Operator Theory:
Advances and Applications, Vol. 267, 185–198

Spectral Algorithms for MRA Orthonormal Wavelets

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Abstract. Operator techniques lead to spectral algorithms to compute scaling functions and wavelets associated with multiresolution analyses (MRAs). The spectral algorithms depend on the choice of pairs of suitable orthonormal bases (ONBs). This work presents the spectral algorithms for three different pairs of ONBs: Haar bases, Walsh–Paley bases and trigonometric bases. The Walsh–Paley bases connect wavelet theory and dyadic harmonic analysis. The results for trigonometric bases are the first viable attempt to do a discrete Fourier analysis of the problem.

Mathematics Subject Classification (2010). Primary 42C40; Secondary 47N40.

Keywords. Orthonormal wavelets, multiresolution analysis, spectral algorithms.

1. Introduction

On the Hilbert space $L^2(\mathbb{R})$ of square integrable complex functions consider the translation and dilation operators, T and D, given by

$$[Tf](x) := f(x-1)\,, \quad [Df](x) := 2^{1/2}\, f(2x)\,, \quad (f \in L^2(\mathbb{R}))\,.$$

By an *orthonormal wavelet on* $\mathbb{R}$ we mean here a function $\psi \in L^2(\mathbb{R})$ such that the family of dilated-translated versions of ψ

$$\{\psi_{j,k}(x) := [D^j T^k \psi](x) = 2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$$

is an orthonormal basis (shortly ONB) of $L^2(\mathbb{R})$.

Certain orthonormal wavelets ψ, in particular, the compactly supported orthonormal wavelets, are related to *multiresolution analyses* (MRAs), i.e., increasing sequences $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that $\cap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$, $f \in V_0 \Leftrightarrow f_{j,0} = D^j f \in V_j$, and there exists a function

This work was partially supported by research projects MTM2012-31439 and MTM2014-57129-C2-1-P (Secretaría General de Ciencia, Tecnología e Innovación, Ministerio de Economía y Competitividad, Spain).

$\varphi \in L^2(\mathbb{R})$, the *scaling function*, whose integer translates $\{\varphi_{0,k} = T^k\varphi\}_{k\in\mathbb{Z}}$ form an ONB of V_0. In such case, $\psi_{j,k} \in V_j$ for every $j, k \in \mathbb{Z}$, and a pair of discrete quadrature mirror filters $\big(h_k\big)_{k\in\mathbb{Z}}$ and $\big((-1)^{1-k}h_{1-k}\big)_{k\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ are involved in the *two-scale relations*

$$\varphi_{j,0} = \sum_{k\in\mathbb{Z}} h_k\,\varphi_{j+1,k}\,, \qquad (j \in \mathbb{Z})\,, \tag{1.1}$$

$$\psi_{j,0} = \sum_{k\in\mathbb{Z}} (-1)^{1-k}h_{1-k}\,\varphi_{j+1,k}\,, \quad (j \in \mathbb{Z})\,. \tag{1.2}$$

The two-scale relations (1.1) and (1.2) are the point of departure to obtain algorithms that compute the graph of MRA scaling functions and wavelets. Examples are the Daubechies [2] cascade algorithm or the Daubechies–Lagarias [5] infinite product matrix representation. In the Fourier domain, under suitable conditions, from relation (1.1) one gets

$$\hat{\varphi}(\theta) = \prod_{a=0}^{\infty} \frac{h(e^{-\pi i\theta/2^a})}{2^{1/2}}\,\hat{\varphi}(0)\,. \tag{1.3}$$

where $h(e^{2\pi i\theta}) := \sum_{k\in\mathbb{Z}} h_k\,e^{2\pi i k\theta}$ is the transfer function of the filter $\big(h_k\big)$. Conditions for the infinite product in (1.3) determining the Fourier transform of a convenient scaling function can be found in [11].

Fourier transform is closely related to translation operators. On the basis of Fourier analysis, operator techniques have been widely used in wavelet theory, see, for example, [1, 9, 14]. Beyond Fourier analysis, different choices of ONBs may be related to spectral and shift representations of translation and dilation operators on $L^2(\mathbb{R})$. Such ONBs allow to describe every orthonormal wavelet and MRA in terms of rigid operator-valued functions defined on the corresponding functional spectral spaces. By-products of the picture are "spectral formulas" useful to design computational algorithms for orthonormal wavelets and MRAs.

In particular, in [7] the authors propose to consider arbitrary ONBs $\{L_i^{(0)}\}_{i\in\mathbb{I}}$ of $L^2[0,1)$ and $\{K_{\pm,j}^{(0)}\}_{j\in\mathbb{J}}$ of $L^2[\pm1,\pm2)$, and extend them to a pair of ONBs of $L^2(\mathbb{R})$ by means of translations of $\{L_i^{(0)}\}_{i\in\mathbb{I}}$ and dilations of $\{K_{\pm,j}^{(0)}\}_{j\in\mathbb{J}}$. The two resultant ONBs of $L^2(\mathbb{R})$,

$$\{L_i^{(n)} := T^n L_i^{(0)}\}_{i\in\mathbb{I},n\in\mathbb{Z}}\,, \quad \{K_{s,j}^{(m)} := D^m K_{s,j}^{(0)}\}_{s=\pm,j\in\mathbb{J},m\in\mathbb{Z}}\,, \tag{1.4}$$

lead to shift and spectral representations of the operators T and D, respectively. The computational usefulness of this approach has been investigated in [6, 8] for a particular choice of the ONBs $\{L_i^{(0)}\}$ and $\{K_{\pm,j}^{(0)}\}$, the so-called Haar bases.

This work explores two new choices of the ONBs $\{L_i^{(0)}\}$ and $\{K_{\pm,j}^{(0)}\}$ and the corresponding spectral algorithms for compactly supported MRA orthonormal wavelets. The first new bases considered are related with the Walsh–Paley system and the dyadic harmonic analysis [15]. The second bases are related with the trigonometric system and the usual harmonic analysis. For the sake of completeness