Dirac-Weyl equation on a hyperbolic graphene surface under perpendicular magnetic fields

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Abstract. In this paper the Dirac-Weyl equation on a hyperbolic surface of graphene under magnetic fields is considered. In order to solve this equation analytically for some cases, we will deal with vector potentials symmetric under rotations around the z axis. Instead of using tetrads we will get this equation from a more intuitive point of view by restriction from the Dirac-Weyl equation of an ambient space. The eigenvalues and corresponding eigenfunctions for some magnetic fields are found by means of the factorization method. The existence of a zero energy ground level and its degeneracy is also analysed in relation to the Aharonov-Casher theorem valid for flat graphene.

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1. Introduction

The low energy electrons in flat graphene behave in the continuum limit as massless Dirac particles. Based on this property, there has been a considerable amount of work on the electronic properties of graphene and other allotropes of carbon under different magnetic or electric fields by making use of the (2+1) dimensional Dirac-Weyl equation [1, 2, 3, 4, 5]. Another attractive field of research has been the study of Dirac electrons in non-flat surfaces specially fullerenes (or nanotubes) which can be addressed by expressing the Dirac-Weyl equation on the sphere (or on the cylinder by means of appropriate boundary conditions). This approach has also been applied to nano-ribbons [6, 7, 8, 9, 10, 11, 12, 13, 14].

In the same way, the electronic properties of massless Dirac electrons in a graphene surface with hyperbolic shape, together with the presence of external electromagnetic fields, can also be studied by means of the Dirac-Weyl equation on this surface. This is the main objective of the present paper, where we will consider only perpendicular magnetic fields with rotational symmetry around the z-axis. Usually the construction of the Dirac-Weyl equation on a curved surface is obtained with the help of 'tetrads' and covariant derivatives with spin connections [11, 12, 13]. Here, we will adopt a simpler approach by means of the restriction from the standard Dirac equation defined in an ambient space to a surface included in this space. This point of view is easier to follow and, in particular for constant curvature surfaces, it allows to keep track of the explicit symmetries of the surface. We remark that this approach leads to equivalent results as those obtained in the usual formulation as it has been checked with the case of the sphere [6, 7].

An application of this study will be the finding of the energy levels of the Landau system on the graphene hyperboloid as well as their degeneracy. In this respect, we will see that the number of energy levels is finite, each one with infinite degeneracy. Another application is that this type of surface can be seen as taking part of a quantum blister. In bilayer graphene some parts are deformed and they lead to quantum blisters [15, 16]. By applying magnetic fields in the hyperbolic surface we can study the confining of Dirac electrons on this type of deformations. The application of electric fields can also be interesting. It is known that massless Dirac electrons can not be confined using electric fields on the graphene because of the Klein tunneling, except for some quasi bound states with finite life time. Recently, it has been shown that electrons can be confined on quantum blisters by applying electrostatic voltage. This would be very important for experimental designs [17].

The organization of this paper is as follows. In Section 2, the Dirac-Weyl equation on the hyperboloid is defined. In Section 3, the factorization method of supersymmetric quantum mechanics is introduced in order to solve this equation and the ground state solutions are characterized. The relation between the existence of good ground states and the magnetic flux is discussed taking as reference the Aharonov-Casher theorem valid for magnetic fields acting on flat graphene. Next, a few solvable cases are worked out in Section 4. Finally, this work is finished with some conclusions and remarks along Section 5.

2. The Dirac-Weyl equation on the hyperboloid

Low energy electrons in graphene behave as massless Dirac electrons with an effective Fermi velocity $\upsilon_F = c/300$, where c is the velocity of light (see for instance [1]). Therefore, they are described by the 2+1 dimensional Dirac-Weyl equation in flat space-time. This description can be extended to other surfaces, in particular there are many recent works devoted to adapt it to fullerenes and nanotubes. Here, we will study the Dirac-Weyl equation on the two–dimensional hyperboloid. Our method consists in formulating the Dirac-Weyl equation in an ambient space, where the spatial components (x, y, z) have metric signature $(-, -, +)$. In this space we will restrict the Dirac-Weyl equation to the hyperboloid $-x^2 - y^2 + z^2 = c$, where $c = r^2$ is the square of the 'radius' of the sheet $z > 0$ of a two–sheeted hyperboloid. In this way, we will get a Dirac-Weyl equation on the hyperboloid which inherits the $SO(1, 2)$ symmetry valid on the whole ambient space.

2.1. Reduction of Dirac-Weyl equation to the hyperboloid

The Dirac-Weyl equation in 3+1 space-time for Cartesian coordinates is given by

$$i\hbar\,\frac{\partial\Phi(x,y,z,t)}{\partial t} = v_F\,(\boldsymbol{\sigma}\cdot\mathbf{p})\Phi(x,y,z,t)\,, \tag{2.1}$$

where $\boldsymbol{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ are the Pauli matrices and $\mathbf{p} = -i\hbar\,(\partial_x,\partial_y,\partial_z)$ is the three dimensional momentum operator. The interaction of a Dirac electron with a magnetic field according to the minimal coupling rule is described by replacing the momentum operator $\mathbf{p}$ in (2.1) by $\mathbf{p}-q\mathbf{A}/c$, where the charge of the electron is $q=-e$. The notation for the vector potential and magnetic field is the usual one

$$\mathbf{A} = (A_x,A_y,A_z), \qquad \mathbf{B} = \nabla\times\mathbf{A}\,. \tag{2.2}$$

The time-independent Dirac-Weyl equation, obtained by replacing $\Phi(x,y,z,t) = \Psi(x,y,z)\,e^{-iEt/\hbar}$ into (2.1), is

$$v_F\,\Big[\boldsymbol{\sigma}\cdot(\mathbf{p}-\frac{q}{c}\,\mathbf{A})\Big]\,\Psi(x,y,z) = E\,\Psi(x,y,z)\,. \tag{2.3}$$

In the following, we will adapt the above equation (2.3) to our present situation. Firstly, in order to keep the formal $SO(1,2)$ symmetry, we must use everywhere the scalar product with signature $(-,-,+)$ represented by the dot " $\cdot$ " instead of the Euclidean inner product.

Let us introduce the hyperbolic (or pseudo–spheric) coordinates (r,u,φ) appropriate to describe a two–sheeted hyperboloid oriented along the z axis satisfying the equation $-x^2-y^2+z^2=r^2$. They are related to the Cartesian coordinates (x,y,z) by

$$x = r\sinh u\cos\varphi, \quad y = r\sinh u\sin\varphi, \quad z = r\cosh u\,, \tag{2.4}$$

where $0<u<\infty$, $0<\varphi<2\pi$ and $0<r<\infty$. The momentum operators in hyperbolic coordinates are

$$p_x = -i\hbar\,\partial_x = -i\hbar(-\sinh u\cos\varphi\,\partial_r - \frac{\sin\varphi}{r\sinh u}\,\partial_\varphi + \frac{\cosh u\cos\varphi}{r}\,\partial_u)\,, \tag{2.5}$$

$$p_y = -i\hbar\,\partial_y = -i\hbar(-\sinh u\sin\varphi\,\partial_r + \frac{\cos\varphi}{r\sinh u}\,\partial_\varphi + \frac{\cosh u\sin\varphi}{r}\,\partial_u)\,, \tag{2.6}$$

$$p_z = -i\hbar\,\partial_z = -i\hbar(\cosh u\,\partial_r - \frac{\sinh u}{r}\,\partial_u)\,. \tag{2.7}$$

In a second step, in (2.3) we must use not arbitrary momenta, but those restricted to the tangent plane of the hyperboloid. They are defined in the way shown in [19, 20]:

$$\begin{aligned}
\tilde p_x &= \tfrac12(r\,p_x+p_x\,r) + \tfrac12\left((\mathbf{r}\cdot\mathbf{p})\tfrac{x}{r} + \tfrac{x}{r}(\mathbf{p}\cdot\mathbf{r})\right)\,,\\
\tilde p_y &= \tfrac12(r\,p_y+p_y\,r) + \tfrac12\left((\mathbf{r}\cdot\mathbf{p})\tfrac{y}{r} + \tfrac{y}{r}(\mathbf{p}\cdot\mathbf{r})\right)\,,\\
\tilde p_z &= \tfrac12(r\,p_z+p_z\,r) - \tfrac12\left((\mathbf{r}\cdot\mathbf{p})\tfrac{z}{r} + \tfrac{z}{r}(\mathbf{p}\cdot\mathbf{r})\right)\,.
\end{aligned} \tag{2.8}$$

These operators are the quantum analog of the projection (according to the pseudo–scalar product) of the momentum vectors on the tangent plane at a point of the hyperboloid. They can be identified as the angular momenta. On the hyperbolic surface where $r = R = const.$ the linear momentum operators are given by dividing by the constant radius, thus leading to the following expressions

$$\hat{p}_x = \frac{-i\hbar}{R}(-\frac{\sin\varphi}{\sinh u}\,\partial_\varphi + \cosh u\cos\varphi\,\partial_u + \sinh u\cos\varphi)\,, \tag{2.9}$$

$$\hat{p}_y = \frac{-i\hbar}{R}(\frac{\cos\varphi}{\sinh u}\,\partial_\varphi + \cosh u\sin\varphi\,\partial_u + \sinh u\sin\varphi)\,, \tag{2.10}$$

$$\hat{p}_z = \frac{-i\hbar}{R}(-\sinh u\,\partial_u - \cosh u)\,. \tag{2.11}$$

Finally, we have to use the Dirac matrices appropriate to the metric. In this case, the time-space metric is $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,1)$. A choice for the Pauli matrices is $\hat{\boldsymbol{\sigma}} = (-\sigma_x, -\sigma_y, i\sigma_z) := (\hat\sigma_1, \hat\sigma_2, \hat\sigma_3)$, so that

$$\hat\sigma_k\hat\sigma_j + \hat\sigma_j\hat\sigma_k = 0 \quad (k \neq j), \qquad \hat\sigma_1^2 = \hat\sigma_2^2 = 1,\ \hat\sigma_3^2 = -1\,. \tag{2.12}$$

The Dirac matrices in terms of the previous ones are constructed, in the Weyl representation, as

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \hat\sigma_i \\ -\hat\sigma_i & 0 \end{pmatrix}. \tag{2.13}$$

For this choice of the Dirac matrices the dot product in the stationary Dirac-Weyl equation in (2.3) should be replaced by

$$\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{p}} = -\sigma_x\,\hat{p}_x - \sigma_y\,\hat{p}_y + i\sigma_z\,\hat{p}_z\,. \tag{2.14}$$

We will consider a magnetic field perpendicular to the surface of the hyperboloid and having a rotational symmetry around the z-axis. Hence, we choose the vector potential in the form

$$\mathbf{A} = A(u)\hat\varphi = A(u)(-\sin\varphi, \cos\varphi, 0)\,, \tag{2.15}$$

where $A(u)$ is a function depending on u.

Using the above definitions, after straightforward computations, the Hamiltonian corresponding to the Dirac electron (2.3) on the surface of the hyperboloid becomes

$$H = \frac{1}{R}\begin{pmatrix} (-\sinh u\,\partial_u - \cosh u) & i\,e^{-i\varphi}(-\frac{i}{\sinh u}\,\partial_\varphi + \cosh u\,\partial_u + \sinh u - \frac{qR}{c\hbar}A(u)) \\ i\,e^{i\varphi}(\frac{i}{\sinh u}\,\partial_\varphi + \cosh u\,\partial_u + \sinh u + \frac{qR}{c\hbar}A(u)) & -(-\sinh u\,\partial_u - \cosh u) \end{pmatrix} \tag{2.16}$$

and the eigenvalue equation for H, after dividing by $\hbar\,v_F$, is

$$H\,\Psi(R,u,\varphi) = \mathcal{E}\,\Psi(R,u,\varphi), \qquad \mathcal{E} = \frac{E}{\hbar\,v_F}\,. \tag{2.17}$$

2.2. Rotational symmetry

Since the radius R is constant, the notation $\Psi(R,u,\varphi) := \Psi(u,\varphi)$ will be used hereafter, where (u,φ) are a kind of polar coordinates on the hyperboloid. Next, let us consider the total angular momentum along the z axis,

$$J_z = -i\hbar\partial_\varphi + \frac{\hbar}{2}\sigma_z\,. \tag{2.18}$$

As there is a geometric rotational symmetry around the z-axis, J_z should commute with H: $[H,J_z]=0$. Therefore, the eigenfunctions (2.17) of H can also be chosen as eigenfunctions of J_z at the same time,

$$J_z\Psi(u,\varphi) = \lambda\,\hbar\,\Psi(u,\varphi)\,. \tag{2.19}$$

So, the two-component spinor wavefunction takes the form

$$\Psi(u,\varphi) = N\begin{pmatrix} e^{i(\lambda-\frac{1}{2})\varphi}f_1(u) \\ e^{i(\lambda+\frac{1}{2})\varphi}f_2(u)\end{pmatrix}, \tag{2.20}$$

where λ is a half-odd number and N is a normalization constant. By substituting (2.20) and (2.16) into the eigenvalue equation (2.17) we get

$$\begin{aligned}\Big[\frac{1}{R}(-\sinh u\,\partial_u - \cosh u)\sigma_z + (\frac{i}{2\,R\sinh u} + \frac{i}{R}\cosh u\,\partial_u + \frac{i}{R}\sinh u)\sigma_x \\ +(\frac{q}{c\,\hbar}A(u) - \frac{\lambda}{R\,\sinh u})\sigma_y\Big]\,F(u) = \mathcal{E}F(u)\,,\end{aligned} \tag{2.21}$$

where $F(u) = (f_1(u), f_2(u))^T$ is a column matrix; the superindex T is used for matrix transposition.

2.3. Hermitian form

In order to eliminate the term with σ_z, we apply a transformation to the matrix equation (2.21)

$$F(u) = e^{-\frac{u}{2}\sigma_y}\begin{pmatrix}\psi_1(u)\\ \psi_2(u)\end{pmatrix}. \tag{2.22}$$

Then, after using the Baker-Campbell-Hausdorff formula, Eq. (2.21) becomes

$$\begin{pmatrix} 0 & \frac{i}{R}\partial_u + \frac{i}{2\,R}\coth u - i(\frac{q}{c\,\hbar}A(u) - \frac{\lambda}{R\,\sinh u}) \\ \frac{i}{R}\partial_u + \frac{i}{2\,R}\coth u + i(\frac{q}{c\,\hbar}A(u) - \frac{\lambda}{R\,\sinh u}) & 0\end{pmatrix}\begin{pmatrix}\psi_1(u)\\ \psi_2(u)\end{pmatrix} = \mathcal{E}\begin{pmatrix}\psi_1(u)\\ \psi_2(u)\end{pmatrix}. \tag{2.23}$$

This effective Hamiltonian (2.23) is not Hermitian due to the term "$\frac{i}{2\,R}\coth u$". But it can be made Hermitian by writing the wavefunction $(\psi_1(u),\psi_2(u))^T$ as

$$(\psi_1(u),\psi_2(u))^T = \frac{1}{\sqrt{\sinh u}}(g_1(u), ig_2(u))^T = \frac{1}{\sqrt{\sinh u}}\,G(u). \tag{2.24}$$

It is clear that the latter change is related to the surface element $ds = R^2 \sinh u \, du d\varphi$ of the hyperboloid in polar coordinates. Thus, after these transformations we arrive at an effective Hermitian matrix Hamiltonian that can be expressed as

$$\begin{pmatrix} 0 & \frac{i}{R}\partial_u + i(\frac{\lambda}{R\sinh u} - \frac{q}{c\hbar}A(u)) \\ \frac{i}{R}\partial_u - i(\frac{\lambda}{R\sinh u} - \frac{q}{c\hbar}A(u)) & 0 \end{pmatrix} \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix} = \mathcal{E} \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix}. \tag{2.25}$$

3. SUSY partner Hamiltonians and ground states

3.1. The supersymmetry formalism

Let us define the following first order operators

$$L^{\pm} = \mp\partial_u + W(u), \qquad W(u) = -\frac{\lambda}{\sinh u} + \frac{q\,R}{c\,\hbar}A(u), \tag{3.1}$$

where $W(u)$ is called superpotential function. Using these definitions, the matrix equation (2.25) is rewritten as

$$\begin{pmatrix} 0 & -\,i\,L^+ \\ i\,L^- & 0 \end{pmatrix} \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix} = R\,\mathcal{E} \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix} \tag{3.2}$$

and the components g_1, g_2 are connected by these operators,

$$L^+ g_2(u) = R\,\mathcal{E}\, g_1(u)\,, \qquad L^- g_1(u) = R\,\mathcal{E}\, g_2(u)\,. \tag{3.3}$$

From these equations we get a pair of decoupled second order effective Schrödinger equations

$$H_1\, g_1(u) := L^+L^-\, g_1(u) = \varepsilon\, g_1(u)\,, \tag{3.4}$$

$$H_2\, g_2(u) := L^-L^+\, g_2(u) = \varepsilon\, g_2(u)\,, \tag{3.5}$$

where $\varepsilon = R^2\mathcal{E}^2$. Equations (3.4) and (3.5) in matrix form are

$$\begin{pmatrix} L^+L^- & 0 \\ 0 & L^-L^+ \end{pmatrix} \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix} = \varepsilon \begin{pmatrix} g_1(u) \\ ig_2(u) \end{pmatrix}. \tag{3.6}$$

The diagonal elements of the above matrix are the effective Hamiltonians

$$H_1 = -\partial_u^2 + V_1(u), \qquad H_2 = -\partial_u^2 + V_2(u), \tag{3.7}$$

whose effective potentials are given in terms of the superpotential (3.1) in the following way

$$V_1(u) = W(u)^2 - W'(u), \qquad V_2(u) = W(u)^2 + W'(u)\,. \tag{3.8}$$

Here, the prime denotes differentiation with respect to u. The above relations show that the Hamiltonians H_1 and H_2 are one dimensional supersymmetric partner Hamiltonians [18] and $L^{\pm}$ are intertwining operators that link these Hamiltonians as follows:

$$H_2\, L^- = L^-\, H_1, \qquad H_1\, L^+ = L^+\, H_2. \tag{3.9}$$

These intertwining relations imply that if we assume that the spectrum of H_1 (H_2) is known then its partner H_2 (H_1) will have the same spectrum except possibly the ground state.

Let $\{\varepsilon_{1,n}\}$, $n = 0, 1, \ldots$, be the discrete spectrum of H_1 with real eigenfunctions $\{g_{1,n}\}$, and assume that the ground state of H_1 is annihilated by L^-,

$$L^- \, g_{1,0} = 0 \,. \tag{3.10}$$

Then, as a consequence of (3.4) the ground state eigenvalue of H_1 will be

$$\varepsilon_{1,0} = 0 \,.$$

This will be a 'good' ground state as far as the function $g_{1,0}$ is square-integrable in $(0, \infty)$ and satisfies appropriate boundary conditions. Now, the discrete spectrum of H_2 will consist of the eigenvalues $\{\varepsilon_{2,n-1}\}$ and normalized eigenfunctions $\{g_{2,n-1}\}$ given by

$$\varepsilon_{1,n} = \varepsilon_{2,n-1}, \qquad g_{2,n-1}(u) := \frac{1}{\sqrt{\varepsilon_{1,n}}} L^- \, g_{1,n}(u), \qquad n = 1, 2, \ldots \,. \tag{3.11}$$

In this point, it is assumed that the operator L^- does not spoil the physical requirements of the eigenfunctions. Thus, the eigenvalues of the equations (3.6) for g_1 and g_2 consistent with (3.2) are

$$\varepsilon_0 := \varepsilon_{1,0} = 0, \qquad \varepsilon_n := \varepsilon_{1,n} = \varepsilon_{2,n-1}, \qquad n = 1, 2, \ldots \,.$$

Taking into account the above results, the excited eigenfunctions of the reduced Hamiltonian Eq. (2.25) take the form

$$G_{\pm,n}(u) = N \begin{pmatrix} \pm g_{1,n}(u) \\ i \, g_{2,n-1}(u) \end{pmatrix} \tag{3.12}$$

with the corresponding eigenvalues

$$\mathcal{E}_{\pm,n} := \frac{E}{\hbar \, \upsilon_F} = \pm \frac{1}{R} \sqrt{\varepsilon_n}, \qquad n = 1, 2, \ldots \,. \tag{3.13}$$

The ground state wavefunction and its energy are as follows

$$G_0(u) = N \begin{pmatrix} g_{1,0}(u) \\ 0 \end{pmatrix}, \qquad \mathcal{E}_{\pm,0} = \frac{1}{R} \sqrt{\varepsilon_0} = 0 \,. \tag{3.14}$$

There are other possibilities to characterize the ground state besides (3.10), for instance

$$L^+ \, g_{2,0} = 0 \,. \tag{3.15}$$

Or even a ground state not satisfying (3.10) nor (3.15), however in our examples the present assumption will be sufficient.

3.2. The zero energy ground state and the flux of the magnetic field

Let us pay attention to the zero energy ground state wavefunction $g_{1,0}$ defined by (3.10) that according to (3.1) is determined by the equation

$$\left(\partial_u - \frac{\lambda}{\sinh u} + \frac{q\,R}{c\,\hbar}A(u)\right) g_{1,0}(u) = 0, \tag{3.16}$$

whose solution is

$$g_{1,0}(u) = N\left(\tanh\frac{u}{2}\right)^{\lambda} e^{-\frac{q\,R}{c\,\hbar}\int A(u)du}\,. \tag{3.17}$$

In order $g_{1,0}(u)$ to be a physical ground state it should have the following asymptotic behaviour:

$$\begin{aligned} &(a) \quad \text{in} \quad u\to 0, \qquad g_{1,0}(u)\to 0 \text{ (or be bounded)}\,,\\ &(b) \quad \text{in} \quad u\to\infty, \qquad g_{1,0}(u)\to 0\,. \end{aligned} \tag{3.18}$$

Next, we will interpret the integral in the exponent of (3.17). Let us recall that the magnetic field is given by

$$B_{u,\varphi}(u) = \frac{1}{R\,\sinh u}\left[\partial_u(A(u)\,\sinh u)\right]. \tag{3.19}$$

Then, the magnetic flux $\Phi(u)$ in the circle of radius u will be

$$\Phi(u) = \int_0^u B_{u,\varphi}(u) 2\pi R^2 \sinh u\, du = 2\pi R A(u)\sinh u\,, \tag{3.20}$$

therefore,

$$A(u) = \frac{\Phi(u)}{2\pi R\sinh u}\,. \tag{3.21}$$

Now, assume that we have a null magnetic field for $u > u_0$, then in the region $0 < u < u_0$ we have that the flux is Φ_0 and for $u > u_0$ $A(u) = \frac{\Phi_0}{2\pi R\sinh u}$, so that

$$\int_0^u A(u)\,du = \int_0^{u_0} A(u)\,du + \int_{u_0}^u \frac{\Phi_0}{2\pi R\sinh u}\,du\,. \tag{3.22}$$

As a consequence, for $u > u_0$ $g_{1,0}$ given in (3.17) will take the form

$$g_{1,0}(u) = N\left(\tanh\frac{u}{2}\right)^{\lambda-\frac{\Phi_0}{\phi_0}}\,, \tag{3.23}$$

where $\phi_0 = 2\pi c\hbar/q$ is the quantum of flux. Hence, condition (b) of (3.18) will not be satisfied. In conclusion, we see that a magnetic field with a compact support and finite flux can not lead to a physical zero energy ground state on the hyperboloid. This is contrary to what happens in the case of flat graphene, where the existence and degeneracy of the zero ground energy level depends on the finite flux, a property that is known as Aharonov–Casher theorem [21, 22]. We have seen that on the hyperboloid, only when the flux is divergent the ground state can exist, and its degeneracy will be described by some values of λ (we will show some examples in the following section).

4. Solvable cases of magnetic potentials

Now, we will consider some special cases for the function $A(u)$ such that the eigenvalue equation (2.25) with $q = -e$ can be solved analytically:

$$\text{(i)} \qquad A(u) = -\frac{c\,\hbar}{eR}\,A_0\,\coth u\,, \tag{4.1}$$

$$\text{(ii)} \qquad A(u) = \frac{c\,\hbar}{e\,R}\,(-\frac{\lambda'}{\sinh u} + C_1\,\coth u - \frac{D_1}{C_1})\,, \tag{4.2}$$

$$\text{(iii)} \qquad A(u) = \frac{c\,\hbar}{e\,R}\,(-\frac{\lambda'}{\sinh u} - C_2\,\tanh u - \frac{D_2}{C_2})\,, \tag{4.3}$$

$$\text{(iv)} \qquad A(u) = \frac{c\,\hbar}{e\,R}\,(-\frac{\lambda'}{\sinh u} - C_3\,\tanh u - D_3\,\text{sech}u)\,, \tag{4.4}$$

where the parameters λ', D_k and C_k are real constants. For these cases the corresponding magnetic fields are given by (3.19).

The case $A(u) = 0$, of a null magnetic field, does not support bound states, so it will not be considered. The first case (i) leads to constant magnetic field and the second case (ii) to a decaying magnetic field which give rise to bound states, the analytic solutions can be found when the angular momentum λ coincides with the parameter λ' of the potential. The third and the fourth cases lead to magnetic fields and effective potentials that in general have bad boundary conditions at the origin; only for some special values of the parameters they are acceptable and correspond to finite magnetic fields that have a finite limit in $u \to \infty$ and in $u \to 0$. In this section, we will deal with the first and the second cases in detail; the last cases will be briefly commented.

4.1. Case (i)

The vector potential (4.1) leads to a constant magnetic field

$$B_{u,\varphi}(u) = -\frac{B_0}{R^2}\,, \tag{4.5}$$

where $B_0 = A_0(\frac{c\hbar}{e})$ is constant and the sign determines the orientation of the magnetic field. Therefore, this system can be considered as the Landau system on the hyperboloid for massless relativistic particles. Here, the vector potential gives rise to the following superpotential

$$W(u) = A_0\,\coth u - \lambda\,\text{cosech}\,u, \qquad A_0 < \lambda \tag{4.6}$$

and to the partner potentials

$$V_1(u) = A_0^2 + (A_0^2 + \lambda^2 + A_0)\,\text{cosech}^2 u - \lambda(2A_0 + 1)\coth u\,\text{cosech}\,u\,, \tag{4.7}$$

$$V_2(u) = A_0^2 + (A_0^2 + \lambda^2 - A_0)\,\text{cosech}^2 u - \lambda(2A_0 - 1)\coth u\,\text{cosech}\,u\,. \tag{4.8}$$

They are shape invariant potentials [18] satisfying $V_2(u, A_0 + 1) = V_1(u, A_0) + 2A_0 + 1$.

In this case, $g_{1,0}$ is annihilated by L^-, as it was shown in Sect. 4, where

$$L^- = \partial_u - \lambda\,\text{cosech}u + A_0\,\coth u\,. \tag{4.9}$$

Thus, the zero energy ground state wavefunction is

$$g_{1,0}(u) = N(\tanh\frac{u}{2})^{\lambda}\,\frac{1}{(\sinh u)^{A_0}} \tag{4.10}$$

and its asymptotic behaviour is

$$\begin{array}{llll} (a) & \text{in} & u\to 0, & g_{1,0}(u)\approx u^{\lambda-A_0}, \\ (b) & \text{in} & u\to\infty, & g_{1,0}(u)\approx \dfrac{1}{(\sinh u)^{A_0}}. \end{array} \tag{4.11}$$

This means that the ground state so defined is physically acceptable ($g_{1,0}(u)\to 0$) if

$$A_0 > 0 \quad \text{and} \quad \lambda - A_0 \geq 0\,. \tag{4.12}$$

In these conditions the ground state has zero energy and has infinite degeneracy determined by the (half odd) values of the total angular momentum λ such that $\lambda - A_0 \geq 0$. Since the magnetic field is constant for all u, the total flux is infinite, which agree with the previous discussion.

The energy eigenvalues are given by

$$\varepsilon_0 = \varepsilon_{1,0} = 0, \qquad \varepsilon_n = \varepsilon_{1,n} = \varepsilon_{2,n-1} = A_0^2 - (A_0 - n)^2, \qquad n = 1, 2\ldots [A_0]\,, \tag{4.13}$$

where $[A_0]$ is the integer part of A_0 and all of them have the same infinite degeneracy. The eigenfunctions have the form

$$g_{1,n}(w(u)) = (w-1)^{s^-/2}\,(w+1)^{-s^+/2} P_n^{(s^- -1/2, -s^+ -1/2)}(w(u)), \tag{4.14}$$

$$g_{2,n}(w(u)) = (w-1)^{(s^- -1)/2}\,(w+1)^{-(s^+ -1)/2} P_n^{(s^- +1/2, -s^+ +1/2)}(w(u)), \tag{4.15}$$

where $s^- = \lambda - A_0$, $s^+ = \lambda + A_0$ and $P_n^{(a,b)}(w(u))$ are Jacobi polynomials, $a, b > -1$, $w(u) = \cosh u$ [18]. These solutions are acceptable if A_0 satisfies the above conditions (4.12).

Finally, the eigenvalues of the Dirac-Weyl Eq. (2.17) are

$$\mathcal{E}_{\pm,n} = \pm\frac{1}{R}\sqrt{A_0^2 - (A_0 - n)^2} \tag{4.16}$$

and the eigenfunctions can be read from (3.12) substituting the functions $g_{1,n}$ and $g_{2,n-1}$ of (4.14). The effective potentials V_1, V_2 and the functions $g_{1,1}$, $g_{2,0}$ corresponding to the first excited level are displayed in Fig. 1. Some eigenvalues of the Dirac-Weyl Hamiltonian (Landau levels on the hyperboloid) and the susy partner effective Hamiltonians can be seen in Fig. 2. In conclusion, there is a finite number of Landau levels, including the zero energy level, each one with an infinite degeneracy labeled by the values of the total momentum λ satisfying (4.12).

4.2. Case (ii)

The vector potential (4.2) gives the following magnetic field

$$B_{u,\varphi} = \frac{c\,\hbar}{e\,R}\left(\frac{C_1}{R} - \frac{D_1}{R\,C_1}\coth u\right)\,, \tag{4.17}$$

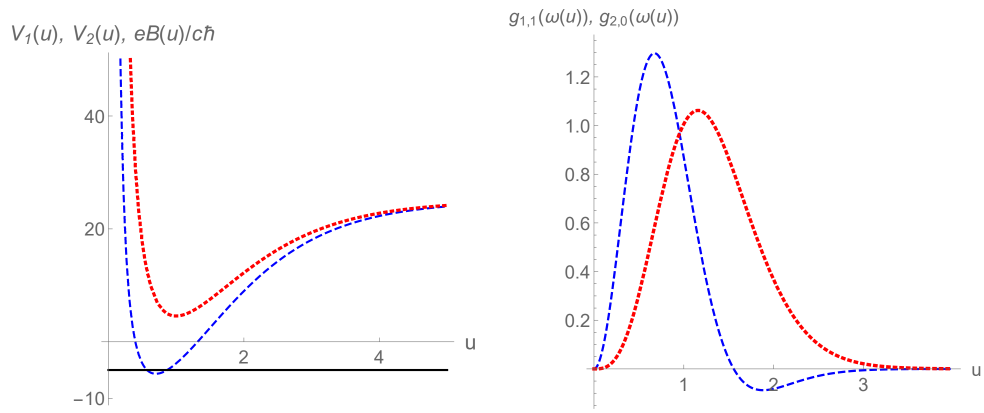

Figure 1. Plot of the potentials V_1, V_2 for case (i) with constant magnetic field (left) and the wavefunctions $g_{1,1}$, $g_{2,0}$ (right) for $A_0 = 5$, $\lambda = 7$. Dashed lines are for V_1, $g_{1,1}$, dotted lines for V_2, $g_{2,0}$ and the continuous line is for the magnetic field.

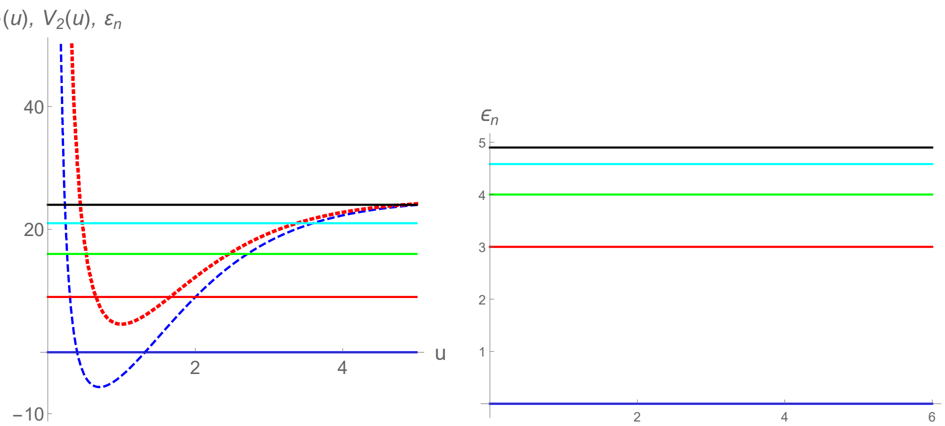

Figure 2. Plot of the potentials V_1 (dashed line), V_2 (dotted line) and the corresponding eigenvalues ε_n for case (i) (left) and the eigenvalues of Dirac-Weyl Hamiltonian $\mathcal{E}_{+,n}$ (right) for $n = 0$ (blue, bottom), $n = 1$ (green), $n = 2$ (red), $n = 3$ (cyan), $n = 4$ (black, top).

where C_1 and D_1 are real constants. This field is singular at the origin and, as it will be shown in our discussion, it goes to a negative constant in $u \to \infty$. If $\lambda = \lambda'$, the corresponding superpotential has the form

$$W(u) = \frac{D_1}{C_1} - C_1 \coth u\,. \tag{4.18}$$

Then, the partner potentials obtained from (3.8) are

$$V_1(u) = \frac{D_1^2}{C_1^2} + C_1^2 + C_1(C_1 - 1)\,\mathrm{cosech}^2 u - 2\,D_1 \coth u\,, \tag{4.19}$$

$$V_2(u) = \frac{D_1^2}{C_1^2} + C_1^2 + C_1(C_1 + 1)\,\mathrm{cosech}^2 u - 2\,D_1 \coth u\,. \tag{4.20}$$

In this case, the ground wavefunction $g_{1,0}$ is also annihilated by L^- provided $D_1 > C_1^2$. These potentials are inside the class of Eckart potentials [18]. Then, the corresponding

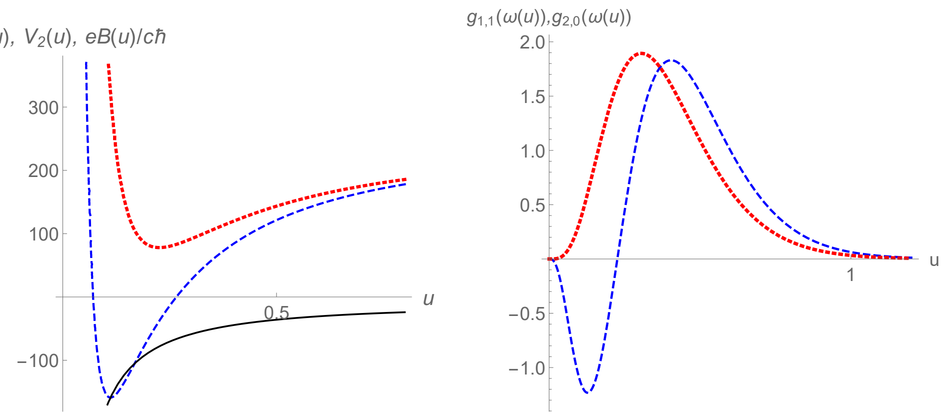

Figure 3. Plot of the Eckart potentials V_1, V_2 for case (ii) (left) and the wavefunctions $g_{1,1}$, $g_{2,0}$ of the first excited state (right) for $C_1 = 3$, $D_1 = 54$. Dashed lines are for V_1, $g_{1,1}$, dotted lines for V_2, $g_{2,0}$ and the continuous line is for the magnetic field.

energy eigenvalues are given by

$$\varepsilon_0 = \varepsilon_{1,0} = 0, \qquad \varepsilon_n = \varepsilon_{1,n} = \varepsilon_{2,n-1} = C_1^2 - (C_1+n)^2 - \frac{D_1^2}{(C_1+n)^2} + \frac{D_1^2}{C_1^2}, \tag{4.21}$$

where $n = 1, 2 \ldots$.

The eigenfunctions have the form

$$g_{1,n}(w(u)) = (w-1)^{s_1^+/2}\,(w+1)^{s_1^-/2} P_n^{(s_1^+,s_1^-)}(w(u)), \tag{4.22}$$

$$g_{2,n}(w(u)) = (w-1)^{s_2^+/2}\,(w+1)^{s_2^-/2} P_n^{(s_2^+,s_2^-)}(w(u)), \tag{4.23}$$

where $P_n^{(a,b)}(w(u))$ are Jacobi polynomials, $a, b > -1$, $w(u) = \coth u$ and $s_1^\pm = \pm\frac{D_1}{(C_1+n)} - (C_1+n)$, $s_2^\pm = \pm\frac{D_1}{(C_1+n+1)} - (C_1+n+1)$ [18]. These solutions are acceptable if D_1 and C_1 satisfy the condition $D_1 > C_1^2$.

Therefore, the eigenvalues of the Dirac-Weyl Eq. (2.17) are

$$\mathcal{E}_{\pm,n} = \pm\frac{1}{R}\sqrt{C_1^2 - (C_1+n)^2 - \frac{D_1^2}{(C_1+n)^2} + \frac{D_1^2}{C_1^2}} \tag{4.24}$$

and the eigenfunctions can be read from (3.12) substituting the functions $g_{1,n}$ and $g_{2,n-1}$ of (4.22)-(4.23). Fig. 3 shows the effective potentials V_1, V_2 and the functions $g_{1,1}$, $g_{2,0}$ corresponding to the first excited level. It can be seen the eigenvalues of susy partner Hamiltonians and the Dirac-Weyl Hamiltonian in Fig. 4.

Let us mention that if $\lambda' \neq \lambda$ the spectrum can not be solved analytically, but the zero energy ground states will exist if $D_1 > C_1^2$ and $\lambda \geq \lambda'$.

4.3. Cases (iii) and (iv)

In these two cases only when the parameters D_2 and D_3 vanish, we will have magnetic fields with reasonable boundary conditions at the origin. For such values the potential will be

$$A(u) = \frac{c\,\hbar}{e\,R}\,(-\frac{\lambda'}{\sinh u} - C_2\,\tanh u)\,. \tag{4.25}$$

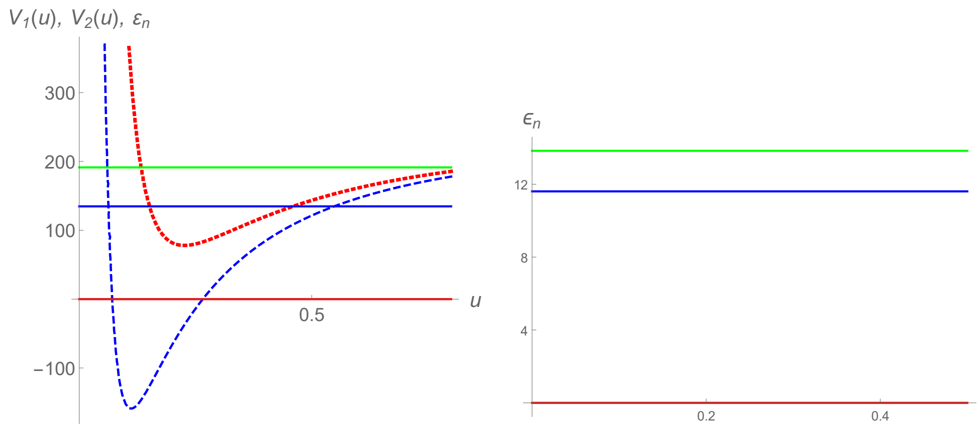

Figure 4. Plot of the Eckart potentials V_1 (dashed line), V_2 (dotted line) and the corresponding eigenvalues ε_n for case (ii) (left) and the eigenvalues of Dirac-Weyl Hamiltonian $\mathcal{E}_{\pm,n}$ (right) for $n = 0$ (red), $n = 1$ (blue), $n = 2$ (green).

Then, the magnetic field will take the form

$$B_{u,\varphi} = -\frac{c\,\hbar}{e\,R}\left(\frac{C_2}{R}(1+\mathrm{sech}^2\mathrm{u})\right). \tag{4.26}$$

When $u \to \infty$, the magnetic field will tend to a constant and at the origin has a minimum. Therefore, the flux will be infinite and the zero energy ground state given by $g_{1,0}$ will be good. If $\lambda = \lambda'$ the corresponding superpotential in this case is

$$W(u) = C_2 \tanh u\,, \tag{4.27}$$

and the partner potentials found from (3.8) are Pöschl–Teller potentials [18],

$$V_1(u) = C_2^2 - C_2(C_2+1)\,\mathrm{sech}^2 u\,, \tag{4.28}$$

$$V_2(u) = C_2^2 - C_2(C_2-1)\,\mathrm{sech}^2 u\,. \tag{4.29}$$

The eigenvalues and eigenfunctions of the Dirac-Weyl equation are obtained as

$$\varepsilon_0 = \varepsilon_{1,0} = 0, \qquad \varepsilon_n = \varepsilon_{1,n} = \varepsilon_{2,n-1} = C_2^2 - (C_2-n)^2\,, \tag{4.30}$$

where $n = 1, 2\ldots$.

The eigenfunctions have the form

$$g_{1,n}(w(u)) = (1-w)^{s_1/2}\,(1+w)^{s_1/2} P_n^{(s_1,s_1)}(w(u)), \tag{4.31}$$

$$g_{2,n}(w(u)) = (1-w)^{s_2/2}\,(1+w)^{s_2/2} P_n^{(s_2,s_2)}(w(u)), \tag{4.32}$$

where $P_n^{(a,b)}(w(u))$ are Jacobi polynomials, $a, b > -1$, $w(u) = \tanh u$ and $s_1 = (C_2 - n)$, $s_2 = (C_2 - n - 1)$ [18]. These solutions are acceptable if $C_2 > 0$.

Therefore, the eigenvalues of the Dirac-Weyl Eq. (2.17) are $\mathcal{E}_{\pm,n} = \pm\frac{1}{R}\sqrt{\varepsilon_n}$ and the eigenfunctions can be read from (3.12) substituting the functions $g_{1,n}$ and $g_{2,n-1}$ of (4.31)-(4.32). Fig. 5 displays the effective potentials V_1, V_2 and the functions $g_{1,1}$, $g_{2,0}$ corresponding to the first excited level in the real line.

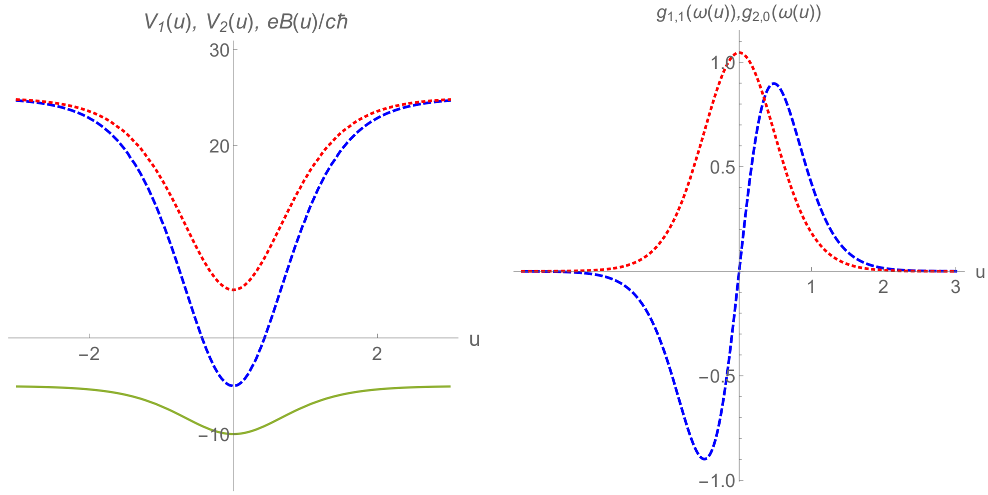

Figure 5. Plot of the Pöschl-Teller potentials V_1, V_2 for case (ii) (left) and the wavefunctions $g_{1,1}$, $g_{2,0}$ of the first excited state (right) for $C_2 = 5$. Dashed lines are for V_1, $g_{1,0}$, dotted lines for V_2, $g_{2,1}$ and the continuous line is for the magnetic field.

5. Conclusions

In this paper, we have studied the system of a massless charged particle on a hyperbolic surface under a rotationally symmetric perpendicular magnetic field. This problem can be identified with that of π–electrons in a deformed graphene sheet having this shape.

Instead of defining the Dirac–Weyl equation on this surface through the metric and spin connection, we have preferred to do it by formulating the Dirac–Weyl equation in an appropriate ambient space and then, restrict it to the hyperbolic surface. In this way, we preserve the rotational symmetry explicitly. This process is carried out in a straightforward way by means of the definition of momentum operators tangent to the surface.

After making use of the rotational symmetry we have arrived to a reduced Dirac–Weyl equation for two–component spinors in the 'radial' variable u, that displays the minimal coupling with the magnetic potential.

One of the points that we addressed was whether the known Aharonov-Casher theorem on the existence and degeneracy of the ground (zero-energy) state applies in this situation. We have shown that for radial symmetric magnetic fields with compact support and finite magnetic flux such zero energy modes don't exist. Indeed, we have considered a few analytically solvable cases where there exist only a finite number of discrete energy levels (besides the continuum spectrum). In these cases the magnetic potential is singular at the origin, such singularity can be compensated by the angular momentum, while the behaviour of the potential far from the origin allows for bound states (see this behaviour in formula (4.11)). In other words, the mechanism for confining massless particles is quite different in the hyperbolic surface than in a flat surface.

Among the analytically solvable cases here studied it is included the constant magnetic field. The discrete spectrum consist of the zero energy ground level plus a finite number of excited levels. The infinite degeneracy of each energy level is characterized by the total angular momentum that regularize the singularity of the potential at the origin. In the other cases the Dirac-Weyl equation is solvable for just one angular momentum ($\lambda = \lambda'$), but the conditions for the existence of zero energy ground state are the same. We should mention that the class of solvable potentials can be extended to other supersymmetric partner potentials by means of Darboux transformations.

We hope that this problem can help in different applications of graphene surfaces deformed in a variety of shapes, including some having a hyperbolic form. In this respect, we remark that recently the confining properties of quantum blisters in graphene have attracted much attention.

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