frontiers in Physics

ORIGINAL RESEARCH
published: 29 January 2019
doi: 10.3389/fphy.2019.00007

# A Note on the Exact Green Function for a Quantum System Decorated by Two or More Impurities

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The exact Green function is constructed for a quantum system, with known Green function, which is decorated by two delta function impurities. It is shown that when two such impurities coincide they behave as a single singular potential with combined amplitude. The results are extended to N impurities and higher dimensions.

**Keywords: quantum green function, delta function potential, contact interaction, Schroedinger equation, quantum graph**
**Pacs: 02.30Gp, 02.30Hq, 03.65Db**

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***Edited by:***
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***Specialty section:***
*This article was submitted to Mathematical Physics, a section of the journal Frontiers in Physics*

***Received:*** *01 October 2018*
***Accepted:*** *11 January 2019*
***Published:*** *29 January 2019*

***Citation:***
*Glasser ML (2019) A Note on the Exact Green Function for a Quantum System Decorated by Two or More Impurities. Front. Phys. 7:7. doi: 10.3389/fphy.2019.00007*

## 1. INTRODUCTION

The one dimensional harmonic oscillator or square well, for example, for which the energy -dependent Green function $G_0(x, x'; E)$ is known, have been taken for many years as solvable models for semi-conductor quantum wells [1]. Frequently delta function potentials are placed at various points to simulate defects or impurities. In the case of a single impurity potential, $V(x) = \lambda\delta(x - a)$, the Green function for the composite system is known to be [2]

$$G(x, x'; E) = G_0(x, x') + \lambda \frac{G_0(x, a; E)G_0(a, x')}{1 - \lambda G_0(a, a; E)} \tag{1}$$

In this note a corresponding formula is derived for the case $V(x) = \lambda\delta(x - a) + \mu\delta(x - b)$ On the basis of the analogy of the algebraic structure the result is extended to N-impurities and for a standard interpretation of the Dirac delta function, to higher dimension.

## 2. CALCULATION

We first note that the same argument can be used for the time dependent-, as well as the energy-dependent Green functions, so we shall omit the third argument and write simply $G(x, x')$.

Beginning with the Dyson equation, noting that $G_0(x, y) = G_0(y, x)$

$$G(x, x') = G_0(x, x') + \int G_0(x, y)V(y)G(y, x')dy, \tag{2}$$

where the integration extends over the system domain, one has the set of equations

$$G(x, x') = G_0(x, x') + \lambda G_0(x, a)G(a, x') + \mu G_0(x, b)G(b, x') \tag{3}$$

$$G(a, x') = G_0(a, x') + \lambda G_0(a, a)G(a, x') + \mu G_0(a, b)G(b, x'), \tag{4}$$

$$G(b, x') = G_0(b, x') + \lambda G_0(a, b)G(a, x') + \mu G_0(b, b)G(b, x'). \tag{5}$$

The linear Equations (4) and (5) are easily solved for $G(a,x')$ and $G(b,x')$ :

$$G(a,x') = \frac{G_0(a,x') + \mu[G_0(b,x')G_0(a,b) - G_0(a,x')G_0(b,b)]}{D} \tag{6}$$

$$G(b,x') = \frac{G_0(b,x') + \lambda[G_0(a,x')G_0(a,b) - G_0(b,x')G_0(a,a)]}{D} \tag{7}$$

with

$$D = [1-\lambda G_0(a,a)][1-\mu G_0(b,b)] - \lambda\mu[G_0(a,b)]^2. \tag{8}$$

By inserting (6) and (7) into (3) we obtain the desired expression

$$\begin{aligned} G(x,x') &= G_0(x,x') \\ &+\frac{1}{D}\{\lambda G_0(x,a)G_0(a,x') + \mu G_0(x,b)G_0(b,x') \\ &+\lambda\mu[G_0(x,a)\left(G_0(a,b)G_0(a,x') - G_0(b,b)G_0(b,x')\right) \\ &+G_0(x,b)\left(G_0(a,b)G_0(a,x') - G_0(a,a)G_0(a,x')\right)]\}. \end{aligned} \tag{9}$$

## 3. DISCUSSION

By setting $\mu$ to 0 (9) reduces to (1), proving this expression as well. The most salient feature of (9) is the denominator $D$ whose zeros form the exact spectrum of the composite system. For example, when $a$ and $b$ coincide, $D$ reduces to $1-(\lambda+\mu)G(a,a)$ and (9) reduces to (1) with $\lambda$ replaced by the amplitude $\lambda+\mu$. I.e., the two impurities combine to form one with combined amplitude. This generalizes the result of Fasssari and Rinaldi [3], for two identical defects symmetrically placed with respect to the center of a harmonic oscillator. An expression similar to (9) has been derived recently by Horing (private communication) for the case of a quantum dot in a magnetic field.

Two further points can be made. Nothing in the derivation of (9) restricts it to the line. If we accept the standard definition $\delta(\vec{x}) = \Pi_{j=1}^{d}\delta(x_j)$, then (9), and its consequences, are valid for $d$-dimensional quantum systems. This has been proven function-theoretically for the three dimensional quantum dot with two symmetrically placed identical impurities by Albeverio et al. [4].

A second observation is that $D$ is simply the Cramer determinant for the pair of simultaneous linear Equations (4) and (5). In the case of impurity potential

$$V(x) = \sum_{j=1}^{N} \lambda_j \delta(x-a_j) \tag{10}$$

there will be $N$ such equations and the determinant is easily evaluated. The general result is

*If a quantum system having Green function $G_0(x,y)$ is decorated with $N$ delta function impurities $\lambda_j\delta(x-a_j)$, $j = 1,2,\cdots N$, then the new energy levels are the roots of*

$$D_N = \prod_{j=1}^{N} A_{jj} - \sum_{j=2}^{N}(-1)^j \sum_{1\le k_1<\cdots k_j\le N} A_{k_1k_2}A_{k_2k_3}\cdots A_{k_jk_1} = 0 \tag{11}$$

*where $A_{lm} = \delta_{lm} - \lambda_l G_0(a_l,a_m)$.*

Thus,

$$\begin{aligned} D_3 &= \prod_{j=1}^{3}[1-\lambda_j G_0(a_j,a_j)] - \sum_{i<j}\lambda_i\lambda_j G_0(a_i,a_j)G_0(a_j,a_i) \\ &+\lambda_1\lambda_2\lambda_3 G_0(a_1,a_2)G(a_2,a_3)G_0(a_3,a_1), \end{aligned} \tag{12} :$$

which reduces to the $N=1$ and $N=2$ cases appropriately and shows that any two coinciding impurities coalesce as indicated above.

Note that if all the $\lambda$s and $a$'s coincide then

$$D_N = (1-\lambda G_0(a,a))^N - \sum_{j=2}^{N}\binom{N}{j}\lambda^j G_0(a,a)^j = (1-N\lambda G_0(a,a)). \tag{13}$$

Equation (11) might offer a new approach to Kronig-Penney-type systems for periodic or random unit cells.

Finally, it should be pointed out that the work in this note is paralleled in the theory of quantum graphs introduced by Linus Pauling about 1930 to describe electrons in molecules which has developed into a sophisticated and important branch of quantum physics [5]. For relations of this discipline to the present work see the papers by Andrade et al. [6] and Andrade and Severini [7].

## AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

## ACKNOWLEDGMENTS

The author thanks Prof. S. Fassari and Prof. L. M. Nieto for helpful comments and acknowledges the financial support of MINECO (Project MTM2014-57129-C2-1-P) and Junta de Castilla y Leon (VA057U16). The author thanks a referee for pointing out the relevance to quantum graphs.

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