alpha_2(t)|. \end{aligned} \quad (42)$$

The expression in the second row in (42) contains four terms. The sum of the second and third terms is called the non-diagonal part that we intend to analyze next.

4.2 Decoherence

Let us call $\rho^{\text{ND}}(t)$ the sum of the second and third term in (42). Under the condition of macroscopicity to be introduced below, the non-diagonal terms in (42) approximately satisfy the following relation:

$$\rho^{\text{ND}}(t) = \rho_{12}^{\text{ND}}(t) |\alpha_1(0)\rangle\langle\alpha_2(0)| + \rho_{21}^{\text{ND}}(t) |\alpha_2(0)\rangle\langle\alpha_1(0)|. \quad (43)$$

This is not true in the general case as one may check by a direct calculation on $|\alpha_1(t)\rangle\langle\alpha_2(t)| = e^{-itH} |\alpha_1(0)\rangle\langle\alpha_2(0)| e^{itH}$.

In order to show (43), we should make the hypothesis of the macroscopicity condition, which means that the peaks of both approximate Gaussians are far from each other. This means that $|\alpha_1^*(0) - \alpha_2(0)|$ is very large (usually, we shall take $\alpha_1(0)$ to be real). Note that, under this hypothesis, the states of the preferred basis $\{|\alpha_1(0)\rangle, |\alpha_2(0)\rangle\}$ are approximately orthogonal or quasi-orthogonal, which means that

$$\langle\alpha_1(0)|\alpha_2(0)\rangle = \langle\alpha_2(0)|\alpha_1(0)\rangle^* \approx 0. \quad (44)$$

To show (44), we first note that after (39) and (41) we have that

$$\langle\alpha_1(0)|\alpha_2(0)\rangle = \left(\sum_{k=0}^N \frac{|\alpha_1(0)|^{2k}}{k!} \right)^{-1/2} \left(\sum_{k=0}^N \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1/2} \sum_{n=0}^N \frac{(\alpha_1^*(0) \alpha_2(0))^n}{n!}. \quad (45)$$

Then, let us use the Cauchy product² and the binomial formula on the two first factors on the right hand side of (45), so as to obtain:

$$\langle\alpha_1(0)|\alpha_2(0)\rangle = \left(\sum_{k=0}^N \frac{(|\alpha_1(0)|^2 + |\alpha_2(0)|^2)^k}{k!} \right)^{-1/2} \sum_{n=0}^N \frac{(\alpha_1^*(0) \alpha_2(0))^n}{n!}. \quad (46)$$

Repeating the procedure, we obtain

$$\begin{aligned} \langle\alpha_1(0)|\alpha_2(0)\rangle &= \sum_{n=0}^N \frac{1}{n!} \left(-\frac{1}{2} (|\alpha_1(0)|^2 + |\alpha_2(0)|^2 - 2\alpha_1^*(0)\alpha_2(0)) \right)^n \\ &= \sum_{n=0}^N \frac{1}{n!} \left(-\frac{(\alpha_1^*(0) - \alpha_2(0))^2}{2} \right)^n. \end{aligned} \quad (47)$$

Clearly after (47), for $|\alpha_1^*(0) - \alpha_2(0)|$ very large, we obtain (44). If this were the case, let us define

$$\rho_{12}^{\text{ND}}(t) := \langle\alpha_1(0)|\rho(t)|\alpha_2(0)\rangle, \quad \rho_{21}^{\text{ND}}(t) := \langle\alpha_2(0)|\rho(t)|\alpha_1(0)\rangle, \quad (48)$$

where $\rho(t)$ has been introduced in (42), from which we have that

²The Cauchy product states that

$$\left(\sum_{n=0}^N a_n \right) \left(\sum_{n=0}^N b_n \right) = \sum_{n=0}^N c_n, \quad \text{with} \quad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

$$\rho_{12}^{\text{ND}}(t) = ab^* \langle \alpha_1(0) | \alpha_1(t) \rangle \langle \alpha_2(t) | \alpha_2(0) \rangle + a^*b \langle \alpha_1(0) | \alpha_2(t) \rangle \langle \alpha_1(t) | \alpha_2(0) \rangle ,$$

$$\rho_{21}^{\text{ND}}(t) = ab^* \langle \alpha_2(0) | \alpha_1(t) \rangle \langle \alpha_2(t) | \alpha_1(0) \rangle + a^*b \langle \alpha_2(0) | \alpha_2(t) \rangle \langle \alpha_1(t) | \alpha_1(0) \rangle . \quad (49)$$

In order to compute (49), let us follow the notation used in (14), (15) and (16). In the sequel, we shall assume that the numbers $\alpha_1(0)$ and $\alpha_2(0)$ are real. If we choose $|\psi\rangle := |\alpha_1(0)\rangle$ and $|\varphi(t)\rangle := |\alpha_1(t)\rangle$, and take into account (38) and (41), we have that the coefficients a_k and b_k in (14) and (15) have the following form:

$$a_k = \frac{(\alpha_1(0))^k}{\sqrt{k!}} e^{-\frac{1}{2} |\alpha_1(0)|^2} , \quad b_k = \frac{(\alpha_1(0))^k}{\sqrt{k!}} e^{-\frac{1}{2} |\alpha_1(0)|^2} , \quad (50)$$

so that,

$$\langle \alpha_1(0) | \alpha_1(t) \rangle = e^{-|\alpha_1(0)|^2} \sum_{n=0}^N \frac{(|\alpha_1(0)|^2)^n}{n!} e^{-iz_n t} . \quad (51)$$

Then, we take $|\psi\rangle := |\alpha_1(0)\rangle$ and $|\varphi(t)\rangle := |\alpha_2(t)\rangle$, which gives after (39) and (41) that

$$a_n = \frac{|\alpha_1(0)|^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_1(0)|^2} , \quad b_n = \frac{|\alpha_2(0)|^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_2(0)|^2} , \quad (52)$$

so that

$$\langle \alpha_1(0) | \alpha_2(t) \rangle = e^{-\frac{1}{2} \{|\alpha_1(0)|^2 + |\alpha_2(0)|^2\}} \sum_{n=0}^N \frac{(|\alpha_1(0)| |\alpha_2(0)|)^n}{n!} e^{-iz_n t} . \quad (53)$$

In the third place, we choose $|\psi\rangle = |\alpha_2(0)\rangle$ and $|\varphi(t)\rangle = |\alpha_1(t)\rangle$, which give

$$a_n = \frac{(\alpha_2(0))^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_2(0)|^2} , \quad b_n = \frac{(\alpha_1(0))^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_1(0)|^2} , \quad (54)$$

so that

$$\langle \alpha_2(0) | \alpha_1(t) \rangle = e^{-\frac{1}{2} \{|\alpha_1(0)|^2 + |\alpha_2(0)|^2\}} \sum_{n=0}^N \frac{(|\alpha_1(0)| |\alpha_2(0)|)^n}{n!} e^{-iz_n t} \quad (55)$$

Finally, the choice $|\psi\rangle := |\alpha_2(0)\rangle$ and $|\varphi(t)\rangle := |\alpha_2(t)\rangle$ gives

$$a_n = \frac{(\alpha_2(0))^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_2(0)|^2} , \quad b_n = \frac{(\alpha_2(0))^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha_2(0)|^2} , \quad (56)$$

so that,

$$\langle \alpha_2(0) | \alpha_2(t) \rangle = e^{-|\alpha_2(0)|^2} \sum_{n=0}^N \frac{(|\alpha_2(0)|^2)^n}{n!} e^{-iz_n t} . \quad (57)$$

So far, we have presented our discussion on the quasi-coherent states based in the properties of the Friedrichs model. In the next sections, we shall present another approach, in such a way that the above discussion can be applied to decoherence and the Loschmidt echo.

In this case, we may redefine (38) and (39), and then

$$|\alpha_i(0)\rangle = e^{-\frac{1}{2}|\alpha_i(0)|^2} \sum_{n=0}^{\infty} \frac{(\alpha_i(0))^n}{\sqrt{n!}} |f_n^D\rangle, \quad i = 1, 2. \quad (58)$$

Following the discussion in [51], we repeat the procedure outlined in Section 3.1 to the present situation. Now, $z_n = nz_0$ and the sum over n goes to infinite. As initial values, we choose

$$\alpha_1(0) = 0, \quad \alpha_2(0) \neq 0. \quad (59)$$

The macroscopicity condition now means that $\alpha_2(0) \gg 1$. Then, equations (51), (53), (55) and (57) now give, respectively,

$$\langle \alpha_1(0) | \alpha_1(t) \rangle = 1, \quad \langle \alpha_1(0) | \alpha_1(t) \rangle = e^{-\frac{1}{2}|\alpha_2(0)|^2} \approx 0, \quad \langle \alpha_2(0) | \alpha_1(t) \rangle = e^{-\frac{1}{2}|\alpha_2(0)|^2} \approx 0, \quad (60)$$

$$\langle \alpha_2(0) | \alpha_2(t) \rangle = \exp \{ -|\alpha_2(0)|^2 \} e^{-iz_0 t}. \quad (61)$$

Consequently, the non-diagonal coefficients defined in (43) and explicitly given in general form in (49) take in our case, the approximate following values:

$$\rho_{12}^{\text{ND}}(t) \approx ab^* \exp \{ -|\alpha_2(0)|^2 \} e^{-iz_0 t}, \quad (62)$$

and

$$\rho_{21}^{\text{ND}} \approx a^* b \exp \{ -|\alpha_2(0)|^2 \} e^{-iz_0 t}. \quad (63)$$

We see that as $t \mapsto \infty$, the non-diagonal terms are of the order of $\exp\{-|\alpha_2(0)|^2\} \approx 0$, which shows decoherence in the preferred basis.

This formalism may be extended to a Friedrichs model with $N + 1$ bound states for H_0 and, hence, with $N + 1$ resonance poles, z_0, z_1, \dots, z_N . In this case, equations (62)-(63) take the following forms, respectively,

$$\rho_{12}^{\text{ND}}(t) \approx ab^* \exp \{ -|\alpha_2(0)|^2 \} \sum_{n=0}^N \frac{(|\alpha_2(0)|^2)^n}{n!} e^{-\frac{i}{\hbar} z_n t} \quad (64)$$

and

$$\rho_{12}^{\text{ND}}(t) \approx a^* b \exp \{ -|\alpha_2(0)|^2 \} \sum_{n=0}^N \frac{(|\alpha_2(0)|^2)^n}{n!} e^{-\frac{i}{\hbar} z_n t}. \quad (65)$$

These results show that the non-diagonal elements of the density matrix go to zero when time goes to infinite. This process is called decoherence, and in previous works [68, 69, 70], more examples of this phenomenon have been studied using the non-hermitian formalism.

As is known, the decoherence is an irreversible process [71], therefore there is some kind of irreversibility in this model. This motivates us to study more aspects of the irreversibility of this model. In particular, in this work we are interested in the Loschmidt echo phenomenon.

4.3 Loschmidt echo

As we have seen in equation (33), $M(\tau)$ can be expressed as

$$M(\tau) = \sum_{n=0}^N e^{-\tau\gamma_n/\hbar} \langle \alpha_2(0) | f_n^D \rangle \langle f_n^G | \alpha_2(0) \rangle. \quad (66)$$

Now, using the identity (39) and relations (13), we have that

$$\langle f_n^G | \alpha_2(0) \rangle = \left(\sum_{k=0}^N \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1/2} \frac{(\alpha_2(0))^n}{\sqrt{n!}}. \quad (67)$$

Analogously, using (41), one has that

$$\langle \alpha_2(0) | f_n^D \rangle = \left(\sum_{k=0}^N \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1/2} \frac{(\alpha_2^*(0))^n}{\sqrt{n!}}, \quad (68)$$

so that

$$\langle \alpha_2(0) | U^\dagger(\tau) U(\tau) | \alpha_2(0) \rangle = A \sum_{n=0}^N \frac{|\alpha_2(0)|^{2n}}{n!} e^{-\tau\gamma_n/\hbar}$$

with

$$A = \left(\sum_{k=0}^N \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1} \approx \left(\sum_{k=0}^{\infty} \frac{|\alpha_2(0)|^{2k}}{k!} \right)^{-1} = \exp\{-|\alpha_2(0)|^2\} \quad (69)$$

and the latter approximation makes sense for N large, so that

$$\langle \alpha_2(0) | U^\dagger(\tau) U(\tau) | \alpha_2(0) \rangle \approx e^{-|\alpha_2(0)|^2} \sum_{n=0}^N \frac{|\alpha_2(0)|^{2n}}{n!} e^{-\tau\gamma_n/\hbar} \quad (70)$$

Comparing equation (70) with equations (64) and (65), it is observed that, in both cases, the decay rate is related to the imaginary part of the Hamiltonian eigenvalues. In the first case the decay is accompanied by an oscillation, while in the second case it is not. In both cases the characteristic times of the exponential decays are the same.

The result shows that there is an intimate relationship between both phenomena. This connection is not surprising since the Loschmidt echo and quantum decoherence can be considered as two expressions of the irreversibility of quantum systems. However, it is of significant importance since it links two phenomena that have been studied separately.

5 Conclusions

The Gamow vectors formalism has been developed in order to give a mathematical description of quantum decay phenomena, such as radioactive phenomena, scattering or decoherence. In this paper we extend the scope of this formalism. We use the Gamow vectors to describe the Loschmidt echo.

The irreversible character of the Loschmidt echo can be studied in two manners. One approach considers that the time evolution is perfectly unitary, but there are imperfections in the implementation of the time reversal operation. The backward and forward evolution process are governed by two slightly different Hamiltonians. This can be attributed to the noise introduced by the environment. The other approach is to assume that the time evolution operator is not unitary, which introduces an intrinsic irreversibility of the time evolution. This kind of evolutions are well represented by the Gamow vector formalism.

In this paper, we have adopted the second approach: we have used Gamow vectors to describe the Loschmidt echo phenomenon. We showed that the characteristic time of this process is directly associated with the resonances of the analytical continuation of the Hamiltonian. Moreover, the type of decay obtained, is related to the number of resonances of the system and their relative weight in the initial state.

Additionally, we have compared this phenomenon with another well known one: the decoherence process. In this case, the resonances are related to the decoherence and relaxation times. We considered a toy model based on the Friedrichs model. We showed that choosing as the initial state a superposition of two quasi-coherent states, and taking into account the macroscopicity condition, decoherence and Loschmidt echo can be described simultaneously. Moreover, the characteristic time of the Loschmidt echo is related with the decoherence time, since both times are determined by the same resonances. Therefore, we have concluded that, under our hypothesis, the Loschmidt echo and decoherence, are two aspects of the same phenomenon, and that there is a mathematical relation between their corresponding characteristic times.

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