Casimir pistons with generalized boundary conditions: a step forward

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Abstract

In this work we study the Casimir effect for massless scalar fields propagating in a piston geometry of the type $I \times N$ where I is an interval of the real line and N is a smooth compact Riemannian manifold. Our analysis represents a generalization of previous results obtained for pistons configurations as we consider all possible boundary conditions that are allowed to be imposed on the scalar fields. We employ the spectral zeta function formalism in the ambit of scattering theory in order to obtain an expression for the Casimir energy and the corresponding Casimir force on the piston. We provide explicit results for the Casimir force when the manifold N is a d-dimensional sphere and a disk.

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1 Introduction

The Casimir effect is undoubtedly one of the most interesting physical phenomena predicted in the ambit of quantum field theory. Since the seminal work of Casimir in 1948 [12], interest on the subject, and more generally on the influence that external conditions have on a quantum system, has steadily increased. In fact the literature regarding the Casimir effect has grown not only in the number of works produced but also in its scope. When it was first theoretically predicted in [12], the Casimir effect focused simply on the attraction between two perfectly conducting neutral plates. Since then the Casimir effect has been studied for a plethora of

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different geometric configurations, quantum systems and boundary conditions (see for instance [8, 9, 36, 40] and references therein for a review on the subject). One of the most interesting and widely analyzed geometric configurations is the piston geometry which was first introduced by Calvalcanti in [13]. While one can find a number of specific piston configurations throughout the literature ([4, 14, 15, 16, 30, 31, 33, 34, 35, 37, 38] represents a, necessarily incomplete, list of examples), the most general one can be described as consisting of two compact manifolds, referred to as chambers, possessing a common boundary of co-dimension one representing the piston.

The reason for the widespread interest enjoyed by piston configurations lies mainly in the following important feature: In general calculations of the Casimir energy for quantum systems propagating in a given geometric configuration and subject to suitable boundary conditions lead, by the very nature of the phenomenon, to divergent quantities. In this case one is confronted with the non-trivial task of extracting, from these divergent results, meaningful physical information about the Casimir effect. In the case of piston configurations these problems are somewhat mitigated. In fact, while the Casimir energy of pistons might be divergent, the Casimir force acting on the piston itself is, is many instances, a well-defined quantity. In this regard, it is worth pointing out that piston configurations with non-vanishing curvature can have a divergent Casimir force acting on the piston [23, 24, 25]. The Casimir force acting on a piston depends not only on the specific geometry of the piston configuration but also on the boundary conditions that are imposed on the quantum field. In fact the Casimir force acting on a piston of a specific geometry can vary substantially as the boundary conditions are changed. For this reason, a precise and comprehensive analysis of the influence that the boundary conditions have on the Casimir force is of paramount importance for a deeper understanding of the Casimir effect. Studying the effect that boundary conditions have on the Casimir force on pistons is not only of theoretical significance but it could also shed some light on the Casimir effect of quantum systems consisting of real, as opposed to idealized, materials. In fact, suitable boundary conditions can be utilized to describe physical properties of real materials. Some results regarding the Casimir effect with general boundary conditions have been obtained, for instance, in [3] in the ambit of parallel plates and in [20, 21, 22] in regards to piston configurations. It is important to mention, for completeness, that real materials could be modeled by smooth potentials with compact support rather than boundaries (see e.g. [1, 5, 27, 26, 19]).

This work is mainly aimed at generalizing the results, obtained in [20, 22], for the Casimir effect in piston configurations. We consider a piston configuration of the type $I \times N$ where $I \subset \mathbb{R}$ is a closed interval of the real line and N is a smooth compact Riemannian manifold with or without boundary ∂N. We analyze a massless scalar field propagating in the aforementioned geometric configuration endowed with the most general boundary conditions for which the Laplace operator describing its dynamics admits strongly consistent selfadjoint extensions. It is important to emphasize, at this point, that the results presented in this paper for the Casimir energy and corresponding force on the piston encompass *all possible* boundary conditions that can be imposed on scalar fields propagating on pistons of the type $I \times N$, and, hence, represent an exhaustive analysis of the Casmir effect for scalar fields propagating on these types of pistons. In order to perform such general analysis we exploit the results obtained in [2] which enable one to characterize all selfadjoint extensions of the Laplacian. By following the techniques employed in [39], we will utilize spectral zeta function regularization methods in order to derive explicit expressions for the desired Casimir energy and the corresponding force on the piston. We perform the analysis of the spectral zeta function of the piston configuration by relying primarily on methods from scattering theory. While there are other methods to obtain the spectral zeta function of the system under consideration, we are of the opinion that the formalism based on scattering theory provides a somewhat more transparent physical interpretation of our results.

The outline of the paper is as follows. In the next section we describe in detail the piston configuration and the general boundary conditions to be imposed on the scalar field. Subsequently, we utilize scattering methods in order to obtain an integral representation of the spectral zeta function. We then analytically continue the representation and derive an expression for the Casimir energy and corresponding force on the piston for the piston under consideration. In the last sections we present some particular cases as examples of our general results. The conclusions provide a summary of our main results and some ideas for possible further studies in this area.

2 The general setup: $U(4)$ boundary conditions

We begin our analysis by considering a direct product manifold M of the type $M = I \times N$. In this setting we define $I = [0, L] \subset \mathbb{R}$ to be a closed interval of the real line and N to be a smooth compact d-dimensional Riemannian manifold with or without a boundary ∂N. It is clear from the above definition that M has dimension $D = d + 1$. The piston configuration can be obtained from the manifold M following the construction detailed in [20, 22]. The two chambers of the piston are realized by dividing the manifold M with a cross-sectional manifold N_a at the point $a \in (0, L)$. The manifold N_a represents the piston itself. The two chambers M_I and M_{II} are, by construction, smooth compact D-dimensional Riemannian manifolds with boundary $\partial M_I = N_0 \cup N_a \cup ([0, a] \times \partial N)$ and $\partial M_{II} = N_a \cup N_L \cup ((a, L] \times \partial N)$, respectively.

Let $\psi(t, x)$ with $x \in M$ denote a massless scalar field propagating on the piston configuration outlined above, and $\phi(x)$ denote the normal modes in which the scalar field decomposes after writing down the Fourier mode decomposition for $\psi(t, x)$ in the time coordinate. Due to the direct product structure of M we can write the equation characterizing the normal modes of the scalar field ϕ as the eigenvalue equation

$$- \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Delta_N \right) \phi = \alpha^2 \phi \, , \tag{2.1}$$

where Δ_N denotes the Laplacian on the manifold N. By using separation of variables we can write the solution ϕ as the product of a longitudinal part and a cross-sectional one, namely $\phi = f(x)Y(\Omega)$ where x is the coordinate in the interval I and Ω denotes the coordinates on N. The functions $Y(\Omega)$ are eigenfunctions of the operator Δ_N with eigenvalue λ

$$- \Delta_N Y(\Omega) = \lambda^2 Y(\Omega) \, , \tag{2.2}$$

while $f(x)$ satisfies the simple second-order differential equation in the space $\mathcal{I} = [0, a) \cup (a, L]$

$$- \frac{\mathrm{d}^2}{\mathrm{d}x^2} f_\lambda(x, k) = k^2 f_\lambda(x, k) \, , \tag{2.3}$$

where, for notational convenience, we have introduced the parameter $k^2 = \alpha^2 - \lambda^2$. The parameter k becomes the eigenvalue once the differential equation (2.3) is augmented by appropriate boundary conditions. As previously stated, we will consider all possible boundary conditions that can be imposed on $f_\lambda(x, k)$ which lead to a selfadjoint boundary value problem. According to the methods developed in [2] this is equivalent to considering all possible non-negative selfadjoint extensions of the operator in (2.3). We would like to point out that we will consider only strongly consistent selfadjoint extensions of the operator in (2.3), that is all the selfadjoint extensions that are non-negative independently on the size of the interval I [39].

2.1 General boundary conditions: $U(4)$

The boundary of $\mathcal{I}$ consists of four points, namely $\partial\mathcal{I} = \{x = 0, x = a^-, x = a^+, x = L\}$. According to the formalism developed in [2, 3, 39], the boundary conditions that characterize a given selfadjoint extension of the differential operator in (2.3) are expressed in the following form

$$\varphi - i\dot{\varphi} = U(\varphi + i\dot{\varphi}) \;, \tag{2.4}$$

where φ is a vector with entries being the boundary values of the function $f_\lambda(x, k)$, and $\dot{\varphi}$ denotes a vector whose entries are the outgoing normal derivative of $f_\lambda(x, k)$ at the boundary (c.f. [3, 39]), that is

$$\varphi = \begin{pmatrix} f_\lambda(0,k) \\ f_\lambda(a^-,k) \\ f_\lambda(a^+,k) \\ f_\lambda(L,k) \end{pmatrix} ; \; \dot{\varphi} = \begin{pmatrix} -f'_\lambda(0,k) \\ f'_\lambda(a^-,k) \\ -f'_\lambda(a^+,k) \\ f'_\lambda(L,k) \end{pmatrix} \Rightarrow \varphi \pm i\dot{\varphi} = \begin{pmatrix} f_\lambda(0,k) \mp i f'_\lambda(0,k) \\ f_\lambda(a^-,k) \pm i f'_\lambda(a^-,k) \\ f_\lambda(a^+,k) \mp i f'_\lambda(a^+,k) \\ f_\lambda(L,k) \pm i f'_\lambda(L,k) \end{pmatrix} \equiv \Psi^\pm \,. \tag{2.5}$$

Since the set of selfadjoint extensions of the differential operator in (2.3) defined over $\mathcal{I}$ is in one-to-one correspondence with the elements of the group $U(4)$ [2], the matrix U in (2.5) must be an element of the unitary group $U(4)$. This means that for any given $U \in U(4)$ we obtain a corresponding selfadjoint extension of the second derivative operator in (2.3) defined on the domain [39]

$$\mathcal{D}_U = \{f_k(x) \in H^2([0, L], \mathbb{C}) : \varphi - i\dot{\varphi} = U(\varphi + i\dot{\varphi})\} \;, \tag{2.6}$$

which is a subspace of the Sobolev space $H^2([0, L], \mathbb{C})$. It must be noted, that not all selfadjoint extensions give rise to a well-defined quantum field theory. Taking into account the fact that the normal modes of the scalar massless quantum field confined in the piston are characterized by the non-relativistic Schödinger eigenvalue problem (2.1), only those selfadjoint extensions that are non-negative can be used to construct a meaningful scalar quantum field theory on the piston.

In order to explicitly implement the boundary conditions (2.4) we need a solution of the differential equation (2.3) which can be easily found to be of the form

$$f_\lambda(x, k) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & 0 \le x \le a^- \\ A_2 e^{ikx} + B_2 e^{-ikx} & a^+ \le x \le L \,, \end{cases} \tag{2.7}$$

where the constants $\{A_1, B_1, A_2, B_2\}$ are to be determined as to satisfy the boundary conditions and the normalization condition. By using the explicit solution (2.7), the boundary vectors $\Psi^{(\pm)}$ defined in (2.5) are given by

$$\Psi^{(\pm)} = \begin{pmatrix} (1\pm k)A_1 + (1\mp k)B_1 \\ e^{iak}(1\mp k)A_1 + e^{-iak}(1\pm k)B_1 \\ e^{iak}(1\pm k)A_2 + e^{-iak}(1\mp k)B_2 \\ e^{ikL}(1\mp k)A_2 + e^{-ikL}(1\pm k)B_2 \end{pmatrix} = M_\pm \cdot \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} , \tag{2.8}$$

where we have introduced the following matrix

$$M_\pm \equiv \begin{pmatrix} 1\pm k & 1\mp k & 0 & 0 \\ e^{iak}(1\mp k) & e^{-iak}(1\pm k) & 0 & 0 \\ 0 & 0 & e^{iak}(1\pm k) & e^{-iak}(1\mp k) \\ 0 & 0 & e^{ikL}(1\mp k) & e^{-ikL}(1\pm k) \end{pmatrix} . \tag{2.9}$$

By substituting (2.8) into (2.4) we obtain the homogeneous linear system

$$(M_- - U\cdot M_+)\cdot \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = 0 . \tag{2.10}$$

The above linear system has a non-trivial solution for the parameters $\{A_1, B_1, A_2, B_2\}$ if and only if the determinant of the matrix $(M_- - U\cdot M_+)$ vanishes, that is

$$\det(M_- - U\cdot M_+) = 0 . \tag{2.11}$$

This expression represents an equation for the parameter k whose solutions determine the eigenvalues of the boundary value problem consisting of the differential equation (2.3) and the boundary conditions associated with $U \in U(4)$. In order to obtain an explicit expression for (2.11) we need an appropriate representation of a generic element $U \in U(4)$. One way of proceeding is to notice that $U(4) \cong (SU(4)\times U(1))/\mathbb{Z}_4$ and , hence, an element $U \in U(4)$ can be written as $U = e^{i\theta}\bar{U}$ where $\theta \in [0, 2\pi]$ and $\bar{U} \in SU(4)$ which, in turn, can be represented in terms of Euler angles and 4×4 Gell-Mann-type matrices as shown in [41]. Since $\dim(U(4)) = 16$, the relation (2.11) would contain sixteen free real parameters. Although with the help of a computer algebra program one could in principle obtain an explicit expression for (2.11) in terms of the required free parameters, it is, in our opinion, more instructive to consider simpler cases. Indeed, the large number of free parameters to follow by considering the full $U(4)$ would certainly obfuscate the main physical properties of the quantum system which represent the focus of our work.

To this end, starting with the next section, we will restrict our attention to boundary conditions that are represented by matrices belonging to the direct product subgroup $U(2)\times U(2) \subset U(4)$.

3 $U(2)\times U(2)$ reductions and topology change

The restriction to the subset $U(2)\times U(2)$ of $U(4)$ allows us to analyze the most general boundary conditions that relate pairs of boundary points of $\mathcal{I}$. If we denote $\mathbf{U}_1 \in U(2)$ and

$\mathbf{U}_2 \in U(2)$ as

$$\mathbf{U}_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \quad \text{and} \quad \mathbf{U}_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} , \tag{3.1}$$

then a generic element of $U(2) \times U(2) \subset U(4)$ describing boundary conditions that relate pairs of boundary points of $\mathcal{I}$ have one of the following forms:

$$V = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} , \quad W = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix} ,$$

$$R = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} . \tag{3.2}$$

It is not very difficult to realize that each of the matrices displayed in (3.2) characterizes a specific class of boundary conditions.

Boundary conditions described by matrices of the form V in (3.2) couple the boundary conditions at $x = 0$ and $x = a^-$ through a $U(2)$ matrix and the boundary conditions at $x = a^+$ and $x = L$ through, in general, another $U(2)$ matrix. In more detail, by using V in (3.2) in the relation (2.4) we get

$$\begin{pmatrix} f_\lambda(0,k) + i f'_\lambda(0,k) \\ f_\lambda(a^-,k) - i f'_\lambda(a^-,k) \end{pmatrix} = \mathbf{U}_1 \begin{pmatrix} f_\lambda(0,k) - i f'_\lambda(0,k) \\ f_\lambda(a^-,k) + i f'_\lambda(a^-,k) \end{pmatrix} , \tag{3.3}$$

$$\begin{pmatrix} f_\lambda(a^+,k) + i f'_\lambda(a^+,k) \\ f_\lambda(L,k) - i f'_\lambda(L,k) \end{pmatrix} = \mathbf{U}_2 \begin{pmatrix} f_\lambda(a^+,k) - i f'_\lambda(a^+,k) \\ f_\lambda(L,k) + i f'_\lambda(L,k) \end{pmatrix} . \tag{3.4}$$

This case represents two disconnected chambers, since the quantum vacuum fluctuations in one chamber are independent from the ones in the other chamber. The spectrum of the boundary value problem (2.3) and (2.4) is given, in this situation, simply by the union of the spectra of the selfadjoint extension defining the dynamics in each of the chambers. The disconnected chamber configuration has been already covered in [20] and, hence, will not be discussed further in this work.

Matrices of the form W in (3.2) describe, instead, the case in which the boundary conditions at $x = 0$ and $x = L$ are coupled through a $U(2)$ matrix and the boundary conditions at $x = a^-$ and $x = a^+$ are coupled, generally, through another $U(2)$ matrix. That is, the condition (2.4) becomes,

$$\begin{pmatrix} f_\lambda(0,k) + i f'_\lambda(0,k) \\ f_\lambda(L,k) - i f'_\lambda(L,k) \end{pmatrix} = \mathbf{U}_1 \begin{pmatrix} f_\lambda(0,k) - i f'_\lambda(0,k) \\ f_\lambda(L,k) + i f'_\lambda(L,k) \end{pmatrix} , \tag{3.5}$$

$$\begin{pmatrix} f_\lambda(a^-,k) - i f'_\lambda(a^-,k) \\ f_\lambda(a^+,k) + i f'_\lambda(a^+,k) \end{pmatrix} = \mathbf{U}_2 \begin{pmatrix} f_\lambda(a^-,k) + i f'_\lambda(a^-,k) \\ f_\lambda(a^+,k) - i f'_\lambda(a^+,k) \end{pmatrix} , \tag{3.6}$$

which can easily be obtained by replacing U in (2.4) with W defined in (3.2). In this case, as it is clear from (3.6), the quantum fluctuations are allowed to travel through the piston itself, a situation which occurs when the piston is not opaque. Using the boundary conditions

(3.6) is equivalent to modeling the piston itself as a point supported potential. This case would complement the analysis of semi-transparent pistons [38] and pistons with transmittal boundary conditions [22]. The boundary conditions in (3.5) can induce a topology change as they allow for the two ends $x = 0$ and $x = L$ of the piston configuration to be identified. In this case the piston configuration would have the topology of a torus.

Matrices of the form R in (3.2) characterize the situation in which boundary conditions at $x = 0$ are coupled, through a $U(2)$ matrix, to the boundary conditions at $x = a^+$ while boundary conditions at $x = a^-$ are coupled to the ones at $x = L$ through another $U(2)$ matrix. Although formally this case leads to a boundary value problem which is strongly selfadjoint, it is not suitable for describing a piston configuration. In fact, fields propagating in the left chamber would be constrained by the boundary at $x = 0$ but would have no constraints on the right boundary of that chamber, namely $x = a^-$. This leads to a scenario which would *de facto* eliminate the left chamber since fields propagating in it would "*feel*" the left boundary but not the right one. A similar argument applies to the fields propagating in the right chamber since, in this case, the piston itself is completely opaque. Because of the remarks above, we will be focusing our analysis on the membrane configuration.

3.1 Membrane configuration

There are basically two approaches that can be applied to the analysis of the membrane configuration. The first consists of writing a solution of the differential equation (2.3) as a linear combination of sine and cosine functions and then impose the boundary conditions in (3.5)-(3.6). The second approach, instead, is based on the formalism of scattering theory where the solutions are written in terms of transmission and reflection amplitudes. In the analysis that will follow we use the latter approach since, in our opinion, it describes the Casimir effect for the membrane configuration in a physically more meaningful way. To carry out the calculation we will follow a procedure consisting of two steps:

1. We start by studying the piston wall over the entire real line. In this case the piston wall can be described as a potential supported at the point $x = a$ defined by the boundary conditions (3.6) through the unitary matrix $\mathbf{U}_2$. The scattering states obtained in this case will satisfy (3.6) independently of the presence of the external walls at $x = 0, L$.

2. Afterwards we built the quantum field normal modes as linear combinations of the previously found scattering states and impose, on them, the boundary condition (3.5), given by the unitary matrix $\mathbf{U}_1$, at the external points of the piston $x = 0, L$.

With this approach we can characterize the spectrum of normal modes of the massless quantum scalar field in terms of non-relativistic scattering data of the piston wall. This characterization enables one to have a better intuition about the phenomena appearing in the Casimir force in terms of the physical properties of the piston that are encoded in the scattering data. To this end, we express the eigenfunctions of (2.3) with the boundary conditions (3.5) and (3.6) as the following linear combination

$$f_\lambda(x, k) = A_\lambda(k)\psi^R_{\lambda,k}(x; \mathbf{U}_2) + B_\lambda(k)\psi^L_{\lambda,k}(x; \mathbf{U}_2) \ , \tag{3.7}$$

where $\psi^R_{\lambda,k}(x;\mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x;\mathbf{U}_2)$ are the left-to-right and the right-to-left scattering states, respectively, and should be determined by the boundary condition (3.6). On the other hand the coefficients $A_\lambda(k)$ and $B_\lambda(k)$ are determined by the boundary condition (3.5).

3.1.1 The piston on the real line

The functions $\psi^R_{\lambda,k}(x;\mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x;\mathbf{U}_2)$ are solutions to the scattering problem consisting of a point supported potential, positioned at $x=a$, described by the unitary matrix $\mathbf{U}_2$. According to standard scattering theory, the left-to-right ($\psi^R_{\lambda,k}(x;\mathbf{U}_2)$) and the right-to-left ($\psi^L_{\lambda,k}(x;\mathbf{U}_2)$) scattering states can be written as

$$\psi^R_{\lambda,k}(x;\mathbf{U}_2)=\begin{cases} e^{-ikx}\tilde{r}_R+e^{ikx} & -\infty<x<a \\ e^{ikx}\tilde{t}_R & a<x<\infty \end{cases} ;\quad \psi^L_{\lambda,k}(x;\mathbf{U}_2)=\begin{cases} e^{-ikx}\tilde{t}_L & -\infty<x<a \\ e^{ikx}\tilde{r}_L+e^{-ikx} & a<x<\infty\ . \end{cases} \tag{3.8}$$

In order to determine the scattering data $\{\tilde{t}_R,\tilde{r}_R,\tilde{t}_L,\tilde{r}_L\}$ we impose the boundary conditions (3.6) on the functions $\psi^R_{\lambda,k}(x;\mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x;\mathbf{U}_2)$ separately.

By using $\psi^R_{\lambda,k}(x;\mathbf{U}_2)$ in (3.6) we obtain

$$\begin{pmatrix} e^{-2ika}\tilde{r}_R(1-k)+(1+k) \\ \tilde{t}_R(1-k) \end{pmatrix}=\mathbf{U}_2\begin{pmatrix} e^{-2ika}\tilde{r}_R(1+k)+(1-k) \\ \tilde{t}_R(1+k) \end{pmatrix} . \tag{3.9}$$

An explicit expression for the linear system that determines the coefficients $\tilde{r}_R$ and $\tilde{t}_R$ can be found by exploiting the Euler parametrization for $\mathbf{U}_2$, that is

$$\mathbf{U}_2=e^{i\theta}\left[\mathbb{I}\cos(\gamma)+i\sin(\gamma)\left(q_1\sigma_1+q_2\sigma_2+q_3\sigma_3\right)\right] , \tag{3.10}$$

where σ_j represents the Pauli matrices, (q_1,q_2,q_3) is a unit vector $q_1^2+q_2^2+q_3^2=1$, and $\theta\in[-\pi,\pi]$ and $\gamma\in[-\pi/2,\pi/2]$. The solution to the linear system (3.9) with the parametrization (3.10) can then be written as $\tilde{t}_R=t_R$ and $\tilde{r}_R=e^{2ika}r_R$ where t_R and r_R are the scattering amplitudes for the case in which the piston is located at $x=0$

$$t_R=\frac{-2ik\left(q_1-iq_2\right)\sin(\gamma)}{D_{\mathbf{U}_2}(k)} ,\quad r_R=\frac{(k^2+1)\cos(\gamma)+(k^2-1)\cos(\theta)+2ikq_3\sin(\gamma)}{D_{\mathbf{U}_2}(k)} , \tag{3.11}$$

where we have introduced the function

$$D_{\mathbf{U}_2}(k)=\left(k^2+1\right)\cos(\theta)+\left(k^2-1\right)\cos(\gamma)+2ik\sin(\theta) . \tag{3.12}$$

By imposing the boundary conditions (3.6) to the right-to-left scattering state $\psi^L_{\lambda,k}(x;\mathbf{U}_2)$ one finds a linear system for the coefficients $\tilde{t}_L$ and $\tilde{r}_L$ similar to the one in (3.9). By using the parametrization (3.10) one can write the solutions for the right-to-left scattering coefficients as $\tilde{t}_L=t_L$ and $\tilde{r}_L=e^{-2ika}r_L$ where

$$t_L=\frac{-2ik\left(q_1+iq_2\right)\sin(\gamma)}{D_{\mathbf{U}_2}(k)} ,\quad r_L=\frac{(k^2+1)\cos(\gamma)+(k^2-1)\cos(\theta)-2ikq_3\sin(\gamma)}{D_{\mathbf{U}_2}(k)} . \tag{3.13}$$

We would like to point out that the scattering coefficients in $\psi^R_{\lambda,k}(x;\mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x;\mathbf{U}_2)$ do satisfy the usual relations $|r_R|^2+|t_R|^2=1$ and $|r_L|^2+|t_L|^2=1$, which imply, in particular, that the function $D_{\mathbf{U}_2}(k)$ cannot vanish for real $k > 0$. However, it is possible for $D_{\mathbf{U}_2}(k)$ to have zeroes on the positive imaginary k-axis. In fact, the solutions of the equation $D_{\mathbf{U}_2}(i\kappa) = 0$ with $\kappa > 0$ determine the bound states of the system [28]. The solutions can be found to be

$$\kappa_{\pm} = -\tan\left(\frac{\theta \pm \gamma}{2}\right) . \tag{3.14}$$

Since $\theta \in [-\pi, \pi]$ and $\gamma \in [-\pi/2, \pi/2]$, it is not difficult to realize that it is possible to have either no bound states, one bound state, or two bound states. The regions in the $\theta - \gamma$ plane leading to no, one, or two bound states is given in Figure 1.

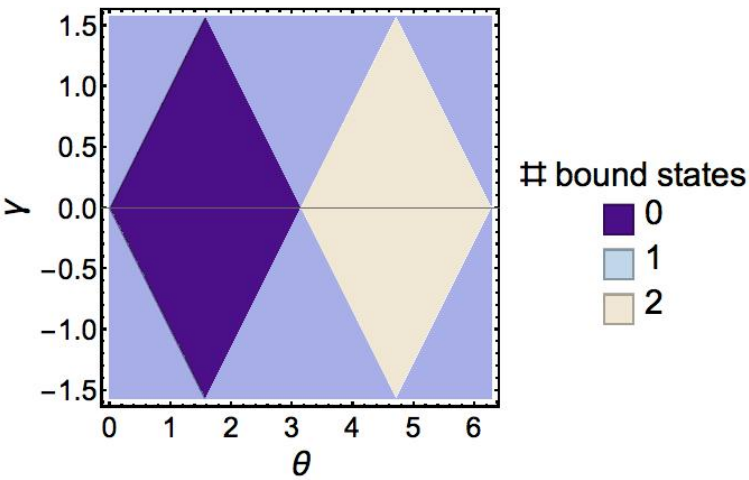

Figure 1: Bound states distribution in the $\theta - \gamma$ plane.

It is important to notice, that in order to have a unitary quantum field theory all the normal modes of the field must have real non-negative frequencies. This means, in particular, that the scattering problem we have just analyzed can not have bound states. Hence we have to restrict ourselves to those unitary matrices $\mathbf{U}_2$ that give rise to non negative selfadjoint extensions, i. e. the dark purple zone in Fig. 1.

3.1.2 The confined piston

The eigenfunctions $f_\lambda(x, k)$ in (3.7) automatically satisfy the boundary conditions on the piston itself when we use $\psi^R_{\lambda,k}(x; \mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x; \mathbf{U}_2)$ in (3.8) with the coefficients found in (3.11) and (3.13). Our next task therefore is to impose the remaining boundary conditions on $f_\lambda(x, k)$, namely the ones at the edges $x = 0$ and $x = L$ of the piston configuration. The scattering states $\psi^R_{\lambda,k}(x; \mathbf{U}_2)$ and $\psi^L_{\lambda,k}(x; \mathbf{U}_2)$ allow us to write the column vector of the boundary data of $f_\lambda(x, k)$ in (3.5) as

$$\begin{pmatrix} f_\lambda(0, k) \pm f'_\lambda(0, k) \\ f_\lambda(L, k) \mp i f'_\lambda(L, k) \end{pmatrix} = \tilde{M}_{\pm} \begin{pmatrix} A_\lambda(k) \\ B_\lambda(k) \end{pmatrix} , \tag{3.15}$$

where we have defined the matrices

$$\tilde{M}_{\pm} = \begin{pmatrix} 1 \mp k\,(1 - \tilde{r}_R) + \tilde{r}_R & (1 \pm k)\tilde{t}_L \\ e^{ikL}(1 \pm k)\tilde{t}_R & e^{-ikL}\left((1 \mp k) + e^{2ikL}(1 \pm k)\tilde{r}_L\right) \end{pmatrix}. \tag{3.16}$$

With this notation the boundary condition (3.5) reads

$$(\tilde{M}_+ - \mathbf{U}_1 \tilde{M}_-) \begin{pmatrix} A_\lambda(k) \\ B_\lambda(k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.17}$$

In order for (3.17) to have non-trivial solutions, the determinant of the coefficients of the linear system must vanish, that is one obtains the secular equation

$$F_\lambda(k, a; S, \mathbf{U}_1) := \det(\tilde{M}_+ - \mathbf{U}_1 \tilde{M}_-) = 0\ . \tag{3.18}$$

The last condition represents an equation in the variable k whose solutions provide, through the relation $\alpha^2 = k^2 + \lambda^2$, the eigenvalues of the problem (2.3) with boundary conditions (3.5) and (3.6). By utilizing (3.18) and after some lengthy but straightforward calculations one obtains an explicit expression for $F_\lambda(k, a; S, \mathbf{U}_1)$ as follows

$$\begin{aligned} F_\lambda(k, a; S, \mathbf{U}_1) &= e^{-ikL}\left[C^+_{\mathbf{U}_1} + k^2 C^-_{\mathbf{U}_1} - 2k(1 - \det(\mathbf{U}_1))\right] \\ &- e^{ikL}\det(S)\left[C^+_{\mathbf{U}_1} + k^2 C^-_{\mathbf{U}_1} + 2k(1 - \det(\mathbf{U}_1))\right] \\ &+ (r_R e^{ik(2a-L)} + r_L e^{-ik(2a-L)})\left[C^+_{\mathbf{U}_1} - k^2 C^-_{\mathbf{U}_1}\right] \\ &+ 2k(r_R e^{ik(2a-L)} - r_L e^{-ik(2a-L)})(u_{11} - u_{22}) + 4k(u_{21} t_R + u_{12} t_L), \end{aligned} \tag{3.19}$$

where u_{ij} are the entries of the matrix $\mathbf{U}_1$, and we have defined, for any 2×2 matrix $\mathbf{m}$, the quantities

$$C^{\pm}_{\mathbf{m}} = C_{\mathbf{m}}(\pm 1) = 1 + \det(\mathbf{m}) \mp \mathrm{tr}(\mathbf{m})\ , \tag{3.20}$$

which are nothing but the characteristic polynomial $C_{\mathbf{m}}(x)$ of $\mathbf{m}$ evaluated at $x = \pm 1$. It is clear from (3.19) that the function $F_\lambda(k, a; S, \mathbf{U}_1)$ depends explicitly on the unitary matrix $\mathbf{U}_1 \in U(2)$, and on the matrix $\mathbf{U}_2 \in U(2)$ through the scattering matrix for the point supported potential described by the unitary matrix $\mathbf{U}_2$

$$S(k; \mathbf{U}_2) = \begin{pmatrix} \tilde{t}_R & \tilde{r}_L \\ \tilde{r}_R & \tilde{t}_L \end{pmatrix}. \tag{3.21}$$

The determinant of the matrix in (3.21) can be computed by using the expressions in (3.11) and (3.13) of the scattering coefficients. One finds explicitly

$$\det(S) = -\frac{D_{\mathbf{U}_2}(-k)}{D_{\mathbf{U}_2}(k)}\ . \tag{3.22}$$

Introducing the notation $\rho_{R,L} \equiv D_{\mathbf{U}_2}(k) r_{R,L}$ and $\tau_{R,L} = D_{\mathbf{U}_2}(k) t_{R,L}$ allows us to rewrite (3.19) as

$$\begin{aligned} F_\lambda(k, a; S, \mathbf{U}_1) \;=\; & \frac{1}{D_{\mathbf{U}_2}(k)}\Big\{D_{\mathbf{U}_2}(k) e^{-ikL}\left[C^+_{\mathbf{U}_1} + k^2 C^-_{\mathbf{U}_1} - 2k(1 - \det(\mathbf{U}_1))\right] \\ & + D_{\mathbf{U}_2}(-k) e^{ikL}\left[C^+_{\mathbf{U}_1} + k^2 C^-_{\mathbf{U}_1} + 2k(1 - \det(\mathbf{U}_1))\right] \\ & + (\rho_R e^{ik(2a-L)} + \rho_L e^{-ik(2a-L)})\left[C^+_{\mathbf{U}_1} - k^2 C^-_{\mathbf{U}_1}\right] \\ & + 2k(\rho_R e^{ik(2a-L)} - \rho_L e^{-ik(2a-L)})(u_{11} - u_{22}) + 4k(u_{21}\tau_R + u_{12}\tau_L)\Big\}\ . \end{aligned} \tag{3.23}$$

By exploiting now Euler's parametrization of the group $U(2)$ for $\mathbf{U}_1$, that is

$$\mathbf{U}_1 = e^{i\alpha}\left[\mathbb{I}\cos(\beta) + i\sin(\beta)\left(n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3\right)\right] \ , \tag{3.24}$$

with $\alpha \in [-\pi, \pi]$ and $\beta \in [-\pi/2, \pi/2]$, one finds the relations

$$C^{\mp}_{\mathbf{U}_1} = 1 + \det(\mathbf{U}_1) \pm \mathrm{tr}(\mathbf{U}_1) = 2e^{i\alpha}(\cos(\alpha) \pm \cos(\beta)), \tag{3.25}$$

$$1 - \det(\mathbf{U}_1) = -2ie^{i\alpha}\sin(\alpha), \quad u_{11} - u_{22} = 2in_3e^{i\alpha}\sin(\beta) \tag{3.26}$$

$$u_{12} = ie^{i\alpha}\sin(\beta)(n_1 - in_2), \quad u_{21} = ie^{i\alpha}\sin(\beta)(n_1 + in_2), \tag{3.27}$$

which can be used in (3.23) to obtain the following expression

$$\begin{aligned} F_\lambda(k, a; S, \mathbf{U}_1) \ = \ & \frac{2e^{i\alpha}}{D_{\mathbf{U}_2}(k)}\big\{D_{\mathbf{U}_2}(k)e^{-ikL}\left[\cos(\alpha) - \cos(\beta) + k^2(\cos(\alpha) + \cos(\beta)) + 2ik\sin(\alpha)\right] \\ & + D_{\mathbf{U}_2}(-k)e^{ikL}\left[\cos(\alpha) - \cos(\beta) + k^2(\cos(\alpha) + \cos(\beta)) - 2ik\sin(\alpha)\right] \\ & + (\rho_R e^{ik(2a-L)} + \rho_L e^{-ik(2a-L)})\left[\cos(\alpha) - \cos(\beta) - k^2(\cos(\alpha) + \cos(\beta))\right] \\ & + 2ikn_3\sin(\beta)(\rho_R e^{ik(2a-L)} - \rho_L e^{-ik(2a-L)}) \\ & + 2ik\sin(\beta)((n_1 + in_2)\tau_R + (n_1 - in_2)\tau_L)\big\} \ . \end{aligned} \tag{3.28}$$

One final remark regards the overall factor in (3.28). It is easy to realize, from the definition in (3.12), that $D_{\mathbf{U}_2}(k)$ has no poles. This implies that the factor $2e^{i\alpha}(D_{\mathbf{U}_2}(k))^{-1}$ does not contribute to the zeroes of the function $F_\lambda(k, a; S, \mathbf{U}_1)$ and can, hence, be safely discarded. We can therefore conclude that the function $F_\lambda(k, a; S, \mathbf{U}_1)$ has the same zeroes as the following function

$$h_\lambda(k, a; S, \mathbf{U}_1) = \frac{e^{-i\alpha}}{2}D_{\mathbf{U}_2}(k)F_\lambda(k, a; S, \mathbf{U}_1) \ , \tag{3.29}$$

which is the one we will utilize in order to analyze the spectral zeta function of our piston configuration.

4 The spectral zeta function and Casimir energy

The function $h_\lambda(k, a; S, \mathbf{U}_1)$ can be used to derive an expression for the spectral zeta function associated with the piston configuration which is defined in terms of the eigenvalues α of our system as follows

$$\zeta(s) = \sum_{\alpha > 0} \alpha^{-2s} \ . \tag{4.1}$$

The above zeta function is known to be convergent for $\Re(s) > D/2$ [17, 18, 32] and can be analytically continued to a meromorphic function in the whole complex plane possessing only simple poles. The spectral zeta function can be utilized to compute the Casimir energy of suitable quantum systems [8, 9, 11, 17, 18, 32], and in particular for the piston configuration under consideration in this work. In this framework, the Casimir energy of a piston is expressed as

$$E_{\mathrm{Cas}}(a) = \lim_{\epsilon \to 0} \frac{\mu^{-2\epsilon}}{2}\zeta\left(\epsilon - \frac{1}{2}, a\right) \ , \tag{4.2}$$

where μ is a parameter with the dimension of mass. In general, the spectral zeta function develops a pole at the point $s = -1/2$. Because of the presence of this pole, the limit in (4.2) leads to result

$$E_{\rm Cas}(a) = \frac{1}{2}\mathrm{FP}\,\zeta\left(-\frac{1}{2},a\right) + \frac{1}{2}\left(\frac{1}{\epsilon} + \ln\mu^2\right)\mathrm{Res}\,\zeta\left(-\frac{1}{2},a\right) + O(\epsilon)\,. \tag{4.3}$$

From the expression for the Casimir energy in (4.3) one obtains the Casimir force acting on the piston by simply differentiating with respect to the position of the piston a, that is

$$F_{\rm Cas}(a) = -\frac{\partial}{\partial a}E_{\rm Cas}(a)\,. \tag{4.4}$$

From the formulas (4.3) and (4.4) it is not very difficult to realize that the Casimir force acting on the piston is a well defined quantity only if the residue of the spectral zeta function at $s = -1/2$ is independent of position of the piston a.

The eigenvalues α of our system are only known implicitly as the positive zeroes of the function $h_\lambda(k, a; S, \mathbf{U}_1)$ through the relation $\alpha^2 = k^2 + \lambda^2$. One can, therefore, employ a contour integral representation, based on Mittag-Leffler's theorem, to write the spectral zeta function as follows [6, 7, 32]

$$\zeta(s,a) = \frac{1}{2\pi i}\sum_\lambda d(\lambda)\int_\gamma \left(k^2+\lambda^2\right)^{-s}\frac{\partial}{\partial k}\ln h_\lambda(k,a;S,\mathbf{U}_1)\,\mathrm{d}k\,, \tag{4.5}$$

valid in the region of the complex plane $\Re(s) > D/2$. Here, γ represents a contour that encloses, in the counterclockwise direction, all the positive zeroes of the function $h_\lambda(k, a; S, \mathbf{U}_1)$. In addition, $d(\lambda)$ denotes the degeneracy of the eigenvalues λ of the Laplacian on the transverse manifold N. In order to analyze the Casimir energy of the system and the corresponding force, the expression in (4.5) for $\zeta(s, a)$ needs to be analytically extended to a neighborhood of the point $s = -1/2$. The first step in the analytic continuation consists of deforming the contour γ to the imaginary axis [32]. Before performing the contour deformation, it is very important to analyze the small-k behavior of the function $h_\lambda(k, a; S, \mathbf{U}_1)$. By using the definition (3.29) and the expression (3.28) one obtains the following asymptotic behavior as $k \to 0$

$$\begin{aligned} h_\lambda(k,a;S,\mathbf{U}_1) &= \big\{8\cos(\theta)\cos(\alpha) - 8\cos(\gamma)\cos(\beta) + 4\sin(\theta)[L(\cos(\alpha)-\cos(\beta)) - 2\sin(\alpha)] \\ &+ 4(\cos(\gamma)-\cos(\theta))[a(a-L)(\cos(\alpha)-\cos(\beta)) + L\sin(\alpha)] \\ &- 4(2a-L)[(\cos(\alpha)-\cos(\beta))q_3\sin(\gamma) + (\cos(\gamma)-\cos(\theta))n_3\sin(\beta)] \\ &+ 8\sin(\beta)\sin(\gamma)(n_1q_1+n_2q_2-n_3q_3)\big\}k^2 + O(k^4)\,. \end{aligned} \tag{4.6}$$

Since $h_\lambda(k, a; S, \mathbf{U}_1)$ is of order k^2 as $k \to 0$, a simple contour deformation to the imaginary axis would allow the integral to acquire an unwanted contribution from the origin $k = 0$. In order to avoid this spurious contribution we replace the representation (4.5) of the spectral zeta function with the following one

$$\zeta(s,a) = \frac{1}{2\pi i}\sum_\lambda d(\lambda)\int_\gamma \left(k^2+\lambda^2\right)^{-s}\frac{\partial}{\partial k}\ln\left[\frac{h_\lambda(k,a;S,\mathbf{U}_1)}{k^2}\right]\mathrm{d}k\,. \tag{4.7}$$

By exploiting the fact that the function $h_\lambda(k, a; S, \mathbf{U}_1)$ satisfies the property

$$h_\lambda(ik, a; S, \mathbf{U}_1) = h_\lambda(-ik, a; S, \mathbf{U}_1)\ , \tag{4.8}$$

which can be proved by noticing that for any $w \in \mathbb{C}$, $\rho_R(-w) = \rho_L(w)$ and $\tau_R(-w) = \tau_L(w)$, the contour deformation to the imaginary axis leads to the expression

$$\zeta(s, a) = \sum_\lambda d(\lambda)\zeta_\lambda(s, a)\ , \tag{4.9}$$

where we have introduced the zeta function

$$\zeta_\lambda(s, a) = \frac{\sin(\pi s)}{\pi}\int_\lambda^\infty \left(z^2 - \lambda^2\right)^{-s} \frac{\partial}{\partial z}\ln\left[\frac{h_\lambda(iz, a; S, \mathbf{U}_1)}{z^2}\right] \mathrm{d}z\ . \tag{4.10}$$

The integral representation (4.10) is valid in the region of the complex plane $1/2 < \Re(s) < 1$. The upper bound on the region of validity is obtained by requiring the integral to be convergent at the lower limit of integration and by noticing that, as $z \to \lambda$, the integrand behaves as

$$(z^2 - \lambda^2)^{-s}\frac{\partial}{\partial z}\ln\left[\frac{h_\lambda(iz, a; S, \mathbf{U}_1)}{z^2}\right] \sim (z - \lambda)^{-s}\ . \tag{4.11}$$

As $z \to \infty$ the function $h_\lambda(iz, a; S, \mathbf{U}_1)$ displays, instead, the following behavior

$$\begin{aligned} h_\lambda(iz, a; S, \mathbf{U}_1) = D_{\mathbf{U}_2}(iz)e^{zL}\Big[&\cos(\alpha) - \cos(\beta) \\ &-z^2(\cos(\alpha) + \cos(\beta)) - 2z\sin(\alpha)\Big][1 + \varepsilon(iz, a)]\ , \end{aligned} \tag{4.12}$$

where $\varepsilon(iz, a)$ represents exponentially small terms. The expression (4.12) allows us to conclude that, as $z \to \infty$, the integrand in (4.10) behaves as

$$(z^2 - \lambda^2)^{-s}\frac{\partial}{\partial z}\ln\left[\frac{h_\lambda(iz, a; S, \mathbf{U}_1)}{z^2}\right] \sim Lz^{-2s}\ , \tag{4.13}$$

which together with the requirement that the integral representation (4.10) be convergent at the upper limit of integration, provides the lower bound $\Re(s) > 1/2$.

In order to analyze the Casimir energy and the corresponding force, we need to extend the definition of the zeta function in (4.10) to the region of the complex plane $\Re(s) \le 1/2$. This is accomplished by simply subtracting and then adding in the integral representation (4.10) a suitable number of terms of the asymptotic expansion as $z \to \infty$ of $\ln\left[z^{-2}h_\lambda(iz, a; S, \mathbf{U}_1)\right]$. By using the definition (3.12) we can write a formula which we can use as a starting point of the asymptotic expansion

$$\ln\left[z^{-2}h_\lambda(iz, a; S, \mathbf{U}_1)\right] \simeq zL - 2\ln z + \ln\Psi(z; \theta, \gamma) + \ln\Psi(z; \alpha, \beta)\ , \tag{4.14}$$

where we have discarded the exponentially small terms and we have introduced, for convenience, the function

$$\Psi(z; x, y) = m_-(x, y) - 2z\sin x - z^2\, m_+(x, y)\ , \tag{4.15}$$

with

$$m_\pm(x,y) = \cos x \pm \cos y \ . \tag{4.16}$$

From the expressions (4.14)-(4.16) it is not difficult to see that the specific form of the asymptotic expansion depends on whether or not the coefficients $m_+(\alpha,\beta)$, $\sin\alpha$, $m_+(\theta,\gamma)$, and $\sin\theta$ vanish. In order to consider all the cases simultaneously we introduce the function

$$\delta(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} , \tag{4.17}$$

and rewrite the logarithm of (4.15) as follows

$$\begin{aligned} \ln\Psi(z;x,y) &= [2-\delta(m_+(x,y))(1+\delta(\sin x))]\ln z + \tau(x,y) \\ &+ [1-\delta(m_+(x,y))]\ln\left[1+\frac{2\sin x}{m_+(x,y)z} - \frac{m_-(x,y)}{m_+(x,y)z^2}\right] \\ &+ \delta(m_+(x,y))[1-\delta(\sin x)]\ln\left[1-\frac{m_-(x,y)}{2z\sin x}\right] , \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \tau(x,y) &= [1-\delta(m_+(x,y))]\, m_+(x,y) + \delta(m_+(x,y))[1-\delta(\sin x)]\ln(2\sin x) \\ &+ \delta(m_+(x,y))\delta(\sin x)\ln[m_-(x,y)] \ . \end{aligned} \tag{4.19}$$

The large-z asymptotic expansion of the quantity in (4.18) can be obtained by following the argument presented in [39]. More explicitly one finds

$$\ln\Psi(z;x,y) = [2-\delta(m_+(x,y))(1+\delta(\sin x))]\ln z + \tau(x,y) + \sum_{n=1}^{\infty}\frac{\omega_n(x,y)}{z^n} , \tag{4.20}$$

where (cf. [39])

$$\begin{aligned} \omega_n(x,y) &= [1-\delta(m_+(x,y))]\,(-1)^{n+1}\sum_{n=0}^{[\frac{n}{2}]}\frac{2^{n-2j}\Gamma(n-j)}{j!\Gamma(n-2j+1)}(\sin x)^{n-2j}\frac{m_-^j(x,y)}{m_+^{n-j}(x,y)} \\ &- \delta(m_+(x,y))[1-\delta(\sin x)]\frac{(\cot x)^n}{n} \ . \end{aligned} \tag{4.21}$$

By exploiting the formula (4.20) it is not very difficult to write the large-z asymptotic expansion of (4.14), that is

$$\begin{aligned} \ln\left[z^{-2}h_\lambda(iz,a;S,\mathbf{U}_1)\right] &\simeq zL + \chi(\theta,\gamma,\alpha,\beta)\ln z + \tau(\theta,\gamma) + \tau(\alpha,\beta) \\ &+ \sum_{n=1}^{\infty}\frac{\omega_n(\theta,\gamma)+\omega_n(\alpha,\beta)}{z^n} , \end{aligned} \tag{4.22}$$

where we have introduced the function

$$\chi(\theta,\gamma,\alpha,\beta) = 2-\delta(m_+(\theta,\gamma))[1+\delta(\sin\theta)] - \delta(m_+(\alpha,\beta))[1+\delta(\sin\alpha)] \ . \tag{4.23}$$

The above asymptotic expansion can now be used to perform the analytic continuation of the spectral zeta function. By subtracting and then adding the first N terms of the asymptotic expansion (4.22) in the integrand of (4.10) we get

$$\zeta(s,a) = Z(s,a) + \sum_{i=-1}^{N} A_i(s,a) \ , \tag{4.24}$$

where $Z(s,a)$ is an analytic function in the region $\Re(s) > (d-N-1)/2$ and has the form

$$\begin{aligned} Z(s,a) &= \frac{\sin(\pi s)}{\pi}\sum_{\lambda} d(\lambda)\int_{\lambda}^{\infty}\left(z^2-\lambda^2\right)^{-s}\frac{\partial}{\partial z}\Bigg\{\ln\left[\frac{h_\lambda(iz,a;S,\mathbf{U}_1)}{z^2}\right] - zL \\ &- \chi(\theta,\gamma,\alpha,\beta)\ln z - \tau(\theta,\gamma) - \tau(\alpha,\beta) - \sum_{n=1}^{N}\frac{\omega_n(\theta,\gamma)+\omega_n(\alpha,\beta)}{z^n}\Bigg\}\mathrm{d}z \ . \end{aligned} \tag{4.25}$$

The remaining quantities in (4.24), i.e. $A_i(s,a)$, are obtained by integrating the terms of the asymptotic asymptotic expansion that have been added back and are meromorphic functions of s possessing only isolated simple poles. It is not difficult to prove that

$$A_{-1}(s,a) = \frac{L}{2\sqrt{\pi}\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)\zeta_N\left(s-\frac{1}{2}\right) \ , \tag{4.26}$$

$$A_0(s,a) = \frac{1}{2}\chi(\theta,\gamma,\alpha,\beta)\zeta_N(s) \ , \tag{4.27}$$

and, for $i \geq 1$,

$$A_i(s,a) = -\frac{\omega_i(\theta,\gamma)+\omega_i(\alpha,\beta)}{\Gamma(s)\Gamma\left(\frac{i}{2}\right)}\Gamma\left(s+\frac{i}{2}\right)\zeta_N\left(s+\frac{i}{2}\right) \ , \tag{4.28}$$

where in the previous expressions we have used the following definition of the spectral zeta function associated with the Laplacian Δ_N on the manifold N

$$\zeta_N(s) = \sum_{\lambda} d(\lambda)\lambda^{-2s} \ . \tag{4.29}$$

Before exploiting these results for the Casimir energy, let us remark that the above equations are also perfectly suited to compute the heat kernel coefficients for the piston setting. It is known that only the $A_j(s,a)$, $j=-1,0,1,\ldots$, contribute to the coefficients and (4.22) and (4.23) clearly show how contributions split into (θ,γ) and (α,β) dependent parts, which have been treated in [39]. Results for heat kernel coefficients will therefore simply be sums of results given in [39] and we will not present more details in this context.

We will now employ the analytically continued expression of the spectral zeta function in (4.24) and the definition in (4.2) to derive a formula for the Casimir energy of the piston. By setting $N=D$ in (4.24) we obtain a representation for the spectral zeta function valid in the region $-1 < \Re(s) < 1$ and, hence, suitable for the calculation of the Casimir energy. According to the definition (4.2) the Casimir energy is computed by setting $s = \epsilon - 1/2$ in (4.24) and by subsequently taking the limit $\epsilon \to 0$. During this limiting process, the meromorphic structure of the spectral zeta function $\zeta_N(s)$ plays an important role. In accordance with the general theory of spectral zeta functions, [29, 32] one has

$$\begin{aligned}
\zeta_N(\epsilon-n) &= \zeta_N(-n)+\epsilon\zeta_N'(-n)+O(\epsilon^2)\;, && (4.30)\\
\zeta_N\left(\epsilon+\frac{d-k}{2}\right) &= \frac{1}{\epsilon}\mathrm{Res}\,\zeta_N\left(\frac{d-k}{2}\right)+\mathrm{FP}\,\zeta_N\left(\frac{d-k}{2}\right)+O(\epsilon)\;, && (4.31)\\
\zeta_N\left(\epsilon-\frac{2n+1}{2}\right) &= \frac{1}{\epsilon}\mathrm{Res}\,\zeta_N\left(-\frac{2n+1}{2}\right)+\mathrm{FP}\,\zeta_N\left(-\frac{2n+1}{2}\right)+O(\epsilon)\;, && (4.32)
\end{aligned}$$

where $n\in\mathbb{N}_0$ and $k=\{0,\dots,d-1\}$. Since $Z(s,a)$ is an analytic function for $-1<\Re(s)<1$, we can simply set $s=-1/2$ in its expression. For the terms $A_i(s,a)$ we find instead (c.f. [20])

$$A_{-1}\left(\epsilon-\frac{1}{2},a\right)=\frac{L\,\zeta_N(-1)}{4\pi\varepsilon}+\frac{L}{4\pi}\left[\zeta_N'(-1)+(2\ln 2-1)\zeta_N(-1)\right]+O(\varepsilon)\;, \qquad (4.33)$$

$$A_0\left(\epsilon-\frac{1}{2},a\right) = \frac{1}{2}\chi(\theta,\gamma,\alpha,\beta)\left[\frac{1}{\epsilon}\mathrm{Res}\,\zeta_N\left(-\frac{1}{2}\right)+\mathrm{FP}\,\zeta_N\left(-\frac{1}{2}\right)\right]+O(\varepsilon)\;, \qquad (4.34)$$

and

$$\begin{aligned}
\sum_{i=1}^{D}A_i\left(\epsilon-\frac{1}{2},a\right) &= \frac{1}{\epsilon}\Bigg[\frac{\omega_1(\theta,\gamma)+\omega_1(\alpha,\beta)}{2\pi}\zeta_N(0)+\sum_{i=2}^{D}\frac{\omega_i(\theta,\gamma)+\omega_i(\alpha,\beta)}{2\sqrt{\pi}\Gamma\left(\frac{i}{2}\right)}\Gamma\left(\frac{i-1}{2}\right)\\
&\times\mathrm{Res}\,\zeta_N\left(\frac{i-1}{2}\right)\Bigg]+\frac{\omega_1(\theta,\gamma)+\omega_1(\alpha,\beta)}{2\pi}\left[\zeta_N'(0)+2(\ln 2-1)\zeta_N(0)\right]\\
&+\sum_{i=2}^{D}\frac{\omega_i(\theta,\gamma)+\omega_i(\alpha,\beta)}{2\sqrt{\pi}\Gamma\left(\frac{i}{2}\right)}\Gamma\left(\frac{i-1}{2}\right)\Bigg[\mathrm{FP}\,\zeta_N\left(\frac{i-1}{2}\right)+\left(2-\gamma-2\ln 2+\Psi\left(\frac{i-1}{2}\right)\right)\\
&\times\mathrm{Res}\,\zeta_N\left(\frac{i-1}{2}\right)\Bigg]+O(\varepsilon)\;. \qquad (4.35)
\end{aligned}$$

The above results together with the formula (4.3) allow us to write an explicit expression for the Casimir energy of the piston configuration as follows

$$\begin{aligned}
E_{\mathrm{Cas}}(a) &= \frac{1}{2}\left(\frac{1}{\varepsilon}+\ln\mu^2\right)\Bigg[\frac{L}{4\pi}\zeta_N(-1)+\frac{1}{2}\chi(\theta,\gamma,\alpha,\beta)\mathrm{Res}\,\zeta_N\left(-\frac{1}{2}\right)+\frac{\omega_1(\theta,\gamma)+\omega_1(\alpha,\beta)}{2\pi}\zeta_N(0)\\
&+ \sum_{i=2}^{D}\frac{\omega_i(\theta,\gamma)+\omega_i(\alpha,\beta)}{2\sqrt{\pi}\Gamma\left(\frac{i}{2}\right)}\Gamma\left(\frac{i-1}{2}\right)\mathrm{Res}\,\zeta_N\left(\frac{i-1}{2}\right)\Bigg]+\frac{1}{2}Z\left(-\frac{1}{2},a\right)\\
&+ \frac{L}{8\pi}\left[\zeta_N'(-1)+(2\ln 2-1)\zeta_N(-1)\right]+\frac{1}{4}\chi(\theta,\gamma,\alpha,\beta)\mathrm{FP}\,\zeta_N\left(-\frac{1}{2}\right)\\
&+ \frac{\omega_1(\theta,\gamma)+\omega_1(\alpha,\beta)}{2\pi}\left[\zeta_N'(0)+2(\ln 2-1)\zeta_N(0)\right]+\sum_{i=2}^{D}\frac{\omega_i(\theta,\gamma)+\omega_i(\alpha,\beta)}{2\sqrt{\pi}\Gamma\left(\frac{i}{2}\right)}\Gamma\left(\frac{i-1}{2}\right)\\
&\times \left[\mathrm{FP}\,\zeta_N\left(\frac{i-1}{2}\right)+\left(2-\gamma-2\ln 2+\Psi\left(\frac{i-1}{2}\right)\right)\mathrm{Res}\,\zeta_N\left(\frac{i-1}{2}\right)\right]+O(\varepsilon)\;. \qquad (4.36)
\end{aligned}$$

The above expression clearly shows that the Casimir energy of the piston configuration is, in general, not a well-defined quantity. The ambiguity in the force is proportional to $\zeta_N(-1)$, $\zeta_N(0)$, and the $\mathrm{Res}\,\zeta_N\left((i-1)/2\right)$ with $i = 0, \dots, D$. These quantities depend, in turn, only on the geometry of the transverse manifold N and the boundary conditions imposed on the fields propagating on N through the coefficients $a^N_{k/2}$ of the asymptotic expansion of the heat kernel associated with Δ_N. This is due to the well-known relations with $n \in \mathbb{N}_0$ [29, 32]

$$\begin{aligned}
\Gamma\left(\frac{d-k}{2}\right) \mathrm{Res}\,\zeta_N\left(\frac{d-k}{2}\right) &= a^N_{\frac{k}{2}}\ , \\
\Gamma\left(-\frac{2n+1}{2}\right) \mathrm{Res}\,\zeta_N\left(-\frac{2n+1}{2}\right) &= a^N_{\frac{d+2n+1}{2}}\ , \\
\frac{(-1)^n}{\Gamma(n+1)}\zeta_N(-n) &= a^N_{\frac{d}{2}+n}\ .
\end{aligned} \tag{4.37}$$

While the Casimir energy is generally ambiguous, the Casimir force acting on the piston is a well-defined quantity since the terms responsible for the ambiguity in the energy do not depend on the position of the piston. In fact, by using (4.36) and the definition provided in (4.4) we obtain the following simple expression for the Casimir force acting on the piston

$$F_{\mathrm{Cas}}(a) = -\frac{1}{2}\frac{d}{da}Z\left(-\frac{1}{2}, a\right) = \frac{1}{2\pi}\sum_{\lambda} d(\lambda)\frac{d}{da}J_\lambda(a). \tag{4.38}$$

where $Z\left(-\frac{1}{2}, a\right)$ is given by formula (4.25), and we have introduced the notation

$$J_\lambda(a) \equiv \int_\lambda^\infty \left(z^2-\lambda^2\right)^{\frac{1}{2}}\partial_z \left[\ln(h_\lambda(iz,a;S,\mathbf{U}_1) - As(z;S,\mathbf{U}_1)\right]\mathrm{d}z\ , \tag{4.39}$$

with $As(z;S,\mathbf{U}_1)$ being the asymptotic terms subtracted in equation (4.25); note, that these terms do not depend on the position of the piston a. If we integrate by parts in $J_\lambda(a)$, and take into account that the boundary terms cancel, we can write

$$J_\lambda(a) = -\int_\lambda^\infty \frac{z}{\left(z^2-\lambda^2\right)^{\frac{1}{2}}}\left[\ln(h_\lambda(iz,a;S,\mathbf{U}_1) - As(z;S,\mathbf{U}_1)\right]\mathrm{d}z. \tag{4.40}$$

Hence the Casimit force can finally be written as

$$F_{\mathrm{Cas}}(a) = -\frac{1}{2\pi}\sum_\lambda d(\lambda)\int_0^\infty \partial_a\left[\ln(h_\lambda(i\sqrt{w^2+\lambda^2},a;S,\mathbf{U}_1)\right]\mathrm{d}w, \tag{4.41}$$

after performing the change of variables $w = \sqrt{z^2-\lambda^2}$. The formula (4.41) for the Casimir force will be used in the next section to generate graphs of the Casimir force on the piston for different geometries and boundary conditions.

5 Casimir force for particular piston geometries

It is clear from the expression (4.41) that the Casimir force acting on the piston can be obtained numerically once the manifold N and the boundary conditions have been specified.

In this section we consider the following two manifolds N: the two-dimensional disk and the d-dimensional sphere. Before proceeding with these two cases we would like to make a remark about the piston configuration constructed from a generalized torus. This particular piston configuration is obtained by imposing periodic boundary conditions at $x = 0$ and $x = L$. In this case the left edge and the right one of the piston configuration are identified. Periodic boundary conditions can be obtained by setting $\alpha = \pi/2$, $\beta = \pm\pi/2$, and $n_1 = \mp 1$ in $\mathbf{U}_1$ [39]. With this particular choice of parameters, it is not difficult to realize that the terms with the dependence on the position of the piston a in $h_\lambda(k, a; S, \mathbf{U}_1)$ in (3.29) vanish identically. This implies that in a generalized torus the piston itself does not incur any force. This result should be expected because identifying the two edges of the piston is equivalent to reducing the piston configuration to a single chamber. More generally, for any configuration where $\alpha = \pi/2, 3\pi/2$, $\beta = \pm\pi/2$, and $n_3 = 0$ all the terms dependent on the position of the piston a that appear in $h_\lambda(k, a; S, \mathbf{U}_1)$ (see equation(3.29)) vanish identically. We can, therefore, conclude that in these situations there is *no* Casimir force acting on the piston. In addition, if the selfadjoint extension that characterises the piston gives rise to an opaque piston wall, i.e. $r_R = r_L = 0$ the Casimir force vanishes as well since all the a-dependent terms in $h_\lambda(k, a; S, \mathbf{U}_1)$ are proportional to either ρ_R or ρ_L. Nevertheless, the special case of $\alpha = \pi/2 = -\beta$ and $n_3 = 0 \Rightarrow n_1 = \cos(\xi)$, $n_2 = \sin(\xi)$ is of great interest when the cross section of the piston geometry degenerates to a point. In this case we interpret the free parameter ξ as the quasi-momentum of a one-dimensional crystal lattice where the lattice points are mimicked by identical point-supported potentials, generalising the result of reference [10]. For the examples that we consider in this section we will assume, for simplicity, that $L = 1$.

5.1 The d-dimensional sphere

In this example we consider the base manifold to be a d-dimensional sphere. The eigenvalues of the Laplacian Δ_N on a d-dimensional sphere are known to be

$$\lambda^2 = l(l + d - 1) \, , \tag{5.1}$$

with $l \in \mathbb{N}_0$, and the associated degeneracy has the form

$$d(\nu) = (2l + d - 1)\frac{(l + d - 2)!}{l!(d - 1)!} \, . \tag{5.2}$$

In order to obtain specific graphs of the Casimir force on the piston as a function of the position a we set $d = 2$ and we use the eigenvalues and degeneracy (5.1) and (5.2) in the expression (4.38). Once particular boundary conditions are chosen, a numerical analysis of the Casimir force (4.41) can be performed. It is important to point out that the dimension $d = 2$ has been chosen only for simplicity and that our formula for the Casimir force (4.38) holds for any dimension d. Figures 2-5 show the behavior of the Casimir force on the piston for specific boundary conditions imposed on the field.

The figures have been generated by utilizing a two colors scheme. The blue and red areas denote those regions in the space of parameters in which the Casimir force is negative, respectively, positive. The shade of the color gives a measure of the magnitude of the force on the piston: The darker the color, the smaller the magnitude, the lighter the color the higher the magnitude. The white areas appearing in graphs are those in which the magnitude of the force exceeds the range of the graph. However, the white areas at the two edges of the piston, $x = 0$ and $x = 1$, reflect the fact that the Casimir force grows without bounds as one approaches the edges. The growth is positive (negative) if the white area near one of the edges appears right next to a red (blue) region.

Taking into account equation (4.41) we observe some remarkable behaviors:

1. In all cases we are considering, except for the ones in Fig. 3, we have that $n_3 = q_3 = 0$. It is not difficult to realize that for $n_3 = q_3 = 0$, the function $h_\lambda(k, a; S, \mathbf{U}_1)$ is proportional to $\cos[k(2a - L)]$, and, hence, is an even function with respect to the midpoint $a = L/2$. Obviously the Casimir force, being the k-integral of the logarithmic derivative of $h_\lambda(k, a; S, \mathbf{U}_1)$, is an odd function with respect to the midpoint $a = L/2$. This implies, in particular, that in these cases the Casimir force is always zero at, at least, $a = L/2$. Let us point out that the force can vanish at other points of the interval, however these points of vanishing Casimir energy need to appear in pairs which are symmetric with respect to $a = L/2$. This behavior can be clearly observed from the graphs. For the cases in Fig. 3 we have, instead, $n_3 = 0$, $q_3 = 1$, and $\gamma = \theta = \pi/2$. In these cases the function $h_\lambda(k, a; S, \mathbf{U}_1)$ becomes proportional to $\sin[k(2a - L)]$. By following the argument outlined in the previous paragraph, the Casimir force is, then, an even function of a with respect to $a = L/2$. This means that the Casimir force at $a = L/2$ does not have to be necessarily zero.

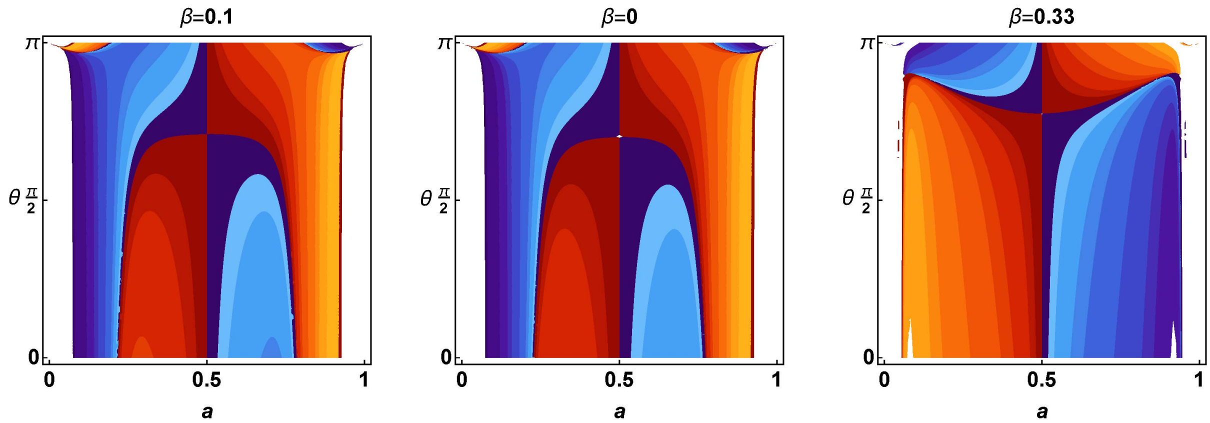

Figure 2: (color online) Behavior of the Casimir force (4.41) as a function of the parameter θ of the piston characterised by $\mathbf{U}_2$ and the position a of the piston, for different values of β. The rest of the parameters are fixed to $L = 1$, $\alpha = 2.8$, $n_1 = q_1 = 1$, and $\gamma = 0$. The curves separating positive force (red color scale) and negative force (blue color scale) correspond to zero Casimir force situations.

2. There exist regions in the space of free parameters for which the resulting Casimir force on the piston is either non-negative or non-positive for all values of the position a. In these situations the Casimir force will tend to move the piston to the right edge (if the

force is non-negative) or to the left edge (if the force is non-positive). An example of this behavior can be seen, for instance, in the first plot of Fig. 3. For $\beta = 0$ the force is always negative and, hence, the piston is moved towards the left edge. In the situation we are considering, if any points of zero force are present, they would represent points of unstable equilibrium for the piston.

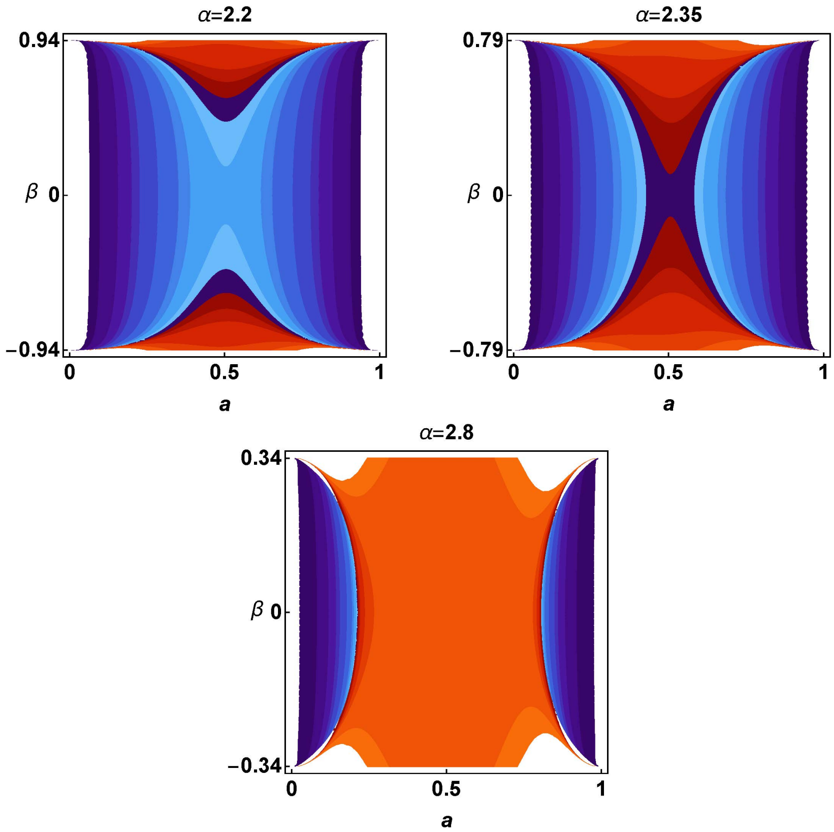

Figure 3: (color online) Behavior of the Casimir force (4.41) as a function of the parameter β of the piston characterised by $\mathbf{U}_1$ and the position a of the piston, for different values of α. The rest of the parameters are fixed to $L = 1$, $\theta = \gamma = \pi/2$, $n_1 = q_3 = 1$, and $\gamma = 0$. The curves separating positive force (red color scale) and negative force (blue color scale) correspond to zero Casimir force situations.

3. In Figs. 2, 4 and 5 the Casimir force is, as explained earlier, an odd function of a with respect to $a = L/2$. In these situations the points of vanishing force, which necessarily exist, can be points of either stable or unstable equilibrium. Let $\epsilon > 0$. If a_0 is a point for which $F_{\rm Cas}(a_0) = 0$, then a_0 is a point of *stable* equilibrium for the piston if $F_{\rm Cas}(a_0 - \epsilon) > 0$ and $F_{\rm Cas}(a_0 + \epsilon) < 0$. On the other hand, if $F_{\rm Cas}(a_0 - \epsilon) < 0$ and $F_{\rm Cas}(a_0 + \epsilon) > 0$ then a_0 is a point of *unstable* equilibrium for the piston. Since the Casimir force in Figs. 2, 4 and 5 is an odd function of a, then we must have an

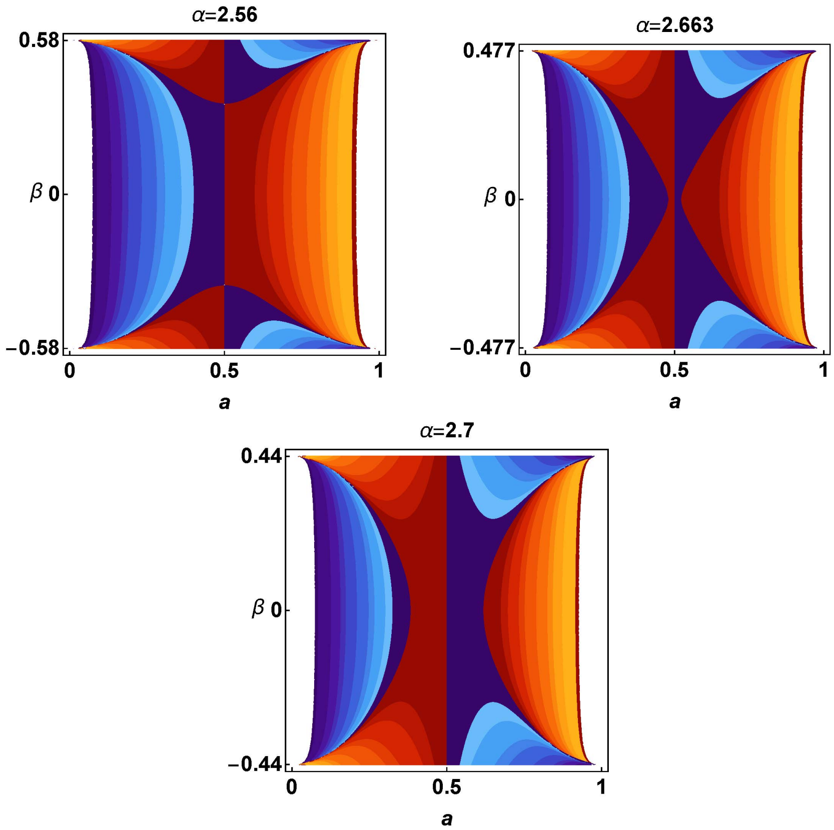

Figure 4: (color online) Behavior of the Casimir force (4.41) as a function of the parameter β of the piston characterised by $\mathbf{U}_1$ and the position a of the piston, for different values of α. The rest of the parameters are fixed to $L = 1$, $\theta = 1.5$, $\gamma = 0$, $n_1 = q_2 = 1$, and $\gamma = 0$. The curves separating positive force (red color scale) and negative force (blue color scale) correspond to zero Casimir force situations.

odd number of points where the force vanishes. These points of stable and unstable equilibrium must alternate as it can be clearly seen in the graphs.

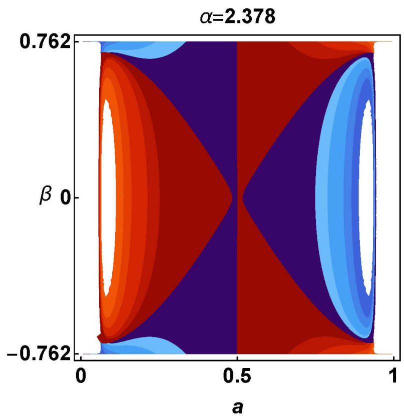

Figure 5: (color online) Behavior of the Casimir force (4.41) as a function of the parameter β of the piston characterised by $\mathbf{U}_1$ and the position a of the piston. The rest of the parameters are fixed to $L = 1$, $\alpha = 2.378$, $\theta = 2$, $\gamma = 1.14$, $n_1 = q_2 = 1$, and $\gamma = 0$. The curves separating positive force (red color scale) and negative force (blue color scale) correspond to zero Casimir force situations.

5.2 Disk

As a further example we consider the transverse manifold N to be a disk of unit radius. The eigenfunctions of the Laplacian Δ_N can be found by using separation of variables once Δ_N is written in polar coordinates (r, ϑ). By imposing periodicity of the solution with respect to the angular variable ϑ and Dirichlet boundary conditions at $r = 1$, the eigenvalues can be easily found to be λ_{kn}^2 which can be determined as the zeroes of the Bessel function of the first kind

$$J_n(\lambda_{kn}) = 0 \; . \tag{5.3}$$

One can show that the degeneracy of the eigenvalues satisfies the relations $d(\lambda_{k0}) = 1$ and $d(\lambda_{kn}) = 2$ for $n \geq 1$. The zeroes of the Bessel function of the first kind with integer order are well known and can be found in tables or with the help of a computer program. The figures for the Casimir force look qualitatively the same as for the sphere presented in the previous subsection and we therefore do not include any more details.

6 Concluding remarks

In this work we have studied the Casimir energy and force for a scalar field propagating in a piston configuration of the type $I \times N$. The field is constrained by boundary conditions that lead to a selfadjoint boundary value problem for the Laplacian on the piston. We have focused, here, primarily on all non-negative selfadjoint extensions that can be described by matrices in the subgroup $U(2) \times U(2)$ of $U(4)$. In particular we have studied the most general boundary conditions that relate the edges $x = 0$ and $x = L$ and the two opposite edges of the piston itself. By using scattering theory we were able to find an expression whose zeroes implicitly determined the eigenvalues of the Laplacian with the general boundary conditions considered. This secular equation has been used as a starting point of an integral representation for the spectral zeta function which was subsequently analytically continued to a larger region of the complex plane. Moreover, the use of non-relativistic scattering theory enables one to understand the physics behind the Casimir force in terms of the well known non-relativistic scattering theory in one-dimension. The Casimir energy associated with the piston configuration and the corresponding force have been computed by exploiting the analytically continued expression of the spectral zeta function. The formula that we found for the Casimir energy and force is written in terms of the spectral zeta function associated with the Laplacian on the transverse manifold N and is valid for any d-dimensional compact Riemannian manifold N with or without boundary. We have found the Casimir energy for a piston configuration is, in general, not a well-defined quantity with the ambiguity depending on the geometry of the manifold N. This aspect of the Casimir energy on piston configuration has already been observed in the literature (see e.g. [8, 20]). While the energy might not be well-defined, the force on the piston is free of ambiguities. The general expression we obtained for the force allowed us to derive the graphs presented in the previous section for specific manifolds N and a number of particular boundary conditions. It is important to point out that our formula (4.38) can be used to perform a numerical analysis of the Casimir force for any suitable transverse manifold N and for any allowed values of the parameters in $\mathbf{U}_1$ and $\mathbf{U}_2$ that characterize the boundary conditions.

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References

[1] A. A. Actor and I. Bender. Casimir effect for soft boundaries. *Phys. Rev.*, D52:3581, 1995.

[2] M. Asorey, A. Ibort, and G. Marmo. Global theory of quantum boundary conditions and topology change. *Int. J. Mod. Phys.*, A20:1001, 2005.

[3] M. Asorey and J. M. Muñoz-Castañeda. Attractive and repulsive Casimir vacuum energy with general boundary conditions. *Nucl. Phys.*, B874:852–876, 2013.

[4] G. Barton. Casimir piston and cylinder, perturbatively. *Phys. Rev.*, D73:065018, 2006.

[5] M. Beauregard, G. Fucci, K. Kirsten, and P. Morales. Casimir effect in the presence of external fields. *J. Phys. A*, 46:115401, 2013.

[6] M. Bordag, E. Elizalde, and K. Kirsten. Heat kernel coefficients of the Laplace operator on the D-dimensional ball. *J. Math. Phys.*, 37:895, 1996.

[7] M. Bordag, B. Geyer, K. Kirsten, and E. Elizalde. Zeta function determinant of the Laplace operator on the D-dimensional ball. *Comm. Math. Phys.*, 179:215, 1996.

[8] M. Bordag, G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko. *Advances in the Casimir effect.* International Series of Monographs in Physics, 145. Oxford University Press, 2009.

[9] M. Bordag, U. Mohideen, and V. M. Mostepanenko. New developments in the Casimir effect. *Phys. Rep.*, 353:1, 2001.

[10] M. Bordag, J. M. Muñoz-Castañeda, and L. Santamaría-Sanz. Vacuum energy for generalized dirac combs at t = 0. *Frontiers in Physics*, 7:38, 2019.

[11] A. A Bytsenko, G. Cognola, and E. Elizalde. *Analytic aspects of quantum fields.* World Scientific, Singapore, 2003.

[12] H. B. Casimir. On the attraction between two perfectly conducting plates. *Proc. K. Ned. Akad. Wet.*, 51:793, 1948.

[13] R. M. Cavalcanti. Casimir force on a piston. *Phys. Rev.*, D69:065015, 2004.

[14] A. Edery. Multidimensional cut-off technique, odd-dimensional Epstein zeta functions and Casimir energy for massless scalar fields. *J. Phys. A*, 39:685, 2006.

[15] A. Edery. Casimir piston for massless scalar field in three dimensions. *Phys. Rev.*, D75:105012, 2007.

[16] A. Edery and V. N. Marachevsky. Compact dimensions and the Casimir effect: the Proca connection. *J. High Energy Phys.*, 12:035, 2008.

[17] E. Elizalde. *Ten Physical Applications of the Spectral Zeta Function.* Springer-Verlag, Berlin, 1995.

[18] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and Zerbini S. *Zeta Regularization Techniques with Applications.* World Scientific, Singapore, 1994.

[19] E. Elizalde and A. Romeo. One-dimensional Casimir effect perturbed by an external field. *J. Phys. A*, 30:5393, 1997.

[20] G. Fucci. Casimir pistons with general boundary conditions. *Nucl. Phys.*, B891:676–699, 2015.

[21] G. Fucci. The Casimir effect for thick pistons. *Int. J. Mod. Phys.*, A31:1680012, 2016.

[22] G. Fucci. The Casimir effect for pistons with transmittal boundary conditions. *Int. J. Mod. Phys.*, A32:1750182, 2017.

[23] G. Fucci and K. Kirsten. Conical Casimir pistons with hybrid boundary conditions. *J. Phys. A*, 44:295403, 2011.

[24] G. Fucci and K. Kirsten. The Casmir effect for conical pistons. *J. High Energy Phys.*, 03:016, 2011.

[25] G. Fucci and K. Kirsten. The Casimir effect for generalized piston geometries. *Int. J. Mod. Phys.*, A27:1260008, 2012.

[26] G. Fucci and K. Kirsten. Functional determinants and Casimir energy in higher dimensional spherically symmetric background potentials. *J. Phys. A*, 49:275203, 2016.

[27] G. Fucci, K. Kirsten, and P. Morales. Pistons modeled by potentials. In S. Odintsov, D. Sáez-Gómez, and S. Xambó, editors, *Cosmology, Quantum Vacuum, and Zeta Functions*, pages 313–322. Springer-Verlag, 2011.

[28] A. Galindo and P. Pascual. *Quantum mechanics I.* Texts and Monographs in Physics. Springer-Verlag, 1990.

[29] P. B. Gilkey. *Invariance theory the heat equation and the Atiyah-Singer index theorem.* CRC Press, Boca Raton, 1995.

[30] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio. Attractive Casimir forces in a closed geometry. *Phys. Rev. Lett.*, 95:250402, 2005.

[31] M. P. Hertzberg, R. L. Jaffe, M. Kardar, and A. Scardicchio. Casimir forces in a piston geometry at zero and finite temperatures. *Phys. Rev.*, D76:045016, 2007.

[32] K. Kirsten. *Spectral Functions in Mathematical Physics.* CRC Press, Boca Raton, 2001.

[33] K. Kirsten and S. A. Fulling. Kaluza-Klein models as pistons. *Phys. Rev.*, D79:065019, 2009.

[34] X.-Z. Li, H.-B. Cheng, J.-M. Li, and X.-H. Zhai. Attractive and repulsive nature of the Casimir force in a rectangular cavity. *Phys. Rev.*, D56:2155, 1997.

[35] V. N. Marachevsky. Casimir interaction of two plates inside a cylinder. *Phys. Rev.*, D75:085019, 2007.

[36] K. A. Milton. *The Casimir effect: physical manifestations of zero-point energy.* World Scientific Publishing, 2001.

[37] K. A. Milton, J. Wagner, and K. Kirsten. Casimir effect for a semitransparent wedge and an annular piston. *Phys. Rev.*, D80:125028, 2009.

[38] P. Morales and K. Kirsten. Semitransparent pistons. *Int. J. Mod. Phys.*, A25:2196, 2010.

[39] J. M. Muñoz-Castañeda, K. Kirsten, and M. Bordag. QFT over the finite line. Heat kernel coefficients, spectral zeta functions and selfadjoint extensions. *Lett. Math. Phys.*, 105(4):523–549, 2015.

[40] G. Plunien, B. Müller, and W. Greiner. The Casimir effect. *Phys. Rep.*, 134:87, 1986.

[41] T. Tilma, M. Byrd, and Sudarshan E. C. G. A parametrization of bipartite systems based on $SU(4)$ Euler angles. *J. Phys. A*, 35:10445, 2002.