A Note on the Riemann $\xi$–Function

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Abstract

This brief note investigates several integrals of Riemann’s $\xi$–function of possible interest with respect to the distribution of its zeros in the critical strip.


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Introduction

The notation used here is:

\[ \xi(s) = (s - 1)\pi^{-s/2}\Gamma(1 + s/2)\zeta(s) \]
\[ \rho = \sigma + i\tau \]
\[ \alpha = \sigma(1 - \sigma) + \tau^2, \quad \beta = (1 - 2\sigma)\tau \]
\[ \Xi(\tau) = \xi(1/2 + i\tau) \]
\[ E_z(a) = \int_1^\infty \frac{dt}{t^z} e^{-at}, \]
\[ \psi(x) = \sum_{n=1}^\infty e^{-\pi n^2x} \quad J(\rho) = \int_0^1 dt [t^{\rho-2} + t^{(1-\rho)-2}] \psi(1/t^2)... \]

\( \gamma \) is the contour consisting of the two parallel lines \([c - i\infty, c + i\infty], [1 - c + i\infty, 1 - c - i\infty], 1 < c < 2 \), which span the critical strip \(0 < \sigma < 1, \rho_n = 1/2 + i\tau_n\) is the \(n\)-th zero of \(\zeta(s)\) on the critical line in the upper half plane.

The function \(\xi(s)\), introduced by Riemann[1], satisfies the simple functional equation \(\xi(1 - s) = \xi(s)\), is analytic and vanishes only in the critical strip. Consequently, by Cauchy’s theorem, one has, for \(0 < \sigma < 1\)

\[ \xi(s) = \int_\gamma \frac{dt}{2\pi i t - s} \] (1)

which, in view of the functional equation, can be written

\[ \xi(s) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t) \left( \frac{1}{t - s} + \frac{1}{t - 1 + s} \right) \] (2)

and expresses the values of \(\xi\) inside the critical strip entirely in terms of its values in a region where \(\zeta(s)\) is completely known from its defining series, say. In the following section this feature will be exploited to obtain several known and some, perhaps, unfamiliar identities.

Calculation

We begin by recalling the tabulated inverse Mellin transform[2]

\[ \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-t} \Gamma(t) \zeta(2t) = \sum_{n=1}^\infty e^{-\pi n^2x} \] (3)

from which, by differentiation, one finds the useful formula

\[ F(x) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-t} \xi(t) = 4\pi^2 x^4 \sum_{n=1}^\infty n^4 e^{-\pi n^2x^2} - 6\pi x^2 \sum_{n=1}^\infty n^2 e^{-\pi n^2x^2}, \] (4)
Parenthetically, we note that if \( f \) is analytic and odd, then \( f(1 - 2t) \xi(t) \) is odd under \( t \rightarrow (1 - t) \) so that

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} f(1 - 2t) \xi(t) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} f(1 - 2t) \xi(t) = 0.
\] (5)

Thus, by noting Romik’s formulas[3] we have, from (4)

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t) = \frac{1}{2} - \frac{\Gamma(5/4)}{\sqrt{2\pi}^{3/4}}
\] (6).

and from (5)

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t) \left[ \frac{1 - 2t}{4\pi^2 + (1 - 2t)^2} \right] = 0
\] (7)

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t)(2t - 1)^{2n+1} = 0, \quad n = 0, 1, 2, \ldots.
\] (8a)

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} t \xi(t) = \frac{1}{2} \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t) = \pi^2 \sum_{n=-\infty}^{\infty} n^2 e^{-\pi^2 n^2} - \frac{3\pi}{2} \sum_{n=-\infty}^{\infty} n^2 e^{-\pi^2 n^2}
\] (8b)

\[
= \frac{\Gamma(5/4)}{128\sqrt{2\pi}^{19/4}} \left[ \Gamma^8(1/4) - 96\pi^4 \right].
\] (8c)

\[
\int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \xi(t) t(1 - t) = \frac{1}{2} \left( 1 - \frac{\pi^{1/4}}{\Gamma(3/4)} \right) = \int_{1-c-i\infty}^{1-c+i\infty} \frac{dt}{2\pi i} \xi(t) t(1 - t)
\] (8d)

None of these appears to have been recorded previously.

Next, by rewriting (2), we have

**Theorem 1**

Within the critical strip Riemann’s function \( \xi(s) \) obeys the integral equation

\[
\xi(s) = 1 - \frac{\pi^{1/4}}{2\Gamma(3/4)} - \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \left[ 2s(1 - s) - t \right] \frac{2s(1 - s) - t}{s(1 - s) - t(1 - t)}, \quad 1 < c < 2.
\] (10)

or

\[
\xi(s) = \frac{1}{2} + \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} \left[ 1 - \frac{2s(1 - s) - t}{s(1 - s) - t(1 - t)} \right].
\] (11)

as well as
Corollary 1

\[ \xi(s) = 2\pi^2 \sum_{n=1}^{\infty} \int_{1}^{\infty} dt \left( t^{s/2} + t^{(1-s)/2} \right) \left( n^4 t - \frac{3}{2\pi^2} n^2 \right) e^{-n^2\pi t}. \quad (12) \]

\[ \frac{\pi^{1/4}}{2\Gamma(3/4)} + \pi \sum_{n=1}^{\infty} n^2 \left[ sE_{(1-s)/2}(\pi n^2) + (1-s)E_{-s/2}(\pi n^2) \right], \quad (13) \]

Here we point out that (10) is equivalent to the very important eq(3.10) in Milgram’s paper[4] and that that (13), apart from having summed a series, is LeClair’s key formula (15) in [5]

To explore further consequences of (2), note that the Mellin transform

\[ \phi(x) = \int_{c-i\infty}^{c+i\infty} \frac{dt}{2\pi i} x^{-s} \xi(t) \]

satisfies the linear differential equation

\[ \phi'(x) + \frac{s}{x} \phi(x) = -\frac{1}{x} F(x), \quad \phi(\infty) = 0 \quad (15) \]

so after a bit of easy analysis

\[ \phi(x) = 2\pi^2 \sum_{n=1}^{\infty} E_{\frac{3}{4}+1}(\pi n^2) - 3\pi \sum_{n=1}^{\infty} E_{\frac{3}{4}}(\pi n^2). \quad (16) \]

By applying (16) to (2) one has (note that \( \tau \) here is not restricted to be real)

**Theorem 2**

In the critical strip Riemann’s function \( \Xi(\tau) \) satisfies the integral equation

\[ \Xi(\tau) = \frac{1}{\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{t \xi(t)}{t^2 - \tau^2} dt, \quad -3/2 < c < -1/2. \quad (17) \]

**Corollary 2**

\[ \Xi(\tau) = 4\pi^2 \sum_{n=1}^{\infty} \int_{1}^{\infty} dt t^{1/4} \cos(\tau \ln \sqrt{t}) \left( n^4 t - \frac{3}{2\pi^2} n^2 \right) e^{-n^2\pi t}. \]

\[ = 4 \int_{1}^{\infty} dt t^{1/4} \cos \left[ \frac{1}{2} \tau \ln t \right] \left[ t\psi''(t) + \frac{3}{2} \psi'(t) \right] \]

or

\[ \Xi(\tau) = \frac{1}{2} - (\tau^2 + 1/4) \sum_{n=1}^{\infty} \Re E_{\frac{3}{4}+1/2}(\pi n^2) \]

\[ \frac{1}{2} - (\tau^2 + 1/4) \sum_{n=1}^{\infty} \Re E_{\frac{3}{4}+1/2}(\pi n^2) \]

\[ \Xi(\tau) = \frac{1}{2} - (\tau^2 + 1/4) \sum_{n=1}^{\infty} \Re E_{\frac{3}{4}+1/2}(\pi n^2) \]
\[ \xi(\rho) = \frac{1}{2} - (\alpha + i\beta) \int_0^1 dt [t^{\rho - 2} + t^{(1-\rho) - 2}] \psi(1/t^2). \]  

For \( \sigma = 1/2 \), thus restricting \( \tau \) to real values, (18) gives

**Corollary 3**

\( \rho \) is a zero of \( \zeta(s) \) on the critical line, \( \sigma = 1/2 \), if and only if,

\[ \text{Re} \int_0^1 t^{\rho - 2} \psi(1/t^2) dt = \frac{1}{4|\rho|^2}. \]  

**Discussion**

It is hard to believe that so many of these simple formulas have not been discovered previously, but as they do not appear in the classic literature, it seems worthwhile to adduce them once more. The author finds (20) particularly attractive and has verified it for the first 100 critical zeros available on Mathematica in its exact form, but even if \( \psi(x) \) is truncated to one exponential (20) is verified to nearly ten decimal place accuracy. In this case, the integral is simply \( E_{3/4+i\pi/2}(\pi) \) and by asymptotic expansion seems capable of producing a formula for \( \tau_n \) similar to LeClair’s[5] and Milgram’s[6], but with fewer complications.

Finally, if the integral \( J(\rho) \) in (19) is broken into real and imaginary parts and \( \rho \) is a zero of \( \zeta \) in the critical strip, but not on the critical line, i.e. \( 0 < \sigma < 1/2 \), then

\[ \alpha \text{Re}[J] - \beta \text{Im}[J] = \frac{1}{2} \]  

\[ \beta \text{Re}[J] + \alpha \text{Im}[J] = 0 \]  

These two relations give us

**Corollary 4**

The Riemann conjecture is true if, and only if,

\[ \text{Re} \int_0^1 dt [t^{\rho - 2} + t^{(1-\rho) - 2}] \psi(1/t^2) = \frac{\alpha}{2(\alpha^2 - \beta^2)} \]  

has no solution for \( 0 < \sigma < 1/2 \). Since the critical strip is known to be free of extraneous zeros to astronomical values of \( |\rho| \), the resolution of this matter might be settled by extracting a low order asymptotic estimate of the Mellin transform \( J(\rho) \) and analyzing the resulting algebraic equation. This is left as an exercise.
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References

[2] Ibid (2.16.1)