Exact results for interaction quench in Wigner phase space

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Abstract

Time evolution of the Wigner distribution function of an initially untraped interacting one-dimensional quantum gas following an interaction quench is examined. Considering the scenario of a quench at \( t = 0 \) from a nonzero to zero (free particles) value of the interaction strength, we derive a simple relationship between the dynamical Wigner distribution function and its initial value. The case of an initially two-particle system interacting through an attractive contact potential is presented. The aforementioned relationship is extended to the case of harmonically trapped quantum gas, showing that if \( \omega \) is the frequency of the confining oscillator trap, the system responds to the interaction quench through a periodic variation on time (with frequency \( \omega_B = 2\omega \)) of its dynamical Wigner distribution function. To exhibit how the interactions affect the phase space dynamics, a particular system of hard core bosons is considered in which we display the ballistic and non-ballistic dynamical Wigner functions after the quantum quench.

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I. INTRODUCTION

The description of time evolution properties of quantum many body systems is presently a fundamental topic of research. The amazing development of trapping and cooling techniques have led to experimental realization of quantum many-body systems consisting of ultracold atomic gases where atoms are confined in traps [1, 2]. These artificial many-body systems can be produced and loaded in various geometric traps. Such experimental progress allowed a full control of the external parameters in the Hamiltonian governing the quantum system dynamics. An interesting issue in the field of ultracold quantum gases is to study the time evolution of a non-equilibrium situation generated through a quantum quench, which consists of a sudden change of the Hamiltonian parameters (for example a change of the harmonic trap frequency or a change in the interaction strength between the atoms of the gas through Feshbach resonance). A quantum quench is the easiest way to drive a system to non-equilibrium: the system is supposed to be in its Hamiltonian ground state until time \( t = 0 \), when the sudden change of a coupling leads to a new Hamiltonian according to which the system evolves for \( t > 0 \). On the theoretical side, significant advances have been carried out in understanding fundamental concepts in the non-equilibrium dynamics of quantum many-body system. Among these ideas is the link between quantum dynamics and quantum chaos [3, 4] and the emergence of a new ensemble in Statistical Mechanics called generalized Gibbs ensemble, which is a more general concept than the usual grand-canonical ensemble and turns out to be a powerful tool in the prediction of relaxation processes for certain integrable one-dimensional systems [5–11].

In the present work we investigate the quantum dynamics of an ultra-cold system of \( N \) atoms with equal mass \( m \) following an interaction quench from finite to zero interaction strength. We start by writing down the underlying Hamiltonians before and after the interaction quench. For \( t < 0 \) the gas is in equilibrium and its many-body Hamiltonian is

\[
H_1 = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \nabla_i^2 \right) + \sum_{i=1}^{N} V_i(\mathbf{r}_i) + \frac{1}{2} \sum_{i \neq j}^{N} \sum_{j=1}^{N} v(\mathbf{r}_i, \mathbf{r}_j),
\]

where \( V_i(\mathbf{r}_i) \) is an external confining potential and \( v(\mathbf{r}_i, \mathbf{r}_j) \) is a two-body atom-atom interaction. We denote the initial reduced one-body-density matrix by \( \rho(\mathbf{r}, \mathbf{r}'; 0) \). At \( t = 0 \), the
interactions are turned off and the many-body Hamiltonian is given by

\[ H_2 = \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \nabla_i^2 \right) + \sum_{i=1}^{N} V_2(r_i), \tag{2} \]

where we are assuming that the external potential may be different before and after the interaction quench. We want to study the subsequent dynamics of the system in phase space. For this quench scenario we relate the so-called dynamical Wigner distribution function to its initial value just before the quench. Our interest in the Wigner distribution function [12] is motivated by the fact that it provides a useful tool to study various properties of many-body systems. Besides, it is well known that it allows a reformulation of quantum mechanics in terms of classical concepts [13, 14], and it is also used to generate semi-classical approximations [15, 16]. Although other distribution functions exist, the Wigner distribution function has the virtue of its mathematical simplicity. Nevertheless, it may take negative values as a manifestation of its quantum nature, and therefore it does not represent a true probability but a quasi-probability distribution [15, 17]. Wigner distribution functions have been used in various contexts, as in cold atomic gases [1, 2], quantum optics [18], quantum information [19], quantum chaos [20], and in the study of non-equilibrium dynamics generated by the perturbation of a Fermi gas system [21]. Also, the Wigner distribution function of the noninteracting limit of a Fermi gas system at zero and nonzero temperatures has been the subject of recent studies [22–24]. The Wigner distribution function is defined as the Fourier transform of the density matrix \( \rho(x_1, x_2; t) \equiv \rho(x+s/2, x-s/2; t) \) with respect to relative variable \( s = x_1 - x_2 \), where \( x = (x_1 + x_2)/2 \) is the centre of mass coordinate. To be precise, we have

\[ W(x, p, t) = \int_{-\infty}^{\infty} ds \rho(x + \frac{s}{2}, x - \frac{s}{2}; t) e^{-ips/\hbar}, \tag{3} \]

which is a function of phase space variables \( (x, p) \). The inverse transformation reads

\[ \rho(x + \frac{s}{2}, x - \frac{s}{2}; t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} W(x, p; t) e^{ips/\hbar}. \tag{4} \]

Setting \( s = 0 \) in Eq. (4), one recovers the spatial local density

\[ \rho(x, t) := \rho(x, x; t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} W(x, p; t), \tag{5} \]

normalized to the total particle number as \( \int_{-\infty}^{\infty} \rho(x, t) dx = N \). Integrating \( W(x, p; t) \) over the whole real line allows to obtain the momentum distribution

\[ n(p, t) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\hbar} W(x, p; t), \quad \text{such that} \quad \int_{-\infty}^{\infty} n(p, t) dp = N. \tag{6} \]
The structure of the paper is as follows. In Section II we consider the case of an initially untraped one dimensional interacting system subjected to an interaction quench. The resulting ballistic dynamics is examined in phase space. We derive a relationship between the dynamical Wigner phase space density at time $t > 0$ and its initial value before the quench. As an application, we analyze the case of a two-particle system initially interacting through two different zero-range interactions of Dirac delta type. In Section III we generalize the study to a harmonic trap interaction quench. A simple relationship is obtained between the Wigner phase space density at arbitrary time $t > 0$ and its initial value before the quench. This relation is of particular interest when evaluated at specific time $t = T/4$, corresponding to a quarter of the oscillator trap period. We will show that, at this time the momentum density of the initially confined interacting system is exactly mapped to a spatial density distribution. In order to observe how the dynamical Wigner function following a quench is affected by interactions, we consider a system of one dimensional Tonks-Girardeau gas suffering a non-ballistic expansion and we compare the resulting Wigner function with the one obtained for a ballistic expansion. Some final conclusions put an end to the paper in Section IV.

II. INTERACTION QUENCH DYNAMICS IN PHASE SPACE FOR AN INITIALLY UNTRAPED INTERACTING SYSTEM

In this section let us consider the situation where both external potentials $V_1$ and $V_2$ are vanishing in Eqs. (1) and (2) and the study is restricted to one-dimensional interacting particles system. The time evolution of the resulting reduced one-body density matrix after a finite time of free expansion $t > 0$ is then given by the transformation law

$$\rho(x_1, x_2; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x_1, \xi_1; t) \rho(\xi_1, \xi_2; 0) U^*(x_2, \xi_2; t) \, d\xi_1 d\xi_2. \quad (7)$$

Here $U(x, \xi; t) = \langle x | e^{-iH_0t/\hbar} | \xi \rangle$ with $H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$. The matrix element $U(x, \xi; t)$ is the free particle Feynman propagator in the configuration space, given by [25]

$$U(x, \xi; t) = \frac{m}{2\pi \hbar t} \exp \left( \frac{i m}{2\hbar t} (x - \xi)^2 \right). \quad (8)$$

If we substitute this expression into Eq. (7) we find

$$\rho(x_1, x_2; t) = \frac{m}{2\pi \hbar t} \int_{-\infty}^{\infty} e^{i \frac{\pi}{\hbar t} [(x_1 - \xi_1)^2 - (x_2 - \xi_2)^2]} \rho(\xi_1, \xi_2; 0) \, d\xi_1 d\xi_2, \quad (9)$$
which can be rewritten as
\[
\rho(x_1, x_2; t) = \frac{m}{2\pi\hbar} e^{i\frac{\hbar}{2m}(x_1^2 - x_2^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{\hbar}{2m}(\xi_1 - \xi_2)(\xi_1 + \xi_2) - 2x_1\xi_1 + 2x_2\xi_2}) \rho(\xi_1, \xi_2; 0) \, d\xi_1 d\xi_2.
\]  
(10)

Now it is more convenient to introduce the center-of-mass and relative coordinates, respectively defined by \(z = (\xi_1 + \xi_2)/2\) and \(z' = \xi_1 - \xi_2\). Then we can write
\[
\rho(x_1, x_2; t) = \frac{m}{2\pi\hbar} e^{i\frac{\hbar}{2m}(x_1^2 - x_2^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{\hbar}{2m}z'} e^{-i\frac{\hbar}{2m}(x_1 - x_2)z'} e^{-i\frac{\hbar}{2m}(x_1 + x_2)z'} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) \, dz \, dz'.
\]  
(11)

In the following we shall examine the time evolution of the above one-body density matrix in the Wigner representation.

To proceed with computing the Wigner distribution function, we insert Eq. (11) into (3), and we may write
\[
W(x, p; t) = \frac{m}{2\pi\hbar} \int_{-\infty}^{\infty} ds \, e^{i\frac{\hbar}{2m}xs} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{\hbar}{2m}zz'} e^{-i\frac{\hbar}{2m}zs} e^{-ips/\hbar} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) \, dz \, dz'.
\]
\[
= \frac{m}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{\hbar}{2m}zz'} e^{-i\frac{\hbar}{2m}z's} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) \, dz \, dz' \int_{-\infty}^{\infty} ds \, e^{i\frac{m}{\hbar}(p - \frac{r}{m})^2}.
\]

Carrying out the integration on the variable \(s\), we obtain
\[
W(x, p; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{\hbar}{2m}zz'} e^{-i\frac{\hbar}{2m}z's} \delta\left(x - \frac{pt}{m} - z\right) \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) \, dz \, dz'.
\]  
(12)

which after integration over \(z\) reduces to
\[
W(x, p; t) = \int_{-\infty}^{\infty} \rho(x - \frac{pt}{m} + \frac{z'}{2}, x - \frac{pt}{m} - \frac{z'}{2}; 0) e^{-ips/\hbar} \, dz'.
\]  
(13)

But according to Eq. (3), the above integral is nothing but the initial Wigner distribution function at \(t = 0\), and at the phase space point \((x - pt/m, p)\), so that for \(t > 0\)
\[
W(x, p; t) = W(x - \frac{pt}{m}, p; 0).
\]  
(14)

This relation is the Wigner phase space version of Eq. (7). Upon integration over \(x\) and making use of the definition in Eq. (6), we get for \(t > 0\)
\[
n(p, t) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\hbar} W(x, p; t) = \int_{-\infty}^{\infty} \frac{dx}{2\pi\hbar} W(x - \frac{pt}{m}, p; 0) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi\hbar} W(\xi, p; 0) = n(p, 0).
\]  
(15)

Therefore, we find that under ballistic expansion (where interatomic collisions during the expansion are not present) the dynamical momentum density for \(t > 0\) is equal to its
initial value \( n(p,0) \). Some information can be obtained from this result. For example, if one considers an initial quantum gas with short range interaction, whose momentum distribution \( n(p,0) \) exhibits a \( 1/p^4 \) tail at large momentum (see for instance Ref. [26]), equation (15) says that this long tail behavior is preserved in the dynamical momentum density for all positive time.

It is worth noticing that Eq. (7) remains valid for an initial non-vanishing potential \( (V_1 \neq 0) \) provided that one realizes at \( t = 0 \) a simultaneously sudden double quench, where the interactions are turned-off with the release of the trap \( (V_2 = 0) \). As a consequence the results in Eqs. (14) and (15), in this case hold also true.

A. Case of two particles interacting through an attractive \( \delta \) interaction

As a first case study we consider a simple model system of two particles interacting through an attractive Dirac delta potential with highly asymmetric mass imbalance: the particle with mass \( M \) is so heavy that the center-of-mass motion can be ignored. The problem reduces then to a one-body system and the \( m \) mass light particle Hamiltonian is given for \( t \leq 0 \) by

\[
H_a = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - a\delta(x).
\]

Here \( a > 0 \) is the strength of the interaction. The single normalized bound state is \( \phi(x) = \sqrt{\alpha} e^{-\alpha|x|} \) with energy \( E_a = -ma^2/(2\hbar^2) \), and \( \alpha = ma/\hbar^2 \). The corresponding Wigner distribution function for \( t \leq 0 \) can be calculated analytically using (3) and is found to be

\[
W(x,p; 0) = \frac{2\alpha^2 \hbar^2 e^{-2\alpha|x|}}{p^2 + \alpha^2 \hbar^2} \left[ \cos \left( \frac{2p |x|}{\hbar} \right) + \frac{\alpha \hbar}{p} \sin \left( \frac{2p |x|}{\hbar} \right) \right].
\]

A plot of this function is given on the left hand side of Figure 1. At time \( t = 0 \) the interaction is suddenly turned off \( (a = 0) \) and then, from (14) we get the following Wigner function for \( t > 0 \):

\[
W(x,p; t) = \frac{2\alpha^2 \hbar^2 e^{-2\alpha|x-x/mt|}}{p^2 + \alpha^2 \hbar^2} \left[ \cos \left( \frac{2p \left| x - \frac{p}{m} t \right|}{\hbar} \right) + \frac{\alpha \hbar}{p} \sin \left( \frac{2p \left| x - \frac{p}{m} t \right|}{\hbar} \right) \right].
\]

A plot of this function \( W(x,p; t) \) is given for a particular value of \( t > 0 \) on the right hand side of Figure 1. A clear distortion can be observed as time grows: for \( t = 0 \) there are two symmetry axes, \( x = 0 \) and \( p = 0 \); for bigger values of \( t \) the perfect symmetry is lost, although
the axes $p = 0$ and $p = mx/t$ (this one clockwise rotating with time) play an important role; squeezing is more and more pronounced as $t \to \infty$, the axis $p = mx/t$ approaching the axis $p = 0$.

FIG. 1: The Wigner function for an attractive Dirac $\delta$ interaction obtained in Eq. (17), for the values $\hbar = m = a = 1$. On the left for $t = 0$, where two symmetry axes, $x = 0$ and $p = 0$, are evident; on the right after the quenching, for $t = 1$, a clear distortion of the initial function can be appreciated: the perfect symmetry is lost and squeezing appears between the axes $p = 0$ and $p = x$. The white regions correspond to small negative values of $W(x, p; t)$.

Finally, using (15) it is easy to show analytically that, the initial momentum density is

$$n(p, 0) = \frac{2\alpha^3 \hbar^3}{\pi (p^2 + \alpha^2 \hbar^2)^2}, \quad (19)$$

It has been shown in [26] that for a system of zero-range $\delta$–interacting one-dimensional atoms with arbitrary strength, the high-$p$ asymptotic behavior of the momentum distribution for both free and harmonically trapped atoms, exhibits a universal $1/p^4$ dependence. As can be seen in Eq. (19) and at large values of the momentum $p$, we recovered this $1/p^4$ dependence of the momentum distribution.

B. Case of two particles interacting through a $\delta$-$\delta'$ interaction

We are going to consider now an extension of the previous study of two interacting particles that takes into account the presence of an extra point-like interaction term in the potential, proportional to $\delta'$. This type of point or zero-range potentials are a subject of recent study in different contexts [27–30]. The Hamiltonian of the light particle with mass $m$ is now given for $t < 0$ by

$$H_{a,b} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - a\delta(x) + b\delta'(x), \quad a > 0, \quad b \in \mathbb{R}. \quad (20)$$
The associated Schrödinger equation has been carefully analyzed in [27], where it was proven that the above Hamiltonian supports only one bound state of energy

$$E_{a,b} = -\frac{ma^2}{2\hbar^2 (1 + \frac{m^2 b^2}{\hbar^4})} = -\frac{\hbar^2 \kappa^2}{2m}, \quad \kappa > 0.$$  \hspace{1cm} (21)

The normalized wave function is

$$\phi_{a,b}(x) = A e^{-\kappa|x|} (1 + B \text{sign}(x)),$$ \hspace{1cm} (22)

where

$$A = \frac{\sqrt{ma}}{\hbar(1 + B^2)}, \quad B = \frac{mb}{\hbar^2},$$ \hspace{1cm} (23)

and \(\theta(x)\) stands for the Heaviside unit step function. In this case the Wigner distribution function for \(t \leq 0\) can be also determined analytically from (3) and it turns out to be the following expression

$$W(x, p; 0) = \frac{2\hbar^2 A^2 e^{-2\kappa|x|}}{p^2 + \kappa^2 \hbar^2} \left[ (1 - B^2) \kappa \cos \left( \frac{2p|x|}{\hbar} \right) - \frac{(1 - B^2)p^2 - (p^2 + \hbar^2 \kappa^2)(1 + B \text{sign}(x))}{\hbar p} \sin \left( \frac{2p|x|}{\hbar} \right) \right].$$ \hspace{1cm} (24)

Remark that the presence of the sign function on (24) indicates the presence of a discontinuity of the Wigner function \(W(x, p; 0)\) along the line \(x = 0\). Some plots of this function are given in Figure 2 for \(b = -0.6, b = -0.9, b = -1.0\) and \(b = -1.2\).

At time \(t = 0\) the interaction is turned off \((a = b = 0)\) and then, from (14) we get the following Wigner function for \(t > 0\):

$$W(x, p; t) = \frac{2\hbar^2 A^2 e^{-2\kappa|x|}}{p^2 + \kappa^2 \hbar^2} \left[ (1 - B^2) \kappa \cos \left( \frac{2p|x - pt/m|}{\hbar} \right) - \frac{(1 - B^2)p^2 - (p^2 + \hbar^2 \kappa^2)(1 + B \text{sign}(x - pt/m))}{\hbar p} \sin \left( \frac{2p|x - pt/m|}{\hbar} \right) \right].$$ \hspace{1cm} (25)

Plots of this function \(W(x, p; t)\) are given in Figure 3 for \(t = 1\) and values \(b = -0.9\) (left) and \(b = -1.2\) (right). A clear distortion can be observed comparing with Figure 2: the axis \(p = 0\) is preserved, but the axis \(x = 0\) rotates around the origin and becomes \(x = p\) (this one changing with time); squeezing is stronger as \(t \to \infty\), the axis \(x = 0\) approaching the axis \(p = 0\).
FIG. 2: The Wigner function for a Dirac $\delta$-$\delta'$ interaction as in Eq. (20), for the values $\hbar = m = a = 1$. From left to right, and from top to bottom: $b = -0.6$, $b = -0.9$, $b = -1.0$ and $b = -1.2$. The white regions correspond to small negative values of $W(x, p; 0)$. The discontinuity of the surface at $x = 0$ can be clearly seen on the plots.

From (15) and (24) the initial momentum density can be determined analytically:

$$n(p, 0) = \frac{\hbar\alpha}{\pi} \frac{B^2(1 + B^2)^2 p^2 + (2 - B^2)\hbar^2\alpha^2}{((1 + B^2)^2 p^2 + \hbar^2\alpha^2)^2},$$

which coincides with (19) in the limit $B = mb/\hbar^2 \to 0$ and which scales as $1/p^2$ for high $p$ if $B \neq 0$ (if $B = 0$ it was already mentioned after (19) that it scales as $1/p^4$). Remark that in order to have a physical meaning this density $n(p, 0)$ must be a positive function, and therefore we have to impose the following restriction on the coefficient of the $\delta'$ term in the potential: $-\sqrt{2} \leq B \leq \sqrt{2}$, a condition not observed before in the literature. It is interesting to observe that the presence of the additional $\delta'$ interaction term, changes substantially the high momentum tail of the momentum density. In fact, at large values of the momentum $p$, the momentum distribution in Eq. (26) exhibits $1/p^2$ behavior while in the absence of $\delta'$ interaction this density scales as $1/p^4$. 

FIG. 3: The Wigner function $W(x, p; t = 1)$ for a Dirac $\delta$-$\delta'$ interaction with values $\hbar = m = a = 1$, for $b = -0.9$ (left) and $b = -1.2$ (right). Squeezing of the initial function Figure 2 can be appreciated between the axes $p = 0$ and $p = x$. The white regions correspond to small negative values of $W(x, p; t = 1)$. The discontinuity of the surface at $x = p$ can be clearly seen.

III. INTERACTION QUENCH DYNAMICS IN PHASE SPACE FOR HARMONICALLY TRAPPED QUANTUM SYSTEM

In this case the cloud of atoms is let to expand ballistically in a harmonic potential for $t > 0$. Since we are interested in the interaction quench, the confining potential is the same before and after the quench, that is $V_1 = V_2$ in Eqs. (1)-(2). The time dependent one-body density matrix reads then

$$\rho(x_1, x_2; t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_{\text{osc}}(x_1, \xi_1; t) \rho(\xi_1, \xi_2; 0) U_{\text{osc}}^*(x_2, \xi_2; t) d\xi_1 d\xi_2,$$

(27)

where $\rho(\xi_1, \xi_2; 0)$ is the density matrix at $t = 0$ (just before the suppression of the interactions) and $U_{\text{osc}}(x_1, \xi_1; t)$ is the well known propagator associated to the harmonic oscillator well with frequency $\omega$, given by [25]

$$U_{\text{osc}}(x_1, \xi_1; t) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega t}} \exp \left[ i \frac{m\omega}{2\hbar \sin \omega t} \left( (x_1^2 + \xi_1^2) \cos \omega t - 2x_1 \xi_1 \right) \right],$$

(28)

so

$$\rho(x_1, x_2; t) = \frac{m\omega}{2\pi i |\sin \omega t|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi_1, \xi_2; 0) e^{\frac{i m\omega}{2\hbar \sin \omega t} [(x_1^2 + \xi_1^2 - x_2^2 - \xi_2^2) \cos \omega t - 2x_1 \xi_1 + 2x_2 \xi_2]} d\xi_1 d\xi_2.$$

(29)

Before going further we note that recently an experimental technique [31] to directly image the momentum distribution of a strongly interacting two-dimensional quantum gas, was demonstrated and characterized. This method is based on, just after the switching off
the initially confining trap and instead of free expansion the gas is subjected to an external harmonic potential \( V(x) = m\omega^2 x^2 / 2 \) where the gas moves ballistically (the interactions are suppressed). It is shown that after a quarter of the oscillator time period \( T = 2\pi / \omega \) the spatial distribution is related to the momentum density of the initially confined quantum gas. By using Eq. (29), in the following we shall provide a very simple proof of this property for an arbitrary initially confining one-dimensional quantum gas. Using Eq. (29), we can write the local density \( \rho(x,t) \equiv \rho(x,t) \) at time \( t \) as

\[
\rho(x,t) = \frac{m\omega}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi_1, \xi_2; 0) e^{im\omega \frac{\hbar}{\sin \omega t} (\xi_1^2 - \xi_2^2) \cos \omega t - 2z(\xi_1 - \xi_2)} \, d\xi_1 d\xi_2, \tag{30}
\]

and considering this density at a specific time \( t = T/4 = \pi/(2\omega) \), Eq. (30) reduces to

\[
\rho(x,t = T/4) = \frac{m\omega}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi_1, \xi_2; 0) e^{-im\omega \frac{\hbar}{\sin \omega t} (\xi_1 - \xi_2)} \, d\xi_1 d\xi_2 \tag{31}
\]

Since the momentum density \( n(p, t) \) is the Fourier transform of one-body density matrix, we have at \( t = 0 \)

\[
n(p, t = 0) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi_1, \xi_2; 0) e^{-i\frac{\hbar}{\pi} (\xi_1 - \xi_2)} \, d\xi_1 d\xi_2. \tag{32}
\]

Comparing this with the expression in Eq. (31) we can write

\[
\rho(x,t = T/4) = m\omega \, n(p = m\omega x, t = 0), \tag{33}
\]

which clearly shows the mapping of the spatial density at \( t = T/4 \) and the momentum density of the initially confined system, and ends the proof.

Let us now come back to the one-body density matrix in Eq. (29) and use the centre of mass \( z = (\xi_1 + \xi_2)/2 \) and relative \( z' = (\xi_1 - \xi_2) \) coordinates to write

\[
\rho(x_1, x_2; t) = \frac{m\omega}{2\pi\hbar |\sin \omega t|} e^{i\frac{m\omega}{\sin \omega t} (x_1^2 - x_2^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) e^{i\frac{m\omega}{\sin \omega t} [(2z \cos \omega t - (x_1 + x_2))z' - 2z(x_1 - x_2)]} \, dz dz'. \tag{34}
\]

The Wigner transform of this density matrix is given by

\[
W(x, p; t) = \frac{m\omega}{2\pi\hbar |\sin \omega t|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\frac{m\omega}{\sin \omega t} (2z \cos \omega t - 2p)z'} \rho(z + \frac{z'}{2}, z - \frac{z'}{2}; 0) \, dz dz' \times \int_{-\infty}^{\infty} e^\frac{i}{\hbar} (m\omega \cos \omega t x - \frac{m\omega}{\sin \omega t} z - p) \, ds,
\]
with \( x = (\xi_1 + \xi_2)/2 \) and \( s = \xi_1 - x_2 \). Performing the \( s \) integration, using the property \( \delta(ax) = \delta(x)/|a| \) and performing the \( z \) integration, we obtain

\[
W(x, p; t) = \int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}(m_\omega x \sin \omega t + p \cos \omega t) z'} \rho(x \cos \omega t - \frac{p \sin \omega t}{m_\omega} + \frac{z'}{2}, x \cos \omega t - \frac{p \sin \omega t}{m_\omega} - \frac{z'}{2}; 0) \, dz'.
\]

(35)

This expression represents the initial Wigner distribution function at a phase space point \((x \cos \omega t - \frac{p \sin \omega t}{m_\omega}, x \cos \omega t - \frac{p \sin \omega t}{m_\omega})\), so that

\[
W(x, p; t) = W\left(x \cos \omega t - \frac{p \sin \omega t}{m_\omega}, m_\omega x \sin \omega t + p \cos \omega t; 0\right),
\]

(36)

which correspond to a uniform rotation of the initial Wigner function. Notice that if \( \omega \to 0 \), this relation reduces to the result given in Eq. (14), as it should.

Next, let us show that Eq. (33) can be derived from a more general relation in phase space. In fact, if \( T \) denotes the period corresponding to frequency \( \omega \) of the harmonic trap and considering the specific time \( t = T/4 = \pi/2\omega \), Eq. (36) reduces to

\[
W(x, p; T/4) = W\left(-\frac{p}{m_\omega}, m_\omega x; 0\right).
\]

(37)

This equation constitutes an extension in phase space of the result in (33). In fact, Eq. (33) is immediately recovered by first performing an integration over the momentum \( p \) in Eq. (37).

Indeed, if we recall Eq. (5) we may write

\[
\rho(x, T/4) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} W\left(-\frac{p}{m_\omega}, m_\omega x; 0\right) dp = \frac{m_\omega}{2\pi\hbar} \int_{-\infty}^{\infty} W(u, m_\omega x; 0) dp = m_\omega n(p = m_\omega x, 0),
\]

(38)

where we have used Eq. (6), and hence we get the desired result in Eq. (33).

Before closing this section we observe that Eq. (36) is telling us that in the Wigner function representation it is clear that the response of the system after the quenching is a periodic function of time. In addition, if one assumes that the initial density matrix obeys the symmetry property \( \rho(-x_1, -x_2; 0) = \rho(x_1, x_2; 0) \), using the definition (3) we deduce that the initial Wigner function obeys the property \( W(-x, -p; 0) = W(x, p; 0) \). Hence, it follows that the dynamical Wigner function in Eq. (36) has time period \( T' = T/2 \) corresponding to frequency \( 2\omega \), and the dynamical momentum density \( n(p, t) \) is a periodic function of time with frequency \( 2\omega \).
A. Ballistic versus non-ballistic expansions in phase space

So far we have examined ballistic expansion following an interaction quench. However if the interactions are not switched-off, interatomic collisions during the expansion can induce a significant redistribution of momentum. For such a non-ballistic expansion, it is interesting to see how interactions affect the Wigner distribution function. It is instructive to consider a physical situation where analytical calculations are feasible.

As an example, we will consider a strongly interacting system called Tonks-Girardeau gas, which consists in a gas of $N$ identical bosons interacting through very strong repulsive zero-range interactions in one spatial dimension (hard core bosons). The experimental observation of such a one-dimensional hard core gas was reported in [32]. The Bose-Fermi mapping theorem [33] allows to link a gas of bosons in strong interacting regime to a system of noninteracting spin polarized fermions. As a result, and for the case of an external harmonic trap confinement of the form $V(x, t) = m\omega^2(t)x^2/2$ with time dependent frequency $\omega(t)$, it has been proven in [34] that the time dependent one-body density matrix of such Bose gas $\rho(x_1, x_2; t)$ at time $t$ is related to its initial value through the following scaling and gauge transformation, that is

$$\rho(x_1, x_2; t) = \frac{1}{b(t)} \exp \left[ \frac{\bar{b} m}{4\hbar} (x_1^2 - x_2^2) \right] \rho \left( \frac{x_1}{b(t)}, \frac{x_2}{b(t)}; 0 \right),$$

where the scaling factor $b(t)$ is a solution of the second-order ordinary differential equation, $\ddot{b} + \omega^2 b = \omega_0^2/b^3$ with initial conditions $b(0) = 1, \dot{b}(0) = 0$ and $\omega_0 = \omega(t = 0)$. For a quantum quench scenario where the gas is initially confined by harmonic trap with frequency $\omega_0$ and at $t = 0$ the frequency suddenly is changed and becomes $\omega$ such that $\omega < \omega_0$ for $t > 0$, the scaling factor is $b(t) = \sqrt{1 + \epsilon \sin^2 \omega t}$ where $\epsilon = (\omega_0^2/\omega^2 - 1)$ is the quench strength [34]. It should be noted that, very recently this system has been studied within the newly developed approach of conformal field theory [35].

To obtain the dynamical Wigner function in this case we will use the definition (3), and
we multiply both sides of Eq. (39) by $e^{-\frac{i}{\hbar}ps}$. Then, we get

$$W_{\text{non-ballistic}}(x,p; t) = \int_{-\infty}^{\infty} ds \rho(x + \frac{s}{2}, x - \frac{s}{2}; t) e^{-\frac{i}{\hbar}ps}$$

$$= \frac{1}{b} \int_{-\infty}^{\infty} ds \rho \left( \frac{x}{b} + \frac{s}{2}, \frac{x}{b} - \frac{s}{2}; 0 \right) e^{-\frac{i}{\hbar}(p - mbx/b)s}$$

$$= \int_{-\infty}^{\infty} du \rho \left( \frac{x}{b} + \frac{u}{2}, \frac{x}{b} - \frac{u}{2}; 0 \right) e^{-\frac{i}{\hbar}(bp - mbx)u} = W \left( \frac{x}{b}, bp - mbx; 0 \right).$$

$$= W \left( \frac{x}{\sqrt{1 + \epsilon \sin^2 \omega t}}, p \sqrt{1 + \epsilon \sin^2 \omega t} - \frac{m \omega x \sin(2\omega t)}{2\sqrt{1 + \epsilon \sin^2 \omega t}}; 0 \right). \quad (40)$$

We recall that the above relation has been obtained in the case of trap to trap quench, but the interactions are still present. It is instructive to compare the result in Eq. (40) with the one, where simultaneously at time $t = 0$ a trap to trap quench is realized and the interactions are switched-off. The resulting ballistic expansion dynamics in the harmonic trap with frequency $\omega$ is governed by Eq. (27) and therefore the result given in Eq. (36), which we display and denoted here by $W_{\text{ballistic}}$, is still valid

$$W_{\text{ballistic}}(x,p; t) = W \left( x \cos \omega t - \frac{p \sin \omega t}{m \omega}, m \omega x \sin \omega t + p \cos \omega t; 0 \right). \quad (41)$$

Starting from the same initial state of hard core bosons, the dynamical Wigner functions in Eqs. (40) and (41) clearly show a significant difference: Eq. (40) was obtained after a quench that only affects the potential (the interactions are still acting), while Eq. (41) was obtained after a double quench (of the potential and of the interactions) describing the dynamics of noninteracting bosons. Notice that both the above Wigner functions are periodic functions of time having a same period with frequency $2\omega$.

IV. CONCLUDING REMARKS

In this paper we have studied the non-equilibrium dynamics in phase space generated by a sudden change of the Hamiltonian in a quantum system, through the analysis of the Wigner function. For the case of two attractive particles we calculated the corresponding time dependent Wigner function following a switch-off of the interaction.

For possible experimental implementation in ultra-cold quantum gases field, an interaction of the form $-a\delta(x) + b\delta'(x)$, $a > 0$, $b \in \mathbb{R}$ was considered and we have calculated the two-particle Wigner function. At large values of momentum $p$, we have found that the
associated momentum distribution scales as $1/p^2$, while in the absence of $\delta'$ interaction, this density scales, in this case of pure zero-range delta interaction, as the well known law $1/p^4$.

We have also tackled the following problem of ballistic versus non-ballistic expansions in phase space: in other words, how interactions affect the Wigner distribution function during free expansion. Considering a truly dynamical situation of Tonks-Girardeau Bose gas, for which analytical calculations are feasible, we have obtained explicitly the time dependent Wigner distribution functions for ballistic and non-ballistic expansion when the confining initial potential is suddenly released. The two expressions show significant differences exhibiting clearly the effects of interatomic collisions. An interesting extension of this dynamical situation would be to study the problem of a Lieb-Liniger gas at finite repulsion strength. Work in this direction is in progress.

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