The Birman-Schwinger operator for a zero-thickness layer in the presence of an attractive Gaussian impurity

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Abstract

In this note we are concerned with the limiting case of a zero-thickness layer with harmonic confinement along one of the two available dimensions. We investigate the Birman-Schwinger operator for such a model assuming the presence of a Gaussian impurity inside the layer and prove that such an integral operator is Hilbert-Schmidt, which allows the use of the modified Fredholm determinant in order to compute the impurity bound states. Furthermore, we consider the Hamiltonian $H_0 - \lambda\sqrt{\pi} \delta'(x)e^{-y^2}$, that is to say the energy operator with the interaction term having a point interaction in place of the Gaussian along the $x$-direction, and prove that such an operator is self-adjoint as well as that it is the limit in the norm resolvent sense of the sequence $H_0 - \lambda n e^{-(n^2 x^2 + y^2)}$ as $n \to \infty$. 

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1 Introduction

Due to the emergence of new materials, such as the graphene that appears in the form of thin layers that may be one atom thick, the study of two-dimensional quantum systems has drawn intense and renewed interest. This paper attempts to provide a mathematical contribution to the understanding of quantum properties of such two-dimensional systems. In the recent literature [1, 2, 3, 4], given its relevance in Nanophysics, the spectral analysis of the one-dimensional Hamiltonian with a Gaussian potential, namely

$$H := -\frac{1}{2} \frac{d^2}{dx^2} - \lambda e^{-x^2/2}, \quad (1.1)$$

has been carried out, in particular the calculation of the lowest lying eigenvalues. Here, we intend to study a generalisation of (1.1) to two dimensions, with a complication: the free Hamiltonian is not just given by the Laplacian, but instead, we have added a harmonic confinement in one of the dimensions. Thus, the free Hamiltonian has the form:

$$H_0 = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2}\right) - \frac{1}{2} \frac{d^2}{dy^2}, \quad (1.2)$$

to which we add an attractive impurity assumed to be modelled by the potential

$$W(x, y) = -\lambda V(x, y) = -\lambda e^{-(x^2+y^2)}, \quad \lambda > 0, \quad (1.3)$$

so that the total Hamiltonian is

$$H_\lambda = H_0 + W(x, y) = H_0 - \lambda V(x, y) = H_0 - \lambda e^{-(x^2+y^2)}, \quad \lambda > 0. \quad (1.4)$$

At this point, it is interesting to recall that a three-dimensional material with confinement in only one dimension is said to be a quantum well [5], while a material with a two-dimensional confinement is called a quantum wire. In the limiting case of a perfectly two-dimensional material, namely a layer with zero thickness, it makes sense to consider the model in which a confining potential acts in either dimension, which is the object of our study. It may be worth pointing out that the Hamiltonian (1.2) has been studied by Dell’Antonio and collaborators [6] to model the quantum system consisting of two one-dimensional particles, one of which is harmonically bound to its equilibrium position.

Solving the eigenvalue problem for this kind of Hamiltonians is not, in general, an easy task and often requires rather sophisticated tools. One of the most widely used is the Birman-Schwinger operator, namely the integral operator

$$B_E = (\text{sgn} W)|W|^\frac{1}{2}(H_0 - E)^{-1}|W|^\frac{1}{2}, \quad (1.5)$$

and the related technique: as in most applications $B_E$ can be shown to be compact, the solutions of the eigenvalue problem for the Hamiltonian are given by those values of $E$ for which $B_E$ has an eigenvalue equal to -1 (see [7, 8, 9] and references therein as well as [10], p. 99). Therefore, the detailed study of the properties of the Birman-Schwinger operator arising from our model is quite relevant. In the present note, we show that the Birman-Schwinger operator is Hilbert-Schmidt. We also show that $H_\lambda$ is self-adjoint and bounded from below.

In addition, $H_\lambda$ has a special relation with a kind of two-dimensional contact operator. This is the Hamiltonian,

$$H_\delta = H_0 - \lambda \sqrt{\pi} \delta(x) e^{-y^2}, \quad (1.6)$$
where \( \delta(x) \) is the Dirac delta. We show that \( H_\delta \) is self-adjoint and can be obtained as a limit in the norm resolvent sense as \( n \to \infty \) of the following version of \( H_\lambda \):

\[
H_{n,\lambda} := H_0 - \lambda n e^{-(n^2 x^2 + y^2)}.
\]

The Hamiltonian \( H_\lambda \) (resp. \( H_\delta \)) is bounded from below and its lower bound can be obtained using a certain transcendental equation.

## 2 The Birman-Schwinger operator for our model

Starting from the Hamiltonian \( H_0 \) in (1.2), it is rather straightforward to infer that the associated Green function, namely the integral kernel of the resolvent operator, reads for any \( E < \frac{1}{2} \):

\[
(H_0 - E)^{-1}(x, x', y, y') = \sum_{n=0}^{\infty} e^{-\sqrt{2(n+\frac{1}{2}-E)}|y-y'|} \frac{\sqrt{2(n+\frac{1}{2}-E)}}{2(n+\frac{1}{2}-E)} \phi_n(x)\phi_n(x'),
\]

(2.1)

where \( \phi_n(x) \) is the normalised \( n \)-th eigenfunction of the one-dimensional harmonic oscillator.

Once the above attractive Gaussian perturbation (1.3) is added, the total Hamiltonian is \( H_\lambda \) in (1.4). Therefore, its associated Birman-Schwinger integral kernel [1, 2, 7, 8, 9, 11, 12] given by (1.5) is:

\[
B_E = -\lambda \tilde{B}_E (x, x_1, y_1) = -\lambda |V|^2 (H_0 - E)^{-1} |V|^2 (x, x_1, y_1) =
\]

\[
= -\lambda e^{-(x^2+y^2)/2} \sum_{n=0}^{\infty} e^{-\sqrt{2(n+\frac{1}{2}-E)}|y-y'|} \phi_n(x)\phi_n(x_1) e^{-(x_1^2+y_1^2)/2},
\]

(2.2)

The main goal of this brief note is to rigorously prove that such an integral operator is Hilbert-Schmidt, that is to say \( \text{tr}(B_E^2) < \infty \), given the evident positivity of the operator \( B_E \).

As the kernel of the positive operator \( \tilde{B}_E^2 \) is clearly

\[
\tilde{B}_E^2 (x, x_2, y, y_2) = e^{-(x^2+y^2)/2} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-\sqrt{2(n+\frac{1}{2}-E)}|y-y'|} \phi_n(x)\phi_n(x_2) \right\} dx_2 dy_2 
\]

\[
\times e^{-(x_2^2+y_2^2)} \sum_{n=0}^{\infty} e^{-\sqrt{2(n+\frac{1}{2}-E)}|y'-y_2'|} \phi_n(x')\phi_n(x_2) \right\} \right\} dx dy e^{-(x^2+y^2)/2} \sum_{n=0}^{\infty} \phi_n(x)\phi_n(x') \phi_n(x)\phi_n(x').
\]

(2.3)

its trace reads

\[
\text{tr}(\tilde{B}_E^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{B}_E^2 (x, x, y, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdydxd'y' e^{-(x^2+y^2)} e^{-(x^2+y^2)}
\]

\[
\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-\left(\sqrt{2(m+\frac{1}{2}-E)+\sqrt{2(n+\frac{1}{2}-E)}}|y-y'|\right)} \phi_m(x)\phi_m(x')\phi_n(x)\phi_n(x').
\]

(2.4)

The latter multiple integral can be rewritten as:

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{-\left(\sqrt{2(m+\frac{1}{2}-E)+\sqrt{2(n+\frac{1}{2}-E)}}|y-y'|\right)} \frac{dydy}{2\sqrt{(m+\frac{1}{2}-E)(n+\frac{1}{2}-E)}} \right] \left( \phi_m, e^{-(\cdot)^2} \phi_n \right)^2
\]

(2.5)
where $(f, g)$ denotes the standard scalar product of the two functions.

Let us consider the double integral in (2.4). With the notation,

$$f(y - y') := e^{-\left(\sqrt{2(m + \frac{1}{2} - E)} + \sqrt{2(n + \frac{1}{2} - E)}\right)|y - y'|}, \quad g(y') = e^{-y'^2},$$

the second integral becomes the convolution $(f \ast g)(y)$, so that the double integral may be written as

$$\frac{1}{2\sqrt{(m + \frac{1}{2} - E)(n + \frac{1}{2} - E)}} \int_{-\infty}^{\infty} e^{-y'^2} [(f \ast g)(y)] \, dy.$$  \hspace{1cm} (2.7)

Using the Schwartz inequality (2.7) is smaller than or equal to

$$\frac{1}{2\sqrt{(m + \frac{1}{2} - E)(n + \frac{1}{2} - E)}} ||e^{-\cdot}||_2 ||f \ast g||_2,$$

where $|| \cdot ||_p$ denotes the norm in $L^p(\mathbb{R})$. Young’s inequality [13] shows that

$$||f \ast g||_r \leq ||f||_p ||g||_q,$$

with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.  \hspace{1cm} (2.9)

Therefore, with $p = r = 2$ and $q = 1$, it follows that (2.8) is smaller than or equal to

$$\frac{1}{2\sqrt{(m + \frac{1}{2} - E)(n + \frac{1}{2} - E)}} ||e^{-\cdot}||^2_2 ||f||_1.$$  \hspace{1cm} (2.10)

The two norms in (2.9) yield two integrals which can be easily computed, so as to obtain

$$\sqrt{\frac{\pi}{4(m + \frac{1}{2} - E)(n + \frac{1}{2} - E)}} \left(\sqrt{2(m + \frac{1}{2} - E)} + \sqrt{2(n + \frac{1}{2} - E)}\right) \leq \sqrt{\frac{\pi}{4(m + \frac{1}{2} - E)(n + \frac{1}{2} - E)}}.$$  \hspace{1cm} (2.11)

Hence, the trace (2.4) is bounded by:

$$\text{tr}(\tilde{B}^2_E) \leq \sqrt{\frac{\pi}{4}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle \phi_m, e^{-\cdot} \phi_n \rangle^2}{(m + \frac{1}{2} - E)^{\frac{3}{2}}(n + \frac{1}{2} - E)^{\frac{3}{2}}} = \frac{\pi^{\frac{3}{2}}}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\langle \phi_m \phi_0, \phi_0 \phi_n \rangle^2}{(m + \frac{1}{2} - E)^{\frac{3}{2}}(n + \frac{1}{2} - E)^{\frac{3}{2}}}. \hspace{1cm} (2.12)$$

The scalar products inside the double series can be estimated using Wang’s results on integrals of products of eigenfunctions of the harmonic oscillator [14]. While the scalar product clearly vanishes if $m + n = 2s + 1$, when both indices are either even or odd we get:

$$\langle \phi_{2m} \phi_0, \phi_0 \phi_{2n} \rangle^2 = \frac{1}{2\pi} \left[\frac{2(m + n)!}{(m + n)!}\right]^2 \frac{1}{2^{4(m+n)(2m)!}(2n)!} \leq \frac{1}{2\pi} \left[\frac{2(m + n)!}{2^{2(m+n)!}(m + n)!}\right]^2 = \frac{\phi^2_{2(m+n)}(0)}{2}, \hspace{1cm} (2.13)$$

$$\langle \phi_{2m+1} \phi_0, \phi_0 \phi_{2n+1} \rangle^2 = \frac{1}{2\pi} \left[\frac{2(m + n + 1)!}{(m + n + 1)!}\right]^2 \frac{1}{2^{4(m+n+1)(2m+1)!}(2n+1)!} \leq \frac{1}{2\pi} \left[\frac{2(m + n + 1)!}{2^{2(m+n+1)!}(m + n + 1)!}\right]^2 = \frac{\phi^4_{2(m+n+1)}(0)}{2}, \hspace{1cm} (2.14)$$
the final equalities in (2.13) and (2.14) resulting from [15] and [16]. Therefore, the r.h.s. of (2.12) is bounded from above by:

\[
\text{tr}(\tilde{B}_E^2) \leq \frac{3}{8} \left[ \sum_{m,n=0}^{\infty} \frac{\phi^4_{2(m+n)}(0)}{(2m+\frac{1}{2} - E)^{\frac{3}{2}}(2n+\frac{1}{2} - E)^{\frac{3}{2}}} + \sum_{m,n=0}^{\infty} \frac{\phi^4_{2(m+n+1)}(0)}{(2m+\frac{3}{2} - E)^{\frac{3}{2}}(2n+\frac{3}{2} - E)^{\frac{3}{2}}} \right].
\]

As can be gathered from [16] using Stirling’s formula,

\[
\begin{align*}
\phi^4_{2n}(0) &\leq \frac{1}{\pi^2 n}, \quad n \geq 1, \\
\phi^4_{2(m+n)}(0) &\leq \frac{1}{\pi^2 (m+n)}, \quad m,n \geq 1, \\
\phi^4_{2(m+n+1)}(0) &\leq \frac{1}{\pi^2 (m+n+1)}, \quad m,n \geq 0,
\end{align*}
\]

which implies that (2.15) is bounded by

\[
\text{tr}(\tilde{B}_E^2) \leq \frac{1}{8\pi^2} \left[ \sum_{n=1}^{\infty} \frac{1}{n(2n+\frac{1}{2} - E)^{\frac{3}{2}}} + \sum_{n=1}^{\infty} \frac{1}{n(2n+\frac{3}{2} - E)^{\frac{3}{2}}} \right] + \sum_{m,n=0}^{\infty} \frac{1}{m(2m+\frac{1}{2} - E)^{\frac{3}{2}}n(2n+\frac{1}{2} - E)^{\frac{3}{2}}} + \sum_{m,n=0}^{\infty} \frac{1}{m(2m+\frac{3}{2} - E)^{\frac{3}{2}}n(2n+\frac{3}{2} - E)^{\frac{3}{2}}}
\]

\[
\leq \frac{1}{8\pi^2} \left[ \sum_{n=1}^{\infty} \frac{1}{n(2n+\frac{1}{2} - E)^{\frac{3}{2}}} + \sum_{n=1}^{\infty} \frac{1}{n(2n+\frac{3}{2} - E)^{\frac{3}{2}}} \right] + \frac{1}{8\pi^2} \left[ \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2(2n+\frac{1}{2} - E)^{\frac{3}{2}}} \right]^2 < \infty,
\]

since both series involved in the final expression are clearly absolutely convergent given that the summands are positive sequences decaying like \(n^{-\frac{3}{2}}\). Hence, the trace of the square of the Birman-Schwinger operator, i.e. its Hilbert-Schmidt norm, is finite for any \(E < \frac{1}{2}\).

Our result is not surprising at all since the norm could have been bounded by that of the Birman-Schwinger operator with the same impurity but with the resolvent of our \(H_0\) replaced by that of \(-\frac{A}{2}\) in two dimensions, which is known to be finite [17]. However, it provides us with a far more accurate estimate of the norm, which in turn leads to a more precise determination of the spectral lower bound resulting from the use of the Hilbert-Schmidt norm of the Birman-Schwinger operator in the KLMN theorem [18]. In fact, the latter bound is what we wish to achieve by further estimating the bottom lines of (2.17).

The series in (2.17) can be bounded from above by their respective improper integrals as follows:

\[
S_1 = \sum_{n=1}^{\infty} \frac{1}{n(2n+\frac{1}{2} - E)^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{(2n+\frac{3}{2} - E)^{\frac{3}{2}}} < \int_0^{\infty} \frac{\sqrt{2} dx}{(2x+\frac{1}{2} - E)^{\frac{3}{2}}} = \int_0^{\infty} \frac{2x dx}{(2x+\frac{1}{2} - E)^{\frac{3}{2}}} = \frac{3\sqrt{2}}{4} \int_0^{\infty} \frac{ds}{(s+\frac{1}{2} - E)^{\frac{3}{2}}} \\
= \frac{3\sqrt{2}}{4} \int_0^{\infty} \frac{s^\frac{1}{2} ds}{(s+\frac{1}{2} - E)^{\frac{3}{2}}} \leq \frac{3\sqrt{2}}{4} \int_0^{\infty} \frac{ds}{(s+\frac{1}{2} - E)^{\frac{3}{2}}} = \frac{3\sqrt{2}}{4} \int_0^{\frac{1}{2}} \frac{ds}{(s+\frac{1}{2} - E)^{\frac{3}{2}}}.
\]
\[ S_2 = \sum_{n=0}^{\infty} \left( \frac{n + \frac{1}{2}}{2} \right)^{3/2} (2n + \frac{3}{2} - E)^{3/4} = \frac{\sqrt{2}}{(\frac{1}{2} - E)^{3/4}} + \sum_{n=1}^{\infty} \left( \frac{2n + 1 + \frac{1}{2}}{2} \right)^{3/2} (2n + \frac{3}{2} - E)^{3/4} \]

\[ \leq \frac{\sqrt{2}}{(\frac{1}{2} - E)^{3/4}} + \int_{0}^{\infty} \frac{\sqrt{2} dx}{(2x + \frac{3}{2} - E)^{3/4}} = \frac{\sqrt{2}}{(\frac{1}{2} - E)^{3/4}} - \frac{\sqrt{2}}{(\frac{1}{2} - E)^{3/4}} \]

\[ + \frac{3\sqrt{2}}{4} \int_{0}^{\infty} \frac{(s + \frac{1}{2} + \frac{1}{2}) (2s + \frac{3}{2} - E)^{3/4}}{(s + \frac{3}{2} - E)^{3/4}} \leq \frac{3\sqrt{2}}{4} \int_{0}^{\infty} \frac{ds}{(s + \frac{3}{2} - E)^{3/4}} = \frac{3\sqrt{2}}{(\frac{3}{2} - E)^{3/4}}. \]  

(2.19)

Therefore, the bottom lines of (2.17) are bounded by:

\[ \text{tr}(\tilde{B}_E^2) \leq \frac{1}{8\pi^{3/2}} \left[ \frac{1}{(\frac{1}{2} - E)^{3/4}} + \frac{6\sqrt{2}}{(2 - E)^{3/4}} + \frac{18}{(\frac{1}{2} - E)^{3/4}} \right] + \frac{1}{4\pi^{3/2}} \frac{9}{(\frac{3}{2} - E)^{3/4}}. \]

(2.20)

Hence, our estimate of the Hilbert-Schmidt norm of the Birman-Schwinger operator is:

\[ \text{tr}(\tilde{B}_E^2) = \left\| e^{-\frac{x^2+y^2}{2}} (H_0 - E)^{-1/2} e^{-\frac{x^2+y^2}{2}} \right\|_2^2 \leq \frac{1}{8\pi^{3/2}} \left[ \frac{3\sqrt{2}}{(\frac{1}{2} - E)^{3/4}} + \frac{1}{(\frac{1}{2} - E)^{3/4}} \right]^2 + \frac{1}{4\pi^{3/2}} \frac{9}{(\frac{3}{2} - E)^{3/4}}. \]

As is well known [1, 2, 7, 8, 9, 11, 12], the operator

\[ (H_0 - E)^{-1/2} e^{-(x^2+y^2)} (H_0 - E)^{-1/2} \]

is isospectral to the Birman-Schwinger operator so that their Hilbert-Schmidt norms are identical. Hence, what has been achieved so far can be summarised by means of the following claim.

**Theorem 2.1** The integral operators

\[ (H_0 - E)^{-1/2} e^{-(x^2+y^2)} (H_0 - E)^{-1/2} \quad \text{and} \quad e^{-\frac{x^2+y^2}{2}} (H_0 - E)^{-1/2} e^{-\frac{x^2+y^2}{2}} \]

are Hilbert-Schmidt and

\[ \left\| (H_0 - E)^{-1/2} e^{-(x^2+y^2)} (H_0 - E)^{-1/2} \right\|_2^2 = \left\| e^{-\frac{x^2+y^2}{2}} (H_0 - E)^{-1} e^{-\frac{x^2+y^2}{2}} \right\|_2^2 \]

\[ \leq \frac{1}{8\pi^{3/2}} \frac{1}{(\frac{1}{2} - E)^{3/4}} \left[ 3\sqrt{2} + \frac{1}{(\frac{1}{2} - E)^{3/4}} \right]^2 + \frac{1}{4\pi^{3/2}} \frac{9}{(\frac{3}{2} - E)^{3/4}}. \]

As an immediate consequence of the above theorem we get:

**Corollary 1** The Hamiltonian

\[ H_\lambda = H_0 - \lambda e^{-(x^2+y^2)}, \]

defined in the sense of quadratic forms, is self-adjoint and bounded from below by \( E(\lambda) \), the solution of the equation:

\[ \frac{1}{2(\frac{1}{2} - E)^{3/4}} \left[ 3\sqrt{2} + \frac{1}{(\frac{1}{2} - E)^{3/4}} \right]^2 + \frac{9}{(\frac{3}{2} - E)^{3/4}} = \frac{4\pi^{3/2}}{\lambda^2}. \]

(2.22)
Proof. For any $E < 0$ and $\psi \in Q(H_0)$:

\[
\lambda \left( \psi, e^{-(x^2+y^2)} \psi \right) = \lambda \left( (H_0 - E)^{1/2} \psi, [ (H_0 - E)^{-1/2} e^{-(x^2+y^2)} (H_0 - E)^{-1/2} ] (H_0 - E)^{1/2} \psi \right)
\leq \lambda \left\| (H_0 - E)^{-1/2} e^{-(x^2+y^2)} (H_0 - E)^{-1/2} \right\|_2 \left\| (H_0 - E)^{1/2} \psi \right\|_2
\leq \lambda \left[ \frac{1}{8\pi^{\frac{3}{2}} (\frac{1}{2} - E)^{\frac{7}{2}}} \left[ 3\sqrt{2} + \frac{1}{(\frac{1}{2} - E)^{\frac{7}{2}}} \right]^2 + \frac{1}{4\pi^{\frac{3}{2}} (\frac{3}{2} - E)^{\frac{7}{2}}} \right]^\frac{1}{2} \left[ \psi, H_0 \psi \right] - E \left\| \psi \right\|_2^2 .
\] (2.23)

By taking $E$ sufficiently negative, the first factor in the bottom line of (2.23) can be made arbitrarily small, which ensures that the Gaussian perturbation is infinitesimally small with respect to $H_0$ in the sense of quadratic forms. Hence, we need only invoke the KLMN theorem (see [13]) to infer that $H_\lambda$ is self-adjoint and bounded from below by the quantity

\[
\frac{\lambda}{2\pi^{\frac{7}{4}}} \left[ \frac{1}{2(\frac{1}{2} - E)^{\frac{7}{4}}} \left[ 3\sqrt{2} + \frac{1}{(\frac{1}{2} - E)^{\frac{7}{4}}} \right]^2 + \frac{9}{(\frac{3}{2} - E)^{\frac{7}{4}}} \right]^\frac{1}{2} E, E < 0,
\] (2.24)

so that the supremum of such lower bounds is attained for that particular value of $E$ solving (2.22).

In the following subsections we first consider a Hamiltonian with a point interaction all along the $x$-direction in place of the Gaussian potential and then we investigate in detail the solution of (2.22), that is to say the lower bound of the spectrum of $H_\lambda$.

### 2.1 Hamiltonian with a point interaction along the $x$-direction

Let us consider now the Hamiltonian

\[
H_\delta = H_0 - \lambda \sqrt{\pi} \delta(x) e^{-y^2},
\]

that is to say the energy operator with the interaction term having a point interaction in place of the Gaussian along the $x$-direction. Our goal is to prove that such an operator is self-adjoint and that it is the limit in the norm resolvent sense of the sequence $H_0 - \lambda ne^{-(n^2x^2+y^2)}$ as $n \to \infty$.

**Corollary 2** The Hamiltonian $H_\delta = H_0 - \lambda \sqrt{\pi} \delta(x) e^{-y^2}$, defined in the sense of quadratic forms, is self-adjoint and is the norm resolvent limit of the sequence of Hamiltonians

\[
H_{n,\lambda} = H_0 - \lambda ne^{-(n^2x^2+y^2)}.
\]

Furthermore, $H_\delta$ is bounded from below by $E_\delta(\lambda)$, the solution of the equation:

\[
\frac{1}{(\frac{1}{2} - E)^{\frac{7}{4}}} \left[ \frac{1}{(\frac{1}{2} - E)^{\frac{7}{4}}} + 3\sqrt{2} \right]^2 = \frac{4\pi^{\frac{7}{2}}}{\lambda^2}.
\] (2.25)

**Proof.** First of all, it is quite straightforward to show that the integral operator

\[
(H_0 - E)^{-1/2} \sqrt{\pi} \delta(x) e^{-y^2} (H_0 - E)^{-1/2}, \quad E < 0,
\]
is Hilbert-Schmidt with the square of the Hilbert-Schmidt norm given by:

\[
\pi \left\| (H_0 - E)^{-1/2} \delta(x) e^{-y^2} (H_0 - E)^{-1/2} \right\|^2
= \pi \left\| (\delta(x) e^{-y^2})^{1/2} (H_0 - E)^{-1} (\delta(x) e^{-y^2})^{1/2} \right\|^2
= \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{-y'^2} \left[ (H_0 - E)^{-1} (0, 0, y, y') \right]^2 \, dy' \, dy
= \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{-y'^2} \left[ \sum_{n=0}^{\infty} \frac{\varepsilon_n^{2n} e^{-(2n+\frac{1}{2} - E)|y-y'|}}{\sqrt{2(2n + \frac{1}{2} - E)}} \phi_{2n}^2 (0) \right]^2 \, dy' \, dy
= \frac{\pi}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_{2m}^2 (0) \phi_{2n}^2 (0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\varepsilon_n^{2n} e^{-(2n+\frac{1}{2} - E)|y-y'|} e^{-\varepsilon_n^{2n} e^{-(2n+\frac{1}{2} - E)|y-y'|}} e^{-y'^2} \, dy' \, dy
\]

As the double integral involving the convolution has already been estimated in (2.11), the latter expression is bounded by:

\[
\frac{\pi^{3/2}}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_{2m}^2 (0) \phi_{2n}^2 (0) (2m + \frac{1}{2} - E)^{\frac{3}{2}} (2n + \frac{1}{2} - E)^{\frac{3}{2}} \leq \frac{\pi^{3/2}}{4} \left[ \sum_{n=0}^{\infty} \phi_{2n}^2 (0) \right]^2 (2n + \frac{1}{2} - E)^{\frac{3}{2}} \]

which, using (2.17), is bounded by:

\[
\frac{1}{4\pi^{1/2}} \left[ \frac{1}{(\frac{1}{2} - E)^{\frac{3}{2}}} + \sum_{n=1}^{\infty} \frac{1}{n^2 (2n + \frac{1}{2} - E)^{\frac{3}{2}}} \right]^2 \leq \frac{1}{4\pi^{1/2}} \left[ \frac{1}{(\frac{1}{2} - E)^{\frac{3}{2}}} + \frac{3\sqrt{2}}{(\frac{1}{2} - E)^{\frac{3}{2}}} \right]^2
= \frac{1}{4\pi^{1/2}} \left[ \frac{1}{(\frac{1}{2} - E)^{\frac{3}{2}}} + 3\sqrt{2} \right]^2,
\]

having taken advantage of (2.18). As the right hand side of (2.27) can be made arbitrarily small by taking \( E < 0 \) large in absolute value, the KLMN theorem ensures, as was done previously in the case of \( H_\lambda \), the self-adjointness of \( H_\delta \) as well as the existence of the spectral lower bound \( E_\delta (\lambda) \) given by the solution of (2.25).

As to the convergence of \( H_{n,\lambda} \) to \( H_\delta \), we start by noting that, for any \( E < 0 \), the operator \((H_0 - E)^{-1/2} n e^{-(n^2 x^2 + y^2)} (H_0 - E)^{-1/2}\) converges weakly to \((H_0 - E)^{-1/2} \sqrt{\pi} \delta(x) e^{-y^2} (H_0 - E)^{-1/2}\) as \( n \to \infty \). Furthermore,

\[
\left\| (H_0 - E)^{-1/2} n e^{-(n^2 x^2 + y^2)} (H_0 - E)^{-1/2} \right\|^2 = \left\| n e^{-(n^2 x^2 + y^2)} (H_0 - E)^{-1} e^{-n^2 x^2 + y^2} \right\|^2
= \sum_{l,m=0}^{\infty} \left\| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-y^2} e^{-\left[ \left( \sqrt{2(l + \frac{1}{2} - E)} + \sqrt{2(m + \frac{1}{2} - E)} \right) \right] y^2} e^{-y'^2} \, dy' \, dy \right\|^2 \phi_l, n e^{-(n^2)} \phi_m \right\|^2.
\]

Since

\[
\left\langle \phi_l, n e^{-(n^2)} \phi_m \right\rangle = n \int_{-\infty}^{\infty} e^{-n^2 x^2} \phi_l (x) \phi_m (x) dx = \int_{-\infty}^{\infty} e^{-x^2} \phi_l (x/n) \phi_m (x/n) dx \to \sqrt{\pi} \phi_l (0) \phi_m (0),
\]


as \( n \to \infty \), the right hand side of (2.28) converges to

\[
\frac{\pi}{2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \phi_{2l}(0)^2 \phi_{2m}(0) \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\left(\frac{2}{2l+\frac{1}{2}}-E\right)} - e^{-\left(\frac{2}{2l+\frac{1}{2}}-E\right)}}{\sqrt{(2l + \frac{1}{2} - E)(2m + \frac{1}{2} - E)}} e^{-y^2} dy \right] 
\]

\[
= \pi \left[ \left( \delta(x)e^{-y^2} \right)^2 (H_0 - E)^{-1/2} \right]^{2/2} = \pi \left[ \left( H_0 - E \right)^{-1/2} (H_0 - E)^{-1/2} \right]^{2/2}.
\]

Hence, the Hilbert-Schmidt norm of \((H_0 - E)^{-1/2} e^{-n^2 x^2 + y^2} (H_0 - E)^{-1/2}\) converges to the Hilbert-Schmidt norm of \((H_0 - E)^{-1/2} \sqrt{\pi} \delta(x) e^{-y^2} (H_0 - E)^{-1/2}\) as \( n \to \infty \). Due to Theorem 2.21 in [19], this fact and the previous weak convergence imply that the convergence actually takes place in the Hilbert-Schmidt norm. Thus, the norm convergence of these integral operators ensures the norm resolvent convergence of \( H_{n,\lambda} \) to \( H_\delta \), as guaranteed by Theorem VIII.25 in [18], which completes our proof.

### 2.2 The lower bound of \( \sigma(H_0 - \lambda e^{-(x^2 + y^2)}) \)

As anticipated earlier, the lower bound of the spectrum of \( H_\lambda \) in (1.4) is the function \( E(\lambda) \) given implicitly by the equation (2.22). From this expression, some approximate results can be easily obtained in two different regimes. For example, it is possible to prove that the asymptotic behaviour of (2.22) for large values of both variables is

\[
E = -\frac{81}{4\pi^2} \lambda^4.
\]

(2.29)

On the other hand, for small values of \( \lambda \) we can prove that (2.22) behaves approximately as follows

\[
E = \frac{1}{2} - \frac{1}{4\pi^{2/3}} \lambda^{4/3}.
\]

(2.30)

A plot of the \( \lambda \)-dependence (\( \lambda \) being the strength of the potential of \( H_\lambda \)) of the lower bound of the energy \( E \), resulting from the solution of (2.22), as well as those of the two approximations given by (2.29) and (2.30), are given in Figure 1.

### 3 Final remarks

In this note we have analysed in detail the Birman-Schwinger operator of the two-dimensional Hamiltonian \( H_\lambda = H_0 - \lambda e^{-(x^2 + y^2)} \), namely the integral operator \(-\lambda e^{-\frac{x^2 + y^2}{2}} (H_0 - E)^{-1} e^{-\frac{x^2 + y^2}{2}}\) where \( H_0 = (\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2}) - \frac{1}{2} \frac{d^2}{dy^2} \). In particular, we have rigorously shown that the operator is Hilbert-Schmidt and have estimated its Hilbert-Schmidt norm. This fact has enabled us to use the KLIM theorem to determine a lower bound for \( \sigma(H_\lambda) \), that is to say \( E(\lambda) \), the implicit function representing the solution of an equation involving the energy parameter and the coupling constant. Furthermore, we have investigated the Hamiltonian \( H_\delta = H_0 - \lambda \sqrt{\pi} \delta(x) e^{-y^2} \), having the Gaussian impurity in the direction subjected to the harmonic confinement replaced by a point impurity.

The results of this paper will enable us to study the lowest bound states created by the Gaussian impurity potential of the aforementioned Hamiltonian by means of the modified Fredholm determinant \( \det_2 \left[ 1 - \lambda e^{\frac{x^2 + y^2}{2}} (H_0 - E)^{-1} e^{\frac{x^2 + y^2}{2}} \right] \), the regularised determinant used for Hilbert-Schmidt operators. Work in this direction is in progress.
Figure 1: A plot of the lower bound of the energy $E$ as a function of $\lambda$ resulting from the solution of Equation (2.22) (blue curve), the approximate expression valid for large values of $E$ and $\lambda$ obtained in (2.29) (yellow curve) and the approximation for small values of $\lambda$ as in (2.30) (green curve). In the inset we have enlarged the region where $\lambda$ and $E$ are small. While the similarity between the solution of (2.22) and the function (2.29) is quite acceptable for a wide range of the parameters, the solution of (2.22) is satisfactorily approximated by (2.30) only for very small values of $\lambda$.

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