Admissible vectors, convolution Hilbert algebras, idempotents and weights

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Abstract

Admissible vectors for unitary representations of locally compact groups are the basis for group-frame and coherent state expansions. This work studies the existence and characterization of admissible vectors. Convolution Hilbert algebras, positive functions, square-integrable representations and weights on von Neumann algebras enter into the picture.

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1 Introduction

Let G be a locally compact (lc for brevity) group and let π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. This work studies the existence and description of **admissible vectors** for $\{\pi, \mathcal{H}_\pi\}$, i.e., vectors $\eta \in \mathcal{H}_\pi$ such that the operator

$$L_\eta : \mathcal{H}_\pi \to L^2(G), \quad [L_\eta \psi](x) = (\psi|\pi(x)\eta)\,,$$

is a bounded map and $L_\eta^* L_\eta = I_{\mathcal{H}_\pi}$, where $I_{\mathcal{H}_\pi}$ denotes the identity operator on $\mathcal{H}_\pi$. Admissible vectors lead to (weak) resolutions of the identity in terms of the orbit $\pi(G)\eta \subset \mathcal{H}_\pi$:

$$I_{\mathcal{H}_\pi} = \int_G |\pi(x)\eta)(\pi(x)\eta|\, dx\,,$$

*In memoriam.

where dx is a fixed left Haar measure for G (see Proposition 2 below). Relations of this type are known as **coherent state expansions** and the admissible vector η as *fiducial vector* in mathematical physics; see e.g. [1] and references therein. The orbit $\pi(G)\eta$ may also be interpreted as a **tight frame** for $\mathcal{H}_\pi$, a useful concept in harmonic analysis [2].

Admissible vectors have been explicitly discussed for irreducible representations [3], groups of type I [4, 5], unimodular separable groups [6] and countable discrete groups [7]. On the other hand, the notion of admissible vector is closely related to the subjects of **positive functions** and **square-integrable representations** of both groups and Hilbert algebras, and there is a huge amount of literature on these topics of interest here; see e.g. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. We shall comment the connections with these works all along the exposition.

Next we describe the organization and main results of the work.

Section 2 begins by introducing basic concepts and notation on lc groups G and their unitary representations. In particular, the *left* and *right regular representations*, λ and ρ, of G on $L^2(G)$ are defined, respectively, in (3) and (4). Among other things, in Proposition 2 we recall a well-known result:

> *η is an admissible vector for $\{\pi, \mathcal{H}_\pi\}$ if and only if the range $L_\eta\mathcal{H}_\pi$ of L_η is a closed invariant subspace of $L^2(G)$ for the left regular representation λ of G, π is equivalent to the subrepresentation $\lambda_{|L_\eta\mathcal{H}_\pi}$ and $L_\eta\eta$ is an admissible vector for $\lambda_{|L_\eta\mathcal{H}_\pi}$.*

In what follows, given an admissible vector η for $\{\pi, \mathcal{H}_\pi\}$, we shall write

$$g_\eta := L_\eta\eta, \quad \mathcal{H}_\eta := L_\eta\mathcal{H}_\pi\,.$$

Section 3 translates the concept of admissible vector to the context of the *convolution Hilbert algebras* associated to a lc group G [15, 18]. In some sense, the theory of Hilbert algebras is the non-commutative version of the algebra of all bounded square integrable functions on a measure space. Starting with the modular Hilbert algebra $C_c(G)$ of continuous functions with compact support, one can build the **full right** and **left convolution Hilbert algebras** $\mathcal{U}'$ and $\mathcal{U}''$, given in (14) and (15). Convolution products on the right $f \mapsto f * g$ extend to bounded operators $\pi_r(g)$ on $L^2(G)$ for elements g of $\mathcal{U}'$ and, in a similar way, convolution products on the left $f \mapsto g * f$ extend to bounded operators $\pi_l(g)$ on $L^2(G)$ for elements g of $\mathcal{U}''$, i.e.,

$$\pi_r(g)f := f * g, \quad g \in \mathcal{U}',\ f \in L^2(G)\,,$$

$$\pi_l(g)f := g * f, \quad g \in \mathcal{U}'',\ f \in L^2(G)\,.$$

$\mathcal{U}'$ and $\mathcal{U}''$ generate *right* and *left von Neumann algebras*, $\mathcal{R}_G$ and $\mathcal{L}_G$, as follows:

$$\mathcal{R}_G := \{\pi_r(g) : g \in \mathcal{U}'\}'' = \{\rho(x) : x \in G\}''\,,$$

$$\mathcal{L}_G := \{\pi_l(g) : g \in \mathcal{U}''\}'' = \{\lambda(x) : x \in G\}''\,,$$

(double commutant). One has $\mathcal{L}'_G = \mathcal{R}_G$. Main ingredients in the theory of Hilbert algebras are the *involution operators* $f \mapsto f^\sharp$ and $f \mapsto f^\flat$ with respective domains $\mathcal{D}^\sharp$ and $\mathcal{D}^\flat$, defined by (9) and (10), the **modular operator** Δ, given by (16), and the **modular conjugation** J, defined in (17).

Theorems 9 and 15 characterize admissible vectors in the following terms:

The following assertions are equivalent:

(i) η *is an admissible vector for* $\{\pi, \mathcal{H}_\pi\}$.

(ii) $g_\eta \in \mathcal{U}'$ *and* $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$.

(iii) $Jg_\eta \in \mathcal{U}''$ *and* $\pi_l(Jg_\eta) = P_{J\mathcal{H}_\eta}$.

In such case, $\mathcal{H}_\eta$ and $J\mathcal{H}_\eta$ are *reproducing kernel Hilbert spaces* [19] with respective kernels

$$k_\eta(x,y) := g_\eta(x^{-1}y), \quad \tilde{k}_\eta(x,y) := \delta_G^{-1}(x) Jg_\eta(yx^{-1}),$$

$P_{\mathcal{H}_\eta} \in \mathcal{R}_G$ and $P_{J\mathcal{H}_\eta} \in \mathcal{L}_G$.

Section 4 connects the above results with the classical theory of positive functions and square-integrable representations. Recall that an element $e \in \mathcal{U}'$ is called a **right self-adjoint idempotent** if $e = e^\flat = e^2$. Denote by $\mathcal{E}'$ the set of nonzero right self-adjoint idempotents of $\mathcal{U}'$. $\mathcal{E}'$ is a subset of $\mathcal{P}^\flat$, the convex cone of *right positive elements* of $L^2(G)$ or, in classical terms, the set of functions in $L^2(G)$ of *positive type*. By Theorem 20,

η *is an admissible vector for* $\{\pi, \mathcal{H}_\pi\}$ *if and only if* $g_\eta \in \mathcal{E}'$ *and* $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$.

In Theorem 24 we prove:

The following assertions are equivalent:

(i) $\{\pi, \mathcal{H}_\pi\}$ *has an admissible vector.*

(ii) $\{\pi, \mathcal{H}_\pi\}$ *is equivalent to a subrepresentation of* λ, $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$, *with a cyclic vector* $g \in \mathcal{U}'$ *such that* 0 *does not belong to the spectrum* $\sigma(|\pi_r(g)|)$ *of* $|\pi_r(g)|$ *or* 0 *is an isolated point of* $\sigma(|\pi_r(g)|)$.

In such case, $\{\pi, \mathcal{H}_\pi\}$ *is square-integrable and* $\pi_r(g)|\pi_r(g)|^{-2} g^\flat$ *is an admissible vector for* $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$.

Corollary 26 says:

The following assertions are equivalent:

(i) $\{\pi, \mathcal{H}_\pi\}$ *is irreducible and has an admissible vector.*

(ii) $\{\pi, \mathcal{H}_\pi\}$ *is equivalent to an irreducible subrepresentation of* λ, $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$, *such that* $\mathcal{H}_0 \cap \mathcal{D}^\flat \neq \emptyset$.

In such case, $\mathcal{H}_0 \cap \mathcal{E}' = \{e\}$ *and* e *is an admissible vector for* $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$.

Some results due to Perdrizet [12] and Phillips [13] are included to complete the picture.

Section 5 explores *standard forms* [20, 18] for the study of admissible vectors η in order to avoid the duality between $\mathcal{H}_\eta$ and $J\mathcal{H}_\eta$ and to give a description in terms of *weights*. As Takesaki [18] comments, "weights give a simultaneous generalization of positive linear functionals and traces, which corresponds to infinite measures in the commutative case". Taking into account the modular conjugation J defined in (17) and the self-dual closed convex cone $\mathfrak{P}$ of $L^2(G)$ given by

$$\mathfrak{P} := (\Delta^{-1/4}\mathcal{P}^\flat)^- ,$$

where the bar means closure, the quadruple $\{\mathcal{L}_G, L^2(G), J, \mathfrak{P}\}$ is a **standard form** of the von Neumann algebra $\mathcal{L}_G$. Every element of $L^2(G)$ is represented as a linear combination of four vectors of $\mathfrak{P}$ and to each positive linear functional ω in the predual $[\mathcal{L}_G]_*$ there corresponds a unique $g \in \mathfrak{P}$ with $\omega = \omega_g$, i.e.,

$$\omega(A) = \omega_g(A) := (Ag|g), \quad A \in \mathcal{L}_G .$$

The **Plancherel weight** φ_l on $\mathcal{L}_G$ is given in (25).

Now, let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. By Lemma 29, we can consider the closed subspace $\hat{\mathcal{H}}_\eta$ of $L^2(G)$ defined by

$$\hat{\mathcal{H}}_\eta := \langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle = \langle \mathcal{R}_G \Delta^{-1/4} g_\eta \rangle ,$$

where $\langle\cdot\rangle$ means $\overline{span}\{\cdot\}$. Let $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ denote, respectively, the reduced von Neumann algebras of $\mathcal{L}_G$ and $\mathcal{R}_G$ to $\hat{\mathcal{H}}_\eta$ (see footnote 3 and (23)). Let us consider the **associated weight** $\hat{\omega}_\eta$ on $\hat{\mathcal{L}}_\eta$ given by

$$\hat{\omega}_\eta(A) := \omega_{\Delta^{-1/4} g_\eta}(A) = (A\Delta^{-1/4} g_\eta | \Delta^{-1/4} g_\eta), \quad A \in \hat{\mathcal{L}}_\eta ,$$

In Theorem 30 and Proposition 33 we prove the following facts:

(a) $P_{\hat{\mathcal{H}}_\eta}$ *belongs to the center* $\mathcal{L}_G \cap \mathcal{R}_G$.

(b) $[\hat{\mathcal{L}}_\eta]' = \hat{\mathcal{R}}_\eta$.

(c) $\Delta^{-1/4} g_\eta$ *is a cyclic and separating vector of* $\hat{\mathcal{H}}_\eta$ *for* $\hat{\mathcal{L}}_\eta$.

(d) $[\pi_r(\Delta^{-1/4} g_\eta)]_{|\hat{\mathcal{H}}_\eta} = I_{\hat{\mathcal{H}}_\eta}$.

(e) If $\hat{J}_\eta := J_{|\hat{\mathcal{H}}_\eta}$ *and* $\hat{\mathfrak{P}}_\eta := \mathfrak{P} \cap \hat{\mathcal{H}}_\eta$*, then* $\{\hat{\mathcal{L}}_\eta, \hat{\mathcal{H}}_\eta, \hat{J}_\eta, \hat{\mathfrak{P}}_\eta\}$ *is a standard form of the von Neumann algebra* $\hat{\mathcal{L}}_\eta$.

(f) The modular operator $\hat{\Delta}_\eta$ *on* $\hat{\mathcal{H}}_\eta$ *associated to* $\hat{\omega}_\eta$ *satisfies*

$$\hat{\Delta}_\eta[\lambda(x)\Delta^{-1/4} g_\eta] = \rho(x)\Delta^{-1/4} g_\eta, \quad x \in G .$$

The **modular automorphism groups** $\{\sigma_t^{\varphi_l}\}_{t\in\mathbb{R}}$ and $\{\sigma_t^{\hat{\omega}_\eta}\}_{t\in\mathbb{R}}$ corresponding to the weights φ_l and $\hat{\omega}_\eta$ are of the form

$$\sigma_t^{\varphi_l}(A) := \Delta^{it} A \Delta^{-it}, \quad A \in \mathcal{L}_G\,,$$

$$\sigma_t^{\hat{\omega}_\eta}(A) := \hat{\Delta}_\eta^{it} A \hat{\Delta}_\eta^{-it}, \quad A \in \hat{\mathcal{L}}_\eta\,.$$

As Theorem 34 shows, they are related by the so-called **cocycle derivative** of $\hat{\omega}_\eta$ relative to φ_l, a one parameter family $\{U_t\}$ of partial isometries belonging to $\mathcal{L}_G$ with initial and final space $\hat{\mathcal{H}}_\eta$ in this case:

(a) $U_{s+t} = U_s \sigma_s^{\varphi_l}(U_t)$, $s, t \in \mathbb{R}$.

(b) $U_t U_t^* = U_t^* U_t = P_{\hat{\mathcal{H}}_\eta}$, $t \in \mathbb{R}$.

(c) $\sigma_t^{\hat{\omega}_\eta}(A) = U_t \sigma_t^{\varphi_l}(A) U_t^*$, $A \in \hat{\mathcal{L}}_\eta$, $t \in \mathbb{R}$.

Finally, Theorem 35 includes a sort of **orthogonality relations** in this context.

Some additional remarks in Section 6 close the work.

2 Preliminaries. Admissible vectors

Let G be a locally compact group (*lc group* for brevity). The identity of G will be denoted by e and elements of G by $x, y, \ldots$ From now on we consider a fixed **left Haar measure** dx on G (left Haar measures on G are proportional), that is, a Radon measure on G such that

$$d(xy) = dy, \quad x \in G\,.$$

In what follows $\delta_G : G \to \mathbb{R}^+ = (0,\infty)$ shall denote the **modular function**, a continuous homomorphism from G to the multiplicative group $\mathbb{R}^+$,

$$\delta_G(x) > 0, \quad \delta_G(e) = 1, \quad \delta_G(xy) = \delta_G(x)\delta_G(y)\,,$$

independent of the choice of dx and satisfying the following relations:

$$d(yx) = \delta_G(x)\, dy\,, \tag{1}$$

$$d(y^{-1}) = \delta_G(y^{-1})\, dy\,. \tag{2}$$

The group G is called *unimodular* if $\delta_G = 1$. In particular, abelian, discrete and compact groups are unimodular. See e.g. [21, Chapter 2] for details.

As usual,

$$C(G), \quad C_0(G), \quad C_c(G)$$

denote, respectively, the spaces of continuous, continuous vanishing at infinity and continuous with compact support, complex-valued functions on G. For $1 \le p \le \infty$, for the L^p-spaces associated to the measure dx on G we will write $L^p(G) : L^p(G, dx)$ and for the scalar product in $L^2(G)$,

$$(f|g) := \int_G f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(G)\,,$$

where overline denotes complex-conjugate.

By a **(continuous) unitary representation** of G we mean a homomorphism π from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on some nonzero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong (weak) operator topology. The most basic examples are the *left regular representation* λ of G on $L^2(G)$ defined by

$$[\lambda(x)f](y) := f(x^{-1}y), \quad x \in G,\, f \in L^2(G)\,, \tag{3}$$

and the *right regular representation* ρ of G on $L^2(G)$ given by

$$[\rho(x)f](y) := \delta_G^{1/2}(x)f(yx), \quad x \in G,\, f \in L^2(G)\,. \tag{4}$$

If π_1 and π_2 are unitary representations of G, an *intertwining operator* for π_1 and π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ such that

$$T\pi_1(x) = \pi_2(x)T, \quad x \in G\,.$$

The set of all such operators is denoted by $\mathcal{C}(\pi_1, \pi_2)$. We shall write $\mathcal{C}(\pi)$ for $\mathcal{C}(\pi, \pi)$; it is called the *commutant* or *centralizer* of π. $\mathcal{C}(\pi)$ is a von Neumann algebra.

Two unitary representations π_1 and π_2 of G are said *(unitarily) equivalent* if there exists a unitary operator $U \in \mathcal{C}(\pi_1, \pi_2)$.

A closed subspace $M \subset \mathcal{H}_\pi$ is called an *invariant subspace* for the unitary representation π of G if $\pi M \subset M$. A closed subspace M is invariant under π if and only if the orthogonal projection P_M from $\mathcal{H}_\pi$ onto M belongs to $\mathcal{C}(\pi)$. In such case, the restriction of π to M defines a representation of G on M called a *subrepresentation* of π and denoted by $\pi_{|M}$. If M is invariant under π, then so is the orthogonal complement $M^\perp$ and π is the *direct sum* of $\pi_{|M}$ and $\pi_{M^\perp}$. If π admits an invariant subspace that is nontrivial (i.e., different from $\{0\}$ and $\mathcal{H}_\pi$), then π is called *reducible*, otherwise π is *irreducible*.

In what follows, for any subset R of a Hilbert space $\mathcal{H}$,

$$\langle R \rangle$$

denotes the closed subspace of $\mathcal{H}$ spanned by R.

Clearly, given $f \in \mathcal{H}_\pi$, the closed linear span $\langle \pi(G)f \rangle$ of $\{\pi(x)f : x \in G\}$ is invariant under π. $\langle \pi(G)f \rangle$ is called the *cyclic subspace* generated by f. If $\langle \pi(G)f \rangle = \mathcal{H}_\pi$, f is called a **cyclic vector** for π. π is called a *cyclic representation* if it has a cyclic vector. Every unitary representation is a direct sum of cyclic representations. See e.g. [21, Chapter 3] for details.

Definition 1 *Let G be a lc group and let π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$ (we will write $\{\pi, \mathcal{H}_\pi\}$ for brevity). A vector $\eta \in \mathcal{H}_\pi$ is called an* **admissible vector** *for $\{\pi, \mathcal{H}_\pi\}$ if the operator*

$$L_\eta : \mathcal{H}_\pi \to L^2(G), \quad [L_\eta \psi](x) = (\psi | \pi(x)\eta)\,,$$

is a bounded map and $L_\eta^* L_\eta = I_{\mathcal{H}_\pi}$, where $I_{\mathcal{H}_\pi}$ denotes the identity operator on $\mathcal{H}_\pi$.

The following facts are well-known.

Proposition 2 *Let G be a lc group and let $\{\pi, \mathcal{H}_\pi\}$ be a unitary representation of G. The following are equivalent:*

(i) *η is an admissible vector for $\{\pi, \mathcal{H}_\pi\}$.*

(ii) *L_η is an isometry from $\mathcal{H}_\pi$ into $L^2(G)$.*

(iii) *For every $\psi, \phi \in \mathcal{H}_\pi$,*

$$\int_G (\psi|\pi(x)\eta)(\pi(x)\eta|\phi)\, dx = (\psi|\phi)\,. \tag{5}$$

(iv) *The range $L_\eta\mathcal{H}_\pi$ of L_η is a closed invariant subspace of $L^2(G)$ for the left regular representation λ of G, π is equivalent to the subrepresentation $\lambda_{|L_\eta\mathcal{H}_\pi}$ and $L_\eta\eta$ is an admissible vector for $\lambda_{|L_\eta\mathcal{H}_\pi}$.*

Proof: Since the conditions (1) $L_\eta^* L_\eta = I_{\mathcal{H}_\pi}$, (2) $(L_\eta^* L_\eta \psi|\phi) = (\psi, \phi)$ for all $\psi, \phi \in \mathcal{H}_\pi$, (3) $(L_\eta^* L_\eta \psi|\psi) = (\psi, \psi)$ for all $\psi \in \mathcal{H}_\pi$, (4) $||L_\eta\psi||^2 = ||\psi||^2$ for all $\psi \in \mathcal{H}_\pi$ are mutually equivalent (the equivalence between (2) and (3) follows by polarization), η is admissible if and only if L_η is an isometry. In such case, the range $L_\eta\mathcal{H}_\pi$ of L_η is a closed subspace of $L_2(G)$ and $L_\eta L_\eta^* = P_{L_\eta\mathcal{H}_\pi}$, the orthogonal projection onto $L_\eta\mathcal{H}_\pi$. Condition (2) above is just (iii).

Now, by the definition of L_η, for $\psi \in \mathcal{H}_\pi$ and $x, y \in G$,

$$\begin{aligned}
[\lambda(x)L_\eta\psi](y) &= [L_\eta\psi](x^{-1}y) = (\psi|\pi(x^{-1}y)\eta) = \\
&= (\pi(x)\psi|\pi(y)\eta) = [L_\eta\pi(x)\psi](y)\,,
\end{aligned}$$

so that

$$\lambda(x)L_\eta = L_\eta\pi(x), \quad x \in G\,. \tag{6}$$

Taking adjoints,

$$L_\eta^*\lambda(x) = \pi(x)L_\eta^*, \quad x \in G\,.$$

Thus,

$$P_{L_\eta\mathcal{H}_\pi}\lambda(x) = \lambda(x)P_{L_\eta\mathcal{H}_\pi}, \quad x \in G\,.$$

This means that the range $L_\eta\mathcal{H}_\pi$ is a closed invariant subspace of the left regular representation λ and, moreover, since $L_\eta : \mathcal{H}_\pi \to L_\eta\mathcal{H}_\pi$ is unitary, one has the equivalence between (i) and (iv). □

Corollary 3 *An admissible vector η is a cyclic vector for $\{\pi, \mathcal{H}_\pi\}$.*

Proof: Suppose that η is not a cyclic vector. Then, there exists $0 \neq \psi \in \langle\pi(G)\eta\rangle^\perp$, so that $(\psi|\pi(x)\eta) = 0$ for all $x \in G$ and (5) cannot be satisfied. □

Remark 4 (a) Equation (5) also leads to an expression for L_η^* (the "partial inverse" of L_η): since every $f \in L_\eta \mathcal{H}_\pi \subset L^2(G)$ is of the form $f(x) = (\psi|\pi(x)\eta)$ for some $\psi \in \mathcal{H}_\pi$,

$$L_\eta^*(f) = \begin{cases} \int_G f(x)\,\pi(x)\eta\,dx, & \text{if } f \in L_\eta \mathcal{H}_\pi\,, \\ 0, & \text{if } f \in (L_\eta \mathcal{H}_\pi)^\perp\,, \end{cases}$$

where the integral must be interpreted in weak sense.

(b) From (5) with $\phi = \psi = \eta$,

$$||\eta||_{\mathcal{H}_\pi} = \int_G |(\eta|\pi(x)\eta)|^2\,dx\,.$$

In particular, $\int_G |(\eta|\pi(x)\eta)|^2\,dx < \infty$. This is the original condition in the definition of an admissible vector given by Grossmann, Morlet and Paul in [3]. Moreover, the crucial constants c_η leading to the Duflo-Moore operator C in [3, Th.3.1], are here

$$c_\eta = \frac{1}{||\eta||_{\mathcal{H}_\pi}} \int_G |(\eta|\pi(x)\eta)|^2\,dx = 1\,.$$

(c) Rieffel [11, Th.4.6] proves that a unitary representation $\{\pi, \mathcal{H}_\pi\}$ of G with a cyclic vector η such that $(\eta|\pi(x)\eta) \in L^2(G)$ is a subrepresentation of the left regular representation λ.

(d) Duflo and Moore [22, Th.2] show that, if $\{\pi, \mathcal{H}_\pi\}$ is an irreducible unitary representation of G, then π is equivalent to a subrepresentation of the left regular representation λ if and only if it has a nonzero square integrable coefficient, i.e., there exist $\phi, \psi \in \mathcal{H}_\pi$ such that $(\phi|\pi(x)\psi) \in L^2(G)$.

3 Convolution Hilbert algebras

In this section we translate the concept of admissible vector to the context of the convolution Hilbert algebras associated to the lc group G. The terminology and (partially) the notation are borrowed from Takesaki [18].

Definition 5 *An involutive algebra $\mathcal{U}$ over $\mathbb{C}$ with involution $f \in \mathcal{U} \mapsto f^\sharp \in \mathcal{U}$ (resp. $f \mapsto f^\flat$) is called a* **left** *(resp.* **right***)* **Hilbert algebra** *if $\mathcal{U}$ admits an inner product satisfying the following postulates:*

a) Each fixed $f \in \mathcal{U}$ gives rise to a bounded operator

$$\pi_l(f) : g \in \mathcal{U} \mapsto fg \in \mathcal{U}, \quad (\textit{resp. } \pi_r(f) : g \in \mathcal{U} \mapsto gf \in \mathcal{U}\ )$$

by multiplying from the left (resp. right);

b) $(fg|h) = (g|f^\sharp h)$ (resp. $((fg|h) = (f|hg^\flat)$);

c) The involution $f \in \mathcal{U} \mapsto f^\sharp \in \mathcal{U}$ (resp. $f \mapsto f^\flat$) is preclosed;

d) The subalgebra, denoted $\mathcal{U}^2$, spanned linearly by all possible products fg, $f, g \in \mathcal{U}$, is dense in $\mathcal{U}$ with respect to the inner product.

*If the involution of a left Hilbert algebra $\mathcal{U}$ is an isometry, then it is also a right Hilbert algebra. In this case, we say that $\mathcal{U}$ is a (**unimodular**) Hilbert algebra and the involution of $\mathcal{U}$ is denoted by $f \in \mathcal{U} \mapsto f^* \in \mathcal{U}$.*

If G is a lc group, the space $C_c(G)$ with the convolution product $f * g$ and involution $f \mapsto f^\sharp$ (resp. $f \mapsto f^\flat$) is a left Hilbert algebra (resp. right Hilbert algebra) for the usual inner product $(\cdot|\cdot)$ of $L^2(G)$, where

$$\begin{aligned} [fg](x) = [f * g](x) &:= \int_G f(y) g(y^{-1}x)\, dy, \quad x \in G\,, \\ f^\sharp(x) &:= \delta_G(x^{-1})\, \overline{f(x^{-1})}, \quad x \in G\,, \\ f^\flat(x) &:= \overline{f(x^{-1})}, \quad x \in G\,. \end{aligned}$$

The von Neumann algebra generated by $\{\pi_l(f) : f \in C_c(G)\}$ coincides with that one generated by $\{\lambda(x) : x \in G\}$. It is called the **left von Neumann algebra** of G and denoted by $\mathcal{L}_G$, i.e.,

$$\mathcal{L}_G := \{\pi_l(f) : f \in C_c(G)\}'' = \{\lambda(x) : x \in G\}'' \tag{7}$$

(double commutant). In a similar way, the von Neumann algebra

$$\mathcal{R}_G := \{\pi_r(g) : g \in C_c(G)\}'' = \{\rho(x) : x \in G\}'' \tag{8}$$

is called the **right von Neumann algebra** of G. One has

$$\mathcal{L}_G' = \mathcal{R}_G\,.$$

See e.g. [18, VII.3.1].

The involutions $f \mapsto f^\sharp$ and $f \mapsto f^\flat$ can be extended to the following domains of definition:

$$\begin{aligned} \mathcal{D}^\sharp &:= \left\{ f \in L^2(G) : \textstyle\int_G \delta_G(x) ||f(x)||^2\, dx < \infty \right\}, \\ \mathcal{D}^\flat &:= \left\{ f \in L^2(G) : \textstyle\int_G \delta_G^{-1}(x) ||f(x)||^2\, dx < \infty \right\}. \end{aligned} \tag{9}$$

The corresponding extensions are closed densely defined operators on $L^2(G)$ and bijective involutions on their own domains [18, VI.1.5]. It is usual to denote these extensions by S and F, i.e.,

$$S : \mathcal{D}^\sharp \to \mathcal{D}^\sharp : f \mapsto f^\sharp, \quad F : \mathcal{D}^\flat \to \mathcal{D}^\flat : g \mapsto g^\flat\,, \tag{10}$$

One has $S = S^{-1}$ on $\mathcal{D}^\sharp$ and $F = F^{-1}$ on $\mathcal{D}^\flat$.

First of all we explicit an elementary relation which is the basis of the main results in this section.

Lemma 6 *For $f, g \in L^2(G)$,*

$$(f|\lambda(x)g) = f * g^\flat(x), \quad x \in G\,,$$

g belonging to $\mathcal{D}^\flat$ or not.

Proof: For $x \in G$, one has

$$\begin{aligned}(f|\lambda(x)g_\eta) &= \int_G f(y)\overline{g_\eta(x^{-1}y)}\,dy = \\ &= \int_G f(y)g_\eta^\flat(y^{-1}x)\,dy = f * g_\eta^\flat(x)\,.\end{aligned}$$

□

Let us come back to the context of Definition 1. In what follows, for an admissible vector η for $\{\pi, \mathcal{H}_\pi\}$ we will write

$$g_\eta := L_\eta \eta, \quad \mathcal{H}_\eta := L_\eta \mathcal{H}_\pi\,. \tag{11}$$

The following result is written in terms of the involution F and the convolution product.

Proposition 7 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Then:*

(i) $\mathcal{H}_\eta \subset C(G) \cap L^2(G)$.

(ii) *Let $f \in L^2(G)$. Then, $f \in \mathcal{H}_\eta$ if and only if*

$$f(x) = f * g_\eta^\flat(x), \quad x \in G\,.$$

If $f \in \mathcal{H}_\eta^\perp$, then $f * g_\eta^\flat = 0$.

(iii) $g_\eta = g_\eta * g_\eta^\flat = g_\eta^\flat$.

Proof: (i) Each $f \in \mathcal{H}_\eta \subset L^2(G)$ is of the form $f(x) = L_\eta\psi = (\psi|\pi(x)\eta)$, $x \in G$, for some $\psi \in \mathcal{H}_\pi$. Since the representation π is strong (weak) continuous, $\mathcal{H}_\eta \subset C(G)$

(ii) Let $f \in \mathcal{H}_\eta$. Using the intertwining relation (6) and Lemma 6 one gets, for $x \in G$,

$$\begin{aligned}f(x) &= (\psi|\pi(x)\eta) = (\psi|L_\eta^* L_\eta \pi(x)\eta) = \\ &= (L_\eta\psi|L_\eta\pi(x)\eta) = (L_\eta\psi|\lambda(x)L_\eta\eta) = \\ &= (f|\lambda(x)g_\eta) = f * g_\eta^\flat(x)\,.\end{aligned} \tag{12}$$

If $f \in \mathcal{H}_\eta^\perp$, since $\mathcal{H}_\eta = \langle \lambda(x)g_\eta : x \in G\rangle$, again by Lemma 6,

$$f * g_\eta^\flat(x) = (f|\lambda(x)g_\eta) = 0, \quad x \in G\,.$$

Thus, for $f \in L^2(G)$ with $f \notin \mathcal{H}_\eta$,

$$f * g_\eta^\flat(x) = (f|\lambda(x)g_\eta) = (P_{\mathcal{H}_\eta} f|\lambda(x)g_\eta) = P_{\mathcal{H}_\eta} f(x)$$

and the last expression must be different from $f(x)$ for some $x \in G$, since $P_{\mathcal{H}_\eta} f \neq f$.

(iii) The first equality in (iii) is just (ii) with $f = g_\eta$; the second equality follows from

$$\begin{aligned} f * g_\eta^\flat(x) &= \int_G f(y) g_\eta^\flat(y^{-1}x)\, dy = \\ &= \int_G f(y)\overline{g_\eta(x^{-1}y)}\, dy = \\ &= \int_G f(xy)\overline{g_\eta(y)}\, dy = \\ &= \int_G \overline{f^\flat(y^{-1}x^{-1})}\,\overline{g_\eta(y)}\, dy = \\ &= \overline{g_\eta * f^\flat(x^{-1})} = [g_\eta * f^\flat]^\flat(x) \end{aligned} \tag{13}$$

with $f = g_\eta$. □

Remark 8 (a) Proposition 7.(iii) proves implicitly that $g_\eta \in \mathcal{D}^\flat$. Although we use the symbol $g_\eta^\flat$ in (ii), it is not assumed there, neither in (13), that $g_\eta \in \mathcal{D}^\flat$.

(b) By Proposition 7.(iii),

$$g_\eta \in \mathcal{P}(G) \cap L^2(G) \subset A(G)\,,$$

where $\mathcal{P}(G)$ denotes the set of all continuous functions of positive type on G (see e.g. Godement [8]), and $A(G)$ is the usual *Fourier algebra* of G introduced by Eymard [23]. Recall that $A(G)$ is identified with the predual $[\mathcal{L}_G]_*$ of $\mathcal{L}_G$ [23, Th.3.10]. Se also [18, Section VII.3] and [24, Chapter 3].

(c) Proposition 7 implies that g_η is a *convolution square root* of the matrix element $(\eta|\pi(x)\eta)$, the *characteristic function* of the cyclic representation $\{\pi, \mathcal{H}_\pi\}$. The existence of a square-integrable root of positive type for a continuous square-integrable function of positive type on a lc group is proved by Godement [8, Th.17]. See also Dixmier [15, Th.13.8.6]. Moreover, according to Godement's terminology [8, Sect.29], g_η is a *unit* of G, i.e., a square-integrable function of positive type such that $g_\eta * g_\eta = g_\eta$.

(d) Proposition 7.(ii) says that the range $\mathcal{H}_\eta$ is a *reproducing kernel Hilbert space* [19] with kernel

$$k_\eta(x,y) := g_\eta(x^{-1}y)\,.$$

Obviously, $\mathcal{H}_\eta$ is invariant under the left regular representation λ of G.

Now, we include some definitions in order to characterize the admissible vectors in Theorem 9.

A vector $g \in L^2(G)$ is said to be **right bounded** if

$$\sup\{||f * g|| : f \in C_c(G),\, ||f|| \leq 1\} < \infty$$

The set of all right bounded vectors is denoted by $\mathcal{B}'$. A vector $g \in \mathcal{H}$ is right bounded if and only if $\pi_r(g) \in \mathcal{L}(L^2(G))$, the space of bounded operators on $L^2(G)$. Moreover, $\mathcal{B}'$ is invariant under $\mathcal{R}_G$,

$$\mathfrak{n}_r := \pi_r(\mathcal{B}')$$

is a left ideal of $\mathcal{R}_G$ and $\pi_r(Ag) = A\pi_r(g)$, for $A \in \mathcal{R}_G$ and $g \in \mathcal{B}'$. Let us define

$$\mathcal{U}' := \mathcal{B}' \cap \mathcal{D}^\flat\,. \tag{14}$$

$\mathcal{U}'$ is a (full) right Hilbert algebra, $\pi_r(\mathcal{U}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*$ and the von Neumann algebra generated by $\mathcal{U}'$ coincides with $\mathcal{R}_G$ [18, VI.1.9-15].

The full convolution right Hilbert algebra $\mathcal{U}'$ permit us to identify the admissible vectors. The result is a straightforward consequence of Proposition 7.

Theorem 9 *The following are equivalent:*

(i) η *is an admissible vector for* $\{\pi, \mathcal{H}_\pi\}$.

(ii) $g_\eta \in \mathcal{U}'$ *and* $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$, *where* $P_{\mathcal{H}_\eta}$ *denotes the orthogonal projection from* $L^2(G)$ *onto* $\mathcal{H}_\eta$.

Proof: (i)$\Rightarrow$(ii): If η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$, according to Proposition 7.(iii), $g_\eta = g_\eta^\flat$, so that $g_\eta \in \mathcal{D}^\flat$. Moreover, by Proposition 7.(ii), $g_\eta^\flat \in \mathcal{B}'$ and the orthogonal projection $P_{\mathcal{H}_\eta}$ coincides with

$$\pi_r(g_\eta^\flat) = \pi_r(g_\eta)^* = \pi_r(g_\eta) = \pi_r(g_\eta * g_\eta^\flat) = \pi_r(g_\eta)^2.$$

(ii)$\Rightarrow$(i): If $g_\eta \in \mathcal{U}'$ and $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$, then $\pi_r(g_\eta) = \pi_r(g_\eta)^* = \pi_r(g_\eta^\flat)$ and $\pi_r(g_\eta^\flat)f = f$ for $f \in L_\eta\eta$. Since

$$[\pi_r(g_\eta^\flat)f](x) = [f * g_\eta^\flat](x) = (f|\lambda(x)g_\eta), \quad x \in G,\, f \in \mathcal{H}_\eta\,,$$

we have that g_η is an admissible vector for $\{\lambda_{|\mathcal{H}_\eta}, \mathcal{H}_\eta\}$. The equivalence (i)$\Leftrightarrow$(iv) of Proposition 2 leads to the result. □

The next result is included explicitly to complete the picture.

Corollary 10 *Let* η *be an admissible vector for* $\{\pi, \mathcal{H}_\pi\}$. *Then* $\mathcal{H}_\eta$ *is an invariant subspace of* $L^2(G)$ *for the left regular representation* λ *and* g_η *is a cyclic vector for the subrepresentation* $\{\lambda_{|\mathcal{H}_\eta}, \mathcal{H}_\eta\}$.

Proof: The result follows from Corollary 3 and the equivalence (i)$\Leftrightarrow$(iv) of Proposition 2. $\square$

Remark 11 (a) For a unimodular lc group G, the orthogonal projections in $L^2(G)$ of the form $\pi_r(g)$, $g \in L^2(G)$, are called *finite projections* by Segal [10]. The set of finite projections forms a sub-lattice of the lattice of all projections in $\mathcal{R}_G$ and every projection in $\mathcal{R}_G$ is the least upper bound of the finite projections it bounds [10, Th.2]. Thanks to these results, Segal [10, Th.3] establishes a generalization of the Plancherel formula for separable unimodular lc groups. The Plancherel formula is written in terms of the central decomposition of $\mathcal{R}_G$ (in fact Segal works with $\pi_l(g)$ and $\mathcal{L}_G$). An equivalent approach is given by Ambrose [9] dealing with the so-called *L^2-systems* and *$\mathcal{H}$-systems*, precursors of the concept of Hilbert algebra. A generalization to Hilbert algebras of the theory of square-integrable representations of unimodular lc groups can be found in Rieffel [11], where *self-adjoint idempotents* in Hilbert algebras are treated along the lines developed by Ambrose [9]. Following Rieffel's work [11], a theory of square-integrable representations which works for arbitrary full left Hilbert algebras (and, hence, for arbitrary lc groups) is given by Phillips [13]. The emphasis in [11, 13] is put on an extension to Hilbert algebras of Godement's theorem [8, Th.17] on the existence of convolution square roots; see Remark 8.c. Stetkaer [17] expands the study of square-integrable representations to representations induced from a character of a closed subgroup, mainly for G unimodular. We will enter into some details of these approaches in Section 4.

(b) For a unimodular lc group G, Carey [25, Lemma 2.5] proves that if $\mathcal{H} \subset L^2(G)$ is a reproducing kernel Hilbert space which is invariant under the left regular representation λ of G, then there exists $g \in L^2(G)$ such that $\pi_r(g) = P_{\mathcal{H}}$. See Remark 8.d. For separable and type I unimodular lc groups, Carey [25] uses the natural traces on $\mathcal{L}_G^+$ and $\mathcal{R}_G^+$ (the positive elements of $\mathcal{L}_G$ and $\mathcal{R}_G$) and the Plancherel theorem given by Dixmier [15] to study the Plancherel measure with the aid of the work of Segal [10].

We can *dualize* the above discussion entirely starting from the right Hilbert algebra $\mathcal{U}'$. For it, we shall say that a vector $f \in L^2(G)$ is **left bounded** if

$$\sup\{||\pi_r(g)f|| : g \in \mathcal{U}',\, ||g|| \le 1\} < \infty$$

The set of all left bounded vectors is denoted by $\mathcal{B}$. Clearly, $\mathcal{B}$ contains $C_c(G)$ and to each $f \in \mathcal{B}$ there corresponds a bounded operator $\pi_l(f)$ on $L^2(G)$ determined by

$$\pi_l(f)g = \pi_r(g)f = f * g, \quad g \in \mathcal{U}'\,.$$

$\mathcal{B}$ is invariant under $\mathcal{L}_G$,

$$\mathfrak{n}_l := \pi_l(\mathcal{B})$$

is a left ideal of $\mathcal{L}_G$ and $\pi_l(Af) = A\pi_l(f)$, for $A \in \mathcal{L}_G$ and $f \in \mathcal{B}$. As before, we further extend products of vectors of $L^2(G)$ as follows:

$$f * g = \pi_l(f)g, \quad f \in \mathcal{B},\, g \in L^2(G)\,.$$

We then set

$$\mathcal{U}'' := \mathcal{B} \cap \mathcal{D}^\sharp\,. \tag{15}$$

$\mathcal{U}''$ is a (full) left Hilbert algebra, $C_c(G) \subset \mathcal{U}''$ and the von Neumann algebra generated by $\mathcal{U}''$ coincides with $\mathcal{L}_G$. As before, $\pi_l(\mathcal{U}'') = \mathfrak{n}_l \cap \mathfrak{n}_l^*$.

Two operators play a fundamental role in the theory of convolution Hilbert algebras. They are the **modular operator** Δ defined by

$$\left.\begin{aligned} \Delta : \mathcal{D}_\Delta \to L^2(G), \quad [\Delta f](x) := \delta_G(x) f(x)\,, \\ \mathcal{D}_\Delta = \left\{ f \in L^2(G) : \int_G \delta_G^2(x) ||f(x)||^2\, dx < \infty \right\}, \end{aligned}\right\} \tag{16}$$

and the **modular conjugation** J given by

$$J : L^2(G) \to L^2(G), \quad [Jf](x) := \delta_G^{-1/2}(x) \overline{f(x^{-1})}\,. \tag{17}$$

For the sake of completeness, some of their properties are collected in the following result [18, VI.1.5 and VI.1.19]. Recall the definition of the involutions S and F and their domains given in (9) and (10).

Lemma 12 (Tomita-Takesaki) *(i) $\Delta = FS$ is a linear positive non-singular self-adjoint operator such that $\mathcal{D}(\Delta^{1/2}) = \mathcal{D}^\sharp$ and $\mathcal{D}(\Delta^{-1/2}) = \mathcal{D}^\flat$.*

(ii) J is an antilinear isometry of $L^2(G)$ onto itself such that

a) $(Jf|Jg) = (g|f)$, for $f, g \in \mathcal{H}$,

b) $J = J^{-1}$ or, equivalently, $J^2 = I$,

c) $J\Delta J = \Delta^{-1}$,

d) $S = J\Delta^{1/2} = \Delta^{-1/2}J$,

e) $F = J\Delta^{-1/2} = \Delta^{1/2}J$.

(iii) J maps $\mathcal{U}''$ (resp. $\mathcal{U}'$) onto $\mathcal{U}'$ (resp. $\mathcal{U}''$) anti-isomorphically in the sense that

$$\begin{aligned} \pi_r(Jf) &= J\pi_l(f)J, \quad f \in \mathcal{U}''\,, \\ \pi_l(Jg) &= J\pi_r(g)J, \quad g \in \mathcal{U}'\,, \\ J(f * g) &= (Jg) * (Jf), \quad f, g \in \mathcal{U}''\,. \end{aligned}$$

Moreover, $\mathcal{L}(\mathcal{U}'') := \{\pi_l(\mathcal{U}'')\}'' = \mathcal{L}_G$, $\mathcal{R}(\mathcal{U}') := \{\pi_r(\mathcal{U}')\}'' = \mathcal{R}_G$ and,

$$J\mathcal{L}_G J = \mathcal{R}_G\,, \quad J\mathcal{R}_G J = \mathcal{L}_G\,.$$

The left regular representation λ and the right regular representation ρ of a lc group G, defined in (3) and (4), are related as follows:

Lemma 13 *Let G be a lc group. Then, for $x \in G$,*

$$J\rho(x)J = \lambda(x), \quad J\lambda(x)J = \rho(x)\,.$$

Proof: For $f \in L^2(G)$ and $x, y \in G$,

$$\begin{aligned}
[J\rho(x)Jf](y) &= J\rho(x)[\delta_G^{-1/2}(y)\overline{f(y^{-1})}] = \\
&= J[\delta^{1/2}(x)\delta_G^{-1/2}(yx)\overline{f(y^{-1}x)}] = \\
&= \delta_G^{-1/2}(y)\delta_G^{-1/2}(y^{-1})f(x^{-1}y) = \\
&= [\lambda(x)f](y)\,.
\end{aligned}$$

Now, use $J^2 = I$ to prove the second equality. $\square$

The next results are the dual versions of Proposition 7, Theorem 9 and Corollary 10.

Proposition 14 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Then:*

(i) $\pi_l(Jg_\eta) = J\pi_r(g_\eta)J = JP_{\mathcal{H}_\eta}J$ *is an orthogonal projection. Let us put*

$$\tilde{\mathcal{H}}_\eta := JP_{\mathcal{H}_\eta}JL^2(G)\,.$$

(ii) *J is an antiunitary operator from $\mathcal{H}_\eta$ onto $\tilde{\mathcal{H}}_\eta$.*

(iii) $Jg_\eta \in \mathcal{D}^\sharp$ *and* $[Jg_\eta]^\sharp = SJg_\eta = Jg_\eta$.

(iv) *For $f \in \mathcal{H}_\eta$ and $x \in G$,*

$$Jf(x) = \delta_G^{1/2}(x^{-1})(Jf|\rho(x^{-1})Jg_\eta) = [Jg_\eta * Jf](x)\,.$$

(v) $Jg_\eta = [Jg_\eta]^\sharp * [Jg_\eta] = [Jg_\eta] * [Jg_\eta]^\sharp = [Jg_\eta]^\sharp$.

Proof: (i) Since $JJ = I$, one has

$$[JP_{\mathcal{H}_\eta}J]^2 = JP_{\mathcal{H}_\eta}JJP_{\mathcal{H}_\eta}J = JP_{\mathcal{H}_\eta}J\,.$$

Moreover, for $f, g \in L^2(G)$,

$$(JP_{\mathcal{H}_\eta}Jf|g) = (Jg|P_{\mathcal{H}_\eta}Jf) = (P_{\mathcal{H}_\eta}Jg|Jf) = (f|JP_{\mathcal{H}_\eta}Jg)\,,$$

that is, $[JP_{\mathcal{H}_\eta}J]^* = JP_{\mathcal{H}_\eta}J$.

(ii) For $f \in \mathcal{H}_\eta$ one has $P_{\mathcal{H}_\eta}f = f$ and

$$Jf = JP_{\mathcal{H}_\eta}f = JP_{\mathcal{H}_\eta}JJf = P_{\tilde{\mathcal{H}}_\eta}Jf\,.$$

Now interchange the roles of $\mathcal{H}_\eta$ and $\tilde{\mathcal{H}}_\eta$.

(iii) Since $S = \Delta^{-1/2} J$, $F = \Delta^{1/2} J$ and $F g_\eta = g_\eta$,

$$SJg_\eta = \Delta^{-1/2} g_\eta = \Delta^{-1/2} F g_\eta = J g_\eta \,.$$

(iv) By Lemma 13 and Proposition 7.(ii), for $f \in \mathcal{H}_\eta$ and $x \in G$,

$$\begin{aligned}
\delta_G^{1/2}(x^{-1})(Jf|\rho(x^{-1})Jg_\eta) &= \delta_G^{-1/2}(x)(J\rho(x^{-1})Jg_\eta|f) = \\
&= \delta_G^{-1/2}(x)(\lambda(x^{-1})g_\eta|f) = \\
&= \delta_G^{-1/2}(x)(\lambda(x^{-1})g_\eta|f) = \\
&= \delta_G^{-1/2}(x)\overline{f(x^{-1})} = Jf(x)\,.
\end{aligned}$$

On the other hand, using (1), (2) and Proposition 7.(iii), for $f \in \mathcal{H}_\eta$ and $x \in G$,

$$\begin{aligned}
\delta_G^{1/2}(x^{-1})(Jf|\rho(x^{-1})Jg_\eta) = \delta_G^{-1/2}(x) \int_G Jf(y)\overline{\rho(x^{-1})Jg_\eta(y)}\, dy = \\
= \delta_G^{-1/2}(x) \int_G Jf(y)\overline{\delta^{1/2}(x^{-1})Jg_\eta(yx^{-1})}\, dy = \\
= \int_G Jf(y)\overline{Jg_\eta(yx^{-1})}\, d(yx^{-1}) = \int_G Jf(yx)\overline{Jg_\eta(y)}\, d(y) = \\
= \int_G Jf(yx)\overline{SJg_\eta(y)}\, d(y) = \int_G Jf(yx)\delta_G(y^{-1})Jg_\eta(y^{-1})\, d(y) = \\
= \int_G Jf(yx)Jg_\eta(y^{-1})\, d(y^{-1}) = \int_G Jf(y^{-1}x)Jg_\eta(y)\, d(y) = \\
= [Jg_\eta * Jf](x)\,.
\end{aligned}$$

(v) The result follows from items (iii) and (iv). □

Theorem 15 *The following are equivalent:*

(i) η is an admissible vector for $\{\pi, \mathcal{H}_\pi\}$.

(ii) $Jg_\eta \in \mathcal{U}''$ and $\pi_l(Jg_\eta) = P_{J\mathcal{H}_\eta}$, where $P_{J\mathcal{H}_\eta}$ denotes the orthogonal projection from $L^2(G)$ onto $J\mathcal{H}_\eta$.

Proof: The result is a direct consequence of Theorem 9 and the Tomita-Takesaki formulas in Lemma 12.(iii). It can also be proved using items (ii) and (iv) of Proposition 14. □

Corollary 16 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Then $J\mathcal{H}_\eta$ is an invariant subspace of $L^2(G)$ for the right regular representation ρ and Jg_η is a cyclic vector for the subrepresentation $\{\rho_{|J\mathcal{H}_\eta}, J\mathcal{H}_\eta\}$.*

Proof: By Lemma 13, one has

$$\rho(x)Jf(y) = JJ\rho(x)Jf(y) = J\lambda(x)f(y), \quad f \in \mathcal{H}_\eta,\, x \in G\,.$$

Thus, the invariance under λ of $\mathcal{H}_\eta$ implies the invariance under ρ of $J\mathcal{H}_\eta$. Now, that Jg_η is a cyclic vector for the subrepresentation $\{\rho_{|J\mathcal{H}_\eta}, J\mathcal{H}_\eta\}$ follows from the equality

$$Jf(x) = \delta_G^{1/2}(x^{-1})(Jf|\rho(x^{-1})Jg_\eta), \quad f \in \mathcal{H}_\eta,\, x \in G\,,$$

in Proposition 14.(iv). $\square$

Remark 17 Proposition 14.(iv) implies that $J\mathcal{H}_\eta$ is a *reproducing kernel Hilbert space* [19] with kernel

$$\tilde{k}_\eta(x,y) := \delta_G^{-1}(x)Jg_\eta(yx^{-1})\,.$$

See Remarks 8.d and 11.b.

4 Square roots and idempotents

Let $\mathcal{U}$ be a left Hilbert algebra as in Definition 5, with associated Hilbert space $\mathcal{H}$ and full right and left Hilbert algebras $\mathcal{U}'$ and $\mathcal{U}''$ defined as in (14) and (15). Let $\mathcal{R}(U')$ and $\mathcal{L}(\mathcal{U}'')$ denote the von Neumann algebras generated, respectively, by $\pi_r(\mathcal{U}')$ and $\pi_l(\mathcal{U}'')$.

Given $g \in \mathcal{H}$, let us consider the well-defined operators $\pi_r(g)$ and $\pi_l(g)$ on the respective dense domains $\mathcal{U}''$ and $\mathcal{U}'$ of $\mathcal{H}$ given by

$$\pi_r(g) : \mathcal{U}'' \to \mathcal{H} : f \mapsto fg\,, \quad \pi_l(g) : \mathcal{U}' \to \mathcal{H} : f \mapsto gf$$

Following Perdrizet [12], we introduce the following subsets of $\mathcal{H}$:

$$\begin{aligned}
\mathcal{F}^\flat &:= \left\{g \in \mathcal{H} : \pi_r(g) \text{ is closable on } \mathcal{H}\right\},\\
\mathcal{P}^\flat &:= \left\{g \in \mathcal{F}^\flat : \bar{\pi}_r(g) \text{ is positive}\right\},\\
\mathcal{P}^\flat_a &:= \left\{g \in \mathcal{F}^\flat : \bar{\pi}_r(g) \text{ is positive and essentially self-adjoint}\right\},\\
[\mathcal{U}']^+ &:= \left\{g \in \mathcal{U}' : \pi_r(g) \text{ is positive}\right\},
\end{aligned}$$

where $\bar{\pi}_r(g)$ denotes the closed extension of $\pi_r(g)$. Using $\pi_l(\cdot)$ instead of $\pi_r(\cdot)$, one can also define the corresponding subsets $\mathcal{F}^\sharp$, $\mathcal{P}^\sharp$, $\mathcal{P}^\sharp_a$ and $[\mathcal{U}'']^+$, and

$$\mathfrak{P}^\flat := \left\{gg^\flat : g \in \mathcal{U}'\right\}, \quad \mathfrak{P}^\sharp := \left\{gg^\sharp : g \in \mathcal{U}''\right\}.$$

One has

$$\mathfrak{P}^\flat \subseteq [\mathcal{U}']^+ \subseteq \mathcal{P}^\flat_a \subseteq \mathcal{P}^\flat \subseteq \mathcal{F}^\flat \subseteq \mathcal{H}\,,$$

$$\mathfrak{P}^\sharp \subseteq [\mathcal{U}'']^+ \subseteq \mathcal{P}^\sharp_a \subseteq \mathcal{P}^\sharp \subseteq \mathcal{F}^\sharp \subseteq \mathcal{H}\,.$$

Remark 18 The following results are due to Perdrizet [12]:

(1) Let $g \in \mathcal{H}$. The following conditions are equivalent [12, Prop.2.8]:

  (i) $g \in \mathcal{F}^\flat$.

  (ii) There is a partial isometry $V \in \mathcal{R}(\mathcal{U}')$ such that[1]

$$V^* g \in \mathcal{P}_a^\flat, \quad VV^* g = g\,.$$

  (iii) There is a unique $h \in \mathcal{P}_a^\flat$ such that the normal positive forms ω_g and ω_h coincide on $\mathcal{L}(U'')$, where $\omega_g(A) := (Ag|g)$ for $A \in \mathcal{L}(\mathcal{U}'')$. (Obviously, $h = V^* g$.)

(2) For $g \in \mathcal{F}^\flat$, the operator $\bar{\pi}_r(g)$ is affiliated to $\mathcal{R}(\mathcal{U}')$ [12, Lem.2.2].

(3) Let $g \in \mathcal{H}$. Then, $g \in \mathcal{P}^\flat$ if and only if $g \in \mathcal{D}^\flat$, $g = g^\flat$ and $\bar{\pi}_r(g)$ is positive [12, Prop.2.5].

(4) The subspace $\mathcal{D}^\flat$ is linearly generated by $\mathcal{P}^\flat$ [12, Prop.2.6].

(5) Results similar to (1)-(4) are satisfied for $\mathcal{F}^\sharp$, $\mathcal{P}^\sharp$ and $\mathcal{D}^\sharp$.

(6) $\mathcal{P}^\flat$ and $\mathcal{P}^\sharp$ are mutually dual pointed convex cones[2] in $\mathcal{H}$ and

$$\mathcal{P}^\flat = \overline{\mathfrak{P}^\flat}, \quad \mathcal{P}^\sharp = \overline{\mathfrak{P}^\sharp}\,,$$

where the bar denotes the closure [12, Prop.2.5].

The elements of $\mathcal{P}^\flat$ (resp. $\mathcal{P}^\sharp$) are known as the **right positive elements** (resp. *left positive elements*) of $\mathcal{H}$. According to Remark 18.6, an element $g \in \mathcal{H}$ belongs to $\mathcal{P}^\flat$ if and only if

$$(g|ff^\sharp) \geq 0, \quad f \in \mathcal{U}''\,.$$

Remark 19 Let G be a lc group. The following concept is usual in the literature: A complex function g defined on G is said to be of *positive type* if, for every $f \in C_c(G)$,

$$\int_G g(x)[f^\sharp * f](x)\,dx = \int_G \int_G g(y^{-1}x)\overline{f(y)}f(x)\,dy\,dx \geq 0.$$

When g is continuous, this is equivalent to the usual definition of a *positive definite* function [15, Prop.13.4.4]. For $g \in L^2(G)$, one has that g is of positive type if and only if $g \in \mathcal{P}^\flat$ [13, p.392].

[1] V is the partial isometry appearing in the **polar decomposition** of $\bar{\pi}_r(g)$, i.e., $\bar{\pi}_r(g) = V|\bar{\pi}_r(g)|$, where $|\bar{\pi}_r(g)| = [\bar{\pi}_r(g)^*\bar{\pi}_r(g)]^{1/2}$, the initial space of V is the closure of the range of $|\bar{\pi}_r(g)|$ and the final space of V is the closure of the range of $\bar{\pi}_r(g)$; see e.g. [26, Th.6.1.11]. Perdrizet [12] uses the Friedrichs's extension of $|\bar{\pi}_r(g)|$; see e.g. [27, pp.329–334].

[2] Recall that for a convex cone $\mathfrak{P}$ in $L^2(G)$, the *dual cone* $\mathfrak{P}^\circ$ is defined by $\mathfrak{P}^\circ := \Big\{g \in L^2(G) : (f|g) \geq 0$ for $f \in \mathfrak{P}\Big\}$. If $\mathfrak{P} = \mathfrak{P}^\circ$, then $\mathfrak{P}$ is called **self-dual**.

An element $e \in \mathcal{U}'$ (resp. $e \in \mathcal{U}''$) is called **right self-adjoint idempotent** (resp. *left self-adjoint idempotent*) if $e = e^\flat = e^2$ (resp. $e = e^\sharp = e^2$). Denote by $\mathcal{E}'$ (resp. $\mathcal{E}''$) the set of nonzero right (resp. left) self-adjoint idempotents of $\mathcal{U}'$ (resp. $\mathcal{U}''$). Obviously, $\mathcal{E}' \subset \mathfrak{P}^\flat$ and $\mathcal{E}'' \subset \mathfrak{P}^\sharp$. Moreover, since $\pi_r(g)^* = \pi_r(g^\flat)$ for $g \in \mathcal{U}'$, one has that $e \in \mathcal{E}'$ if and only if $\pi_r(e)$ is an orthogonal projection on $\mathcal{H}$. In a similar way, $e \in \mathcal{E}''$ if and only if $\pi_l(e)$ is an orthogonal projection on $\mathcal{H}$.

Theorem 9 can be rewritten in terms of right self-adjoint idempotents:

Theorem 20 *The following are equivalent:*

(i) η *is an admissible vector for* $\{\pi, \mathcal{H}_\pi\}$.

(ii) g_η *is a right self-adjoint idempotent element of* $L^2(G)$ *such that* $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$.

An element $g \in \mathcal{P}^\flat$ is said to be (right) **integrable** if and only if

$$\sup_{e \in \mathcal{E}''} (g|e) < \infty\,.$$

Let $g, h \in \mathcal{P}^\flat$. Then h is called a (right) **square root** of g if

$$(f|g) = (\pi_l(f)h|h), \quad f \in \mathcal{U}''\,.$$

Remark 21 The following results can be found in Phillips [13]:

(1) $\mathcal{U}'$ (resp. $\mathcal{U}''$) contains a net $\{e_\alpha\}_{\alpha\in A}$ of elements of $\mathcal{E}'$ (resp. $\mathcal{E}''$) such that $\pi_r(e_\alpha)$ (resp. $\pi_l(e_\alpha)$) converges to the identity operator of $\mathcal{H}$ in the strong operator topology [13, Prop.1.7].

(2) Let $g \in \mathcal{P}^\flat$. Then g is integrable if and only if g has a square root $h \in \mathcal{P}^\flat$. If $g \in \mathcal{U}'$, then so is h and in this case $h^2 = g$. See [13, Th.1.10]. This result may be considered an extension to full left Hilbert algebras of Godement's theorem [8, Th.17] and its Rieffel's version for Hilbert algebras [11, Th.3.9].

(3) *Godement's Theorem:* Let G be a lc group and let $g \in L^2(G)$ be positive-definite. If g is essentially bounded in some neighbourhood of the identity of G, then there exists a positive-definite function $h \in L^2(G)$ such that $g = h * h$ a.e. [13, Cor.1.11].

According to Phillips [13], by a **representation** π of a left Hilbert algebra $\mathcal{U}$ is meant a $\sharp$-representation π of $\mathcal{U}$ on a Hilbert space $\mathcal{H}_\pi$ such that there is a self-adjoint net $\{g_\alpha\}_{\alpha\in A} \subset \mathcal{U}$ with the property that $\pi_l(g_\alpha)$ converges strongly to the identity on $\mathcal{H}$ and $\pi(g_\alpha)$ converges strongly to the identity on $\mathcal{H}_\pi$. Obviously, an example is the **left regular representation** $\{\pi_l, \mathcal{H}\}$ of $\mathcal{U}''$ (see Remark 21.1). For each $\xi, \eta \in \mathcal{H}_\pi$, the linear functional $c_{\xi,\eta}$ on $\mathcal{U}$ defined by

$$c_{\xi,\eta}(f) := (\pi(f)\xi|\eta), \quad f \in \mathcal{U}\,,$$

is called a **coordinate functional** of π. A pair of vectors $\xi, \eta \in \mathcal{H}_\pi$ is called a **square-integrable pair** for π if the coordinate functional $c_{\xi,\eta}$ is continuous on $\mathcal{U}$ in its pre-Hilbert space norm. In this case there is an element $g_{\xi,\eta} \in \mathcal{H}$ such that

$$c_{\xi,\eta}(f) = (f|g_{\xi,\eta}), \quad f \in \mathcal{U} .$$

If $\eta \in \mathcal{H}_\pi$ and the pair (η, η) is square-integrable, we say that η is a **square-integrable vector** and write g_η for $g_{\eta,\eta}$. A cyclic representation of $\mathcal{U}$ will be called a **square-integrable representation** if it has a cyclic vector which is square-integrable.

Remark 22 We recall here some results by Phillips [13]:

(1) Let $\{\pi, \mathcal{H}_\pi\}$ be a representation of $\mathcal{U}''$. If $\eta \in \mathcal{H}_\pi$ is a square-integrable vector, then the coordinate element $g_\eta \in \mathcal{H}$ is integrable and positive [13, Prop.3.3].

(2) If $\{\pi, \mathcal{H}_\pi\}$ is a square-integrable representation of $\mathcal{U}''$, then there is an element $g \in \mathcal{P}^\flat$ such that π is unitarily equivalent to the left regular representation π_l of $\mathcal{U}''$ on $\langle \pi_l(\mathcal{U}'')g \rangle$. Conversely, if $\mathcal{H}_0 = \langle \pi_l(\mathcal{U}'')h \rangle$ for some $h \in \mathcal{D}^\flat$, then the left regular representation π_l of $\mathcal{U}''$ on $\mathcal{H}_0$ is square-integrable with a positive cyclic vector $g \in \mathcal{U}'$. See [13, Th.3.5].

Now, let G be a lc group. Recall that unitary representations π of G and nondegenerate $\sharp$-representations of the group algebra $L^1(G)$, on the same Hilbert space $\mathcal{H}_\pi$ and still denoted by π, are in correspondence through the relation

$$\pi(f) := \int_G f(x)\pi(x)\, dx, \quad f \in L^1(G) , \tag{18}$$

where the operator-valued integral is interpreted in the weak sense. Nondegenerate means that for $\xi \neq 0$ in $\mathcal{H}_\pi$ there exists $f \in L^1(G)$ such that $\pi(f)\xi \neq 0$. The von Neumann algebras generated by $\pi(G)$ and $\pi(L^1(G))$ coincide and also the sets of intertwining operators and invariant subspaces for both representations, and the same assertions are true if one considers, instead of $L_1(G)$, the dense modular algebra $C_c(G)$ or the full left Hilbert algebra $\mathcal{U}''$. In particular, for the left regular representation λ of G, $\lambda(f) = \pi_l(f)$. See e.g. [21, Sect.3.2] and [13, 14].

Remark 23 If $\{\pi, \mathcal{H}_\pi\}$ is a unitary representation of a lc group G, for each $\xi, \eta \in \mathcal{H}_\pi$, the *coordinate functional* $c_{\xi,\eta}$ is defined as the function on G given by

$$c_{\xi,\eta}(x) := (\pi(x)\xi|\eta), \quad x \in G ,$$

and the pair of vectors (ξ, η) is called a *square-integrable pair* for π if $c_{\xi,\eta} \in L^2(G)$. Since

$$c_{\xi,\eta}(f) = \int_G f(x)(\pi(x)\xi|\eta)\, dx = \int_G f(x)c_{\xi,\eta}(x)\, dx, \quad f \in C_c(G) ,$$

Schwarz inequality and Riesz theorem imply that (ξ, η) is a square-integrable pair for the unitary representation π of G if and only if it is a square-integrable pair for the corresponding representation of $C_c(G)$ or $\mathcal{U}''$. Furthermore, in such case, $g_{\xi,\eta} = \overline{c_{\xi,\eta}}$ as elements of $L^2(G)$.

As Rieffel [16, p.37] comments, "the most common definitions of *square-integrability* for unitary representations $\{\pi, \mathcal{H}_\pi\}$ of a lc group G just involve the condition that $c_{\xi,\eta} \in L^2(G)$ for some $\xi, \eta \in \mathcal{H}_\pi$ (and $c_{\xi,\eta} \neq 0$)". Duflo and Moore [22] or Grossmann, Morlet and Paul [3] add irreducibility into the definition (in [3] admissible vectors are just square-integrable vectors). For other definitions, see e.g. Rieffel [16, Def.7.8] and comments around.

Similar arguments to those used in the proofs of Takesaki [28, Lem.3.3] and Phillips [13, Th.3.5] lead to the following result. Recall the polar decomposition of a closed operator introduced in footnote 1.

Theorem 24 *Let $\{\pi, \mathcal{H}_\pi\}$ be a unitary representation of a lc group G. The following assertions are equivalent:*

- *(i) $\{\pi, \mathcal{H}_\pi\}$ has an admissible vector.*
- *(ii) $\{\pi, \mathcal{H}_\pi\}$ is equivalent to a subrepresentation of λ, $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$, with a cyclic vector $g \in \mathcal{U}'$ such that 0 does not belong to the spectrum $\sigma(|\pi_r(g)|)$ of $|\pi_r(g)|$ or 0 is an isolated point of $\sigma(|\pi_r(g)|)$.*

In such case, $\{\pi, \mathcal{H}_\pi\}$ is square-integrable and $\pi_r(g)|\pi_r(g)|^{-2}g^\flat$ is an admissible vector for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$, where $|\pi_r(g)|^{-1}$ is the partial inverse of $|\pi_r(g)|$ from $\overline{rang}(|\pi_r(g)|) = rang(|\pi_r(g)|)$ onto $L^2(G)$.

Proof: (i)$\Rightarrow$(ii): This implication follows from Theorem 20. Indeed, put $g = g_\eta$ and $\mathcal{H}_0 = \mathcal{H}_\eta$. Since $\pi_r(g) = |\pi_r(g)| = P_{\mathcal{H}_0}$, one has $\sigma(|\pi_r(g)|) = \{1\}$ or $\sigma(|\pi_r(g)|) = \{0, 1\}$, and, since $g \in \mathcal{H}_0$,

$$\pi_r(g)|\pi_r(g)|^{-2}g^\flat = \pi_r(g)^{-1}g = P_{\mathcal{H}_0}^{-1}g = g\,.$$

Moreover, for $f \in \mathcal{U}''$,

$$c_g(f) = (\pi_l(f)g|g) = (\pi_r(g)f|g) = (f|\pi_r(g)g) = (f|g * g) = (f|g)\,.$$

(ii)$\Rightarrow$(i): Put $A := \pi_r(g)$ and consider its polar decomposition

$$A = VH = KV\,;$$

see footnote 1. Then H and K are positive and affiliated with $\mathcal{R}(\mathcal{U}') = [\mathcal{L}(\mathcal{U}'')]'$. Let ϕ be a bounded positive measurable function of a real variable with compact support. Since $H^2\phi(H)$ is a bounded everywhere defined operator, one can consider

$$g_\phi := A\phi(H)g^\flat\,.$$

Then, $g_\phi \in \overline{\mathrm{rang}}(\pi_r(g)) = \langle \pi_r(g)\mathcal{U}'' \rangle = \langle \pi_l(\mathcal{U}'')g \rangle$. Moreover, $g_\phi \in \mathcal{U}'$ and

$$g_\phi^\flat = g_\phi, \quad \pi_r(g_\phi) = K^2\phi(K) = K^2\phi(K)VV^*\,.$$

See the proof of Lemma 3.3 in Takesaki [28] for details. See also [18, VI.1.12]. In particular, if ϕ is such that $t^2\phi(t) = \chi_\alpha(t)$, the characteristic function of a measurable compact set $\alpha \subset (0,\infty)$ such that $\sigma(\pi_r(g))\backslash\{0\} \subseteq \alpha$, then $g_\phi = g_\phi^\flat = g_\phi^2$, i.e., $g_\phi \in \mathcal{E}'$, and $\pi_r(g_\phi) = P_{\mathcal{H}_\phi}$, where $\mathcal{H}_\phi$ is a closed subspace of $\langle \pi_l(\mathcal{U}'')g\rangle$. That $\mathcal{H}_\phi = \langle \pi_l(\mathcal{U}'')g\rangle = \mathcal{H}_0$ follows from the fact that $\{K^2\phi(K)\}$ coincides with the range projection of K, which is also the range projection of $A = \bar{\pi}_r(g)$. Thus,

$$g_\phi = AH^{-2}g^\flat = \pi_r(g)|\pi_r(g)|^{-2}g^\flat$$

is a right self-adjoint idempotent element of $L^2(G)$ such that $\pi_r(g_\phi) = P_{\mathcal{H}_0}$ and, then, g_ϕ is a cyclic vector for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ and $c_{g_\phi,g_\phi}(f) = (f|g_\phi)$ for $f \in \mathcal{U}''$. Now, apply Theorem 20. □

Next Lemma explicits a fact already implicit in the proofs of Takesaki [28, Lem.3.3] and Phillips [13, Th.3.5 and Prop.4.2].

Lemma 25 *Let G be a lc group and $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ a subrepresentation of the left regular representation λ of G. If $\mathcal{H}_0 \cap \mathcal{D}^\flat \neq \{0\}$, then $\mathcal{H}_0 \cap \mathcal{E}' \neq \emptyset$. If, in addition, $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ is irreducible, then $\mathcal{H}_0 \cap \mathcal{E}' = \{e\}$.*

Proof: Let $0 \neq g \in \mathcal{H}_0 \cap \mathcal{D}^\flat$. By Remark 18.4, $g \in \mathcal{F}^\flat$ and we can consider the closure $A := \bar{\pi}_r(g)$. As before, let $A = VH = KV$ be the polar decomposition of A. Let $\alpha \subset (0,\infty)$ be a measurable compact set such that $\alpha \cap \sigma(H) \neq \emptyset$ and consider the function $\phi(t) = \chi_\alpha(t)/t^2$, $t \in \mathbb{R}$. Then, according to the proof of [28, Lem.3.3], the vector $g_\phi := A\phi(H)g^\flat$ is well-defined and belongs to $\langle \pi_l(\mathcal{U}'')g\rangle \subseteq \mathcal{H}_0$ and also to $\mathcal{U}'$. Furthermore, by spectral theory, $\pi_r(g_\phi)$ is the orthogonal projection onto $\langle \pi_l(\mathcal{U}'')g_\phi\rangle \subseteq \mathcal{H}_0$ and $g_\phi \in \mathcal{E}'$.

Now, assume in addition that $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ is irreducible. If there exist $e_1 \neq e_2$ in $\mathcal{H}_0 \cap \mathcal{E}'$, then, by [12, Lem.2.2], the projections $\pi_r(e_1)$ and $\pi_r(e_2)$ do not coincide, neither the subspaces $\pi_r(e_1)(L^2(G))$ and $\pi_r(e_2)(L^2(G))$. Since $\pi_r(e_i)(L^2(G)) = \langle \pi_l(U'')e_i\rangle$, $i = 1, 2$, are invariant subspaces of $\mathcal{H}_0$ for the left regular representation λ, this contradicts the fact that $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ is irreducible. Thus, $\mathcal{H}_0 \cap \mathcal{E}' = \{e\}$. □

For irreducible representations we have the following result:

Corollary 26 *Let $\{\pi, \mathcal{H}_\pi\}$ be an irreducible unitary representation of a lc group G. The following assertions are equivalent:*

(i) $\{\pi, \mathcal{H}_\pi\}$ has an admissible vector.

(ii) $\{\pi, \mathcal{H}_\pi\}$ is equivalent to an irreducible subrepresentation of λ, $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$, such that $\mathcal{H}_0 \cap \mathcal{D}^\flat \neq \emptyset$

In such case, $\mathcal{H}_0 \cap \mathcal{E}' = \{e\}$ and e is an admissible vector for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$.

Proof: By Lemma 25, one has $\mathcal{H}_0 \cap \mathcal{E}' = \{e\}$. Since $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ is irreducible and $\pi_r(e)(L^2(G)) = \langle \pi_l(U'')e\rangle$ is an invariant subspace of $\mathcal{H}_0$ for λ, one must have $\mathcal{H}_0 = \langle \pi_l(U'')e\rangle$ and, then, e is a right self-adjoint idempotent of $L^2(G)$, which is cyclic for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$. Thus, e is an admissible vector for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$. Now, apply Theorem 20. $\square$

Remark 27 (a) If $e_1, e_2 \in \mathcal{E}'$, then one writes $e_1 \leq e_2$ if $\pi_r(e_1)\pi_r(e_2) = \pi_r(e_2)\pi_r(e_1) = \pi_r(e_1)$. An element $e \in \mathcal{E}'$ is said to be **minimal** if whenever $e_1 \in \mathcal{E}'$ and $e_1 \leq e$, then $e_1 = e$.

(b) In Corollary 26, being $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$ irreducible, the admissible vector e must be minimal.

(c) Phillips [14] proves that any irreducible square-integrable representation $\{\pi, \mathcal{H}_\pi\}$ of a full left Hilbert algebra $\mathcal{U}''$ is equivalent to the left regular representation of $\mathcal{U}''$ on a subspace $\mathcal{H}_0$ of $\mathcal{H}$ of the form $\mathcal{H}_0 = \pi_r(e)\mathcal{H} = [\mathcal{U}'e]^-$, where $e \in \mathcal{E}'$ is minimal. In such case, $\mathcal{F}^\flat \cap \mathcal{H}_0 = \mathcal{D}^\flat \cap \mathcal{H}_0 = \mathcal{U}'e$ and a pair $f, g \in \mathcal{H}_0$, with $f \neq 0$, is square-integrable if and only if $g \in \mathcal{U}'e$.

(d) In his work on integrable and proper actions on C*-algebras, Rieffel [16, Prop.8.7] observes that, for square-integrable irreducible representations $\{\pi, \mathcal{H}_\pi\}$ of a lc group G and normalized π-bounded vectors g, right convolution by the coordinate functional c_g is a projection of $L^2(G)$ onto a closed subspace $\mathcal{H}_0$ consisting entirely of continuous functions, on which λ is unitarily equivalent to π. Obviously, c_g is what we call here an admissible vector for $\{\lambda_{|\mathcal{H}_0}, \mathcal{H}_0\}$.

5 Standard form and weights

Let G be a lc group and let Δ and J be the modular operator and modular conjugation defined, respectively, by (16) and (17). Let $\{\pi, \mathcal{H}_\pi\}$ be a unitary representation of G, let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$ and, as in Section 3, Eq.(11), let us put $g_\eta := L_\eta \eta$ and $\mathcal{H}_\eta := L_\eta \mathcal{H}_\pi$.

Since $\mathcal{H}_\eta$ is invariant under λ and $J\mathcal{H}_\eta$ is invariant under ρ, one has for the corresponding projections that

$$P_{\mathcal{H}_\eta} \in \mathcal{R}_G, \quad P_{J\mathcal{H}_\eta} \in \mathcal{L}_G\,,$$

where $\mathcal{L}_G$ and $\mathcal{R}_G$ are the left and right von Neumann algebras of G given in (7) and (8). Let us consider the *reduced von Neumann algebras*[3] generated, respectively, by λ on $\mathcal{H}_\eta$ and by ρ on $J\mathcal{H}_\eta$:

$$\mathcal{L}_\eta := \left\{\lambda(x)_{|\mathcal{H}_\eta} : x \in G\right\}'', \quad \mathcal{R}_\eta := \left\{\rho(x)_{|J\mathcal{H}_\eta} : x \in G\right\}''.$$

Obviously,

$$J\mathcal{L}_\eta J = \mathcal{R}_\eta, \quad J\mathcal{R}_\eta J = \mathcal{L}_\eta$$

[3] Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If P is a projection in $\mathcal{H}$, belonging to $\mathcal{M}$, the **reduced von Neumann algebra** $\mathcal{M}_P$ is the set of all $A \in \mathcal{M}$ such that $PA = AP = A$, and is a von Neumann algebra on the Hilbert space $P\mathcal{H}$; its commutant will be the induced von Neumann algebra $[\mathcal{M}']_P$, whose elements are all the restrictions to $P\mathcal{H}$ of elements of $\mathcal{M}'$.

and

$$\mathcal{H}_\eta = \langle \mathcal{L}_\eta g_\eta \rangle = \langle \mathcal{L}_G g_\eta \rangle, \quad J\mathcal{H}_\eta = \langle \mathcal{R}_\eta J g_\eta \rangle = \langle \mathcal{R}_G J g_\eta \rangle .$$

Now, as in Section 4, let us consider the mutually dual pointed convex cones in $L^2(G)$

$$\mathcal{P}^\flat := \left\{ g g^\flat : g \in \mathcal{U}' \right\}^-, \quad \mathcal{P}^\sharp := \left\{ g g^\sharp : g \in \mathcal{U}'' \right\}^- ,$$

where the bar means the closure; see Remark 18.6. Then,

$$\mathfrak{P} := (\Delta^{-1/4}\mathcal{P}^\flat)^- = (\Delta^{1/4}\mathcal{P}^\sharp)^- \tag{19}$$

is a self-dual closed convex cone of $L^2(G)$; see footnote 2 for a definition of self-duality. Moreover, every element of $L^2(G)$ is represented as a linear combination of four vectors of $\mathfrak{P}$ and, to each $\omega \in [\mathcal{L}_G]_*^+$, there corresponds a unique $g \in \mathfrak{P}$ with $\omega = \omega_g$, i.e.,

$$\omega(A) = \omega_g(A) := (Ag|g), \quad A \in \mathcal{L}_G .$$

See [18, IX.1.2].

Taking into account the modular conjugation J defined in (17), the quadruple $\left\{\mathcal{L}_G, L^2(G), J, \mathfrak{P}\right\}$ is a **standard form** of the von Neumann algebra $\mathcal{L}_G$, that is, the following requirements are satisfied:

(i) $J\mathcal{L}_G J = \mathcal{L}_G' = \mathcal{R}_G$,

(ii) $JAJ = A^*$, for $A \in \mathcal{L}_G \cap \mathcal{R}_G$,

(iii) $Jg = g$, for $g \in \mathfrak{P}$,

(iv) $AJAJ\mathfrak{P} \subset \mathfrak{P}$, for $A \in \mathcal{L}_G$.

Remark 28 Standard forms are introduced by Haagerup [20]. Every von Neumann algebra $\mathcal{M}$ can be represented on a Hilbert space $\mathcal{H}$ in which it is standard. In such a standard form, for all ω in the predual $\mathcal{M}_*$, there exist $\xi, \eta \in \mathcal{H}$ such that $\omega = \omega_{\xi,\eta}$, i.e.,

$$\omega(A) = \omega_{\xi,\eta}(A) := (A\xi|\eta)_{\mathcal{H}}, \quad A \in \mathcal{M} ;$$

furthermore, every automorphism α of $\mathcal{M}$ is implemented by a unique unitary operator U of $\mathcal{L}(\mathcal{H})$, that is,

$$\alpha(A) = UAU^*, \quad A \in \mathcal{M} ,$$

such that $UJ = JU$ and $U\mathfrak{P} \subset \mathfrak{P}$. See also [18, Sect.IX.1].

Lemma 29 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Then:*

(i) $g_\eta \in \mathcal{P}^\flat$ *and* $Jg_\eta \in \mathcal{P}^\sharp$.

(ii) $g_\eta \in \mathcal{D}(\Delta^{-1/4})$, $Jg_\eta \in \mathcal{D}(\Delta^{1/4})$ *and*

$$\Delta^{-1/4} g_\eta = \Delta^{1/4} J g_\eta = J\Delta^{-1/4} g_\eta . \tag{20}$$

(iii) $\langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle = \langle \mathcal{R}_G \Delta^{1/4} J g_\eta \rangle = \langle \mathcal{R}_G \Delta^{-1/4} g_\eta \rangle$.

Proof: (i) follows from Proposition 7.(iii) and Proposition 14.(v).

(ii) Since $g_\eta \in \mathcal{D}^\flat = \mathcal{D}(\Delta^{-1/2})$ and $\mathcal{D}(\Delta^{-1/2})$ is a core for $\Delta^{-1/4}$ (see e.g. [29, 2.7.7]), then $g_\eta \in \mathcal{D}(\Delta^{-1/4})$. In a similar way, $Jg_\eta \in \mathcal{D}^\sharp = \mathcal{D}(\Delta^{1/2})$ implies $Jg_\eta \in \mathcal{D}(\Delta^{1/4})$. Now, since $g_\eta = g_\eta^\flat = Fg_\eta$ (Proposition 7.iii) and $F = \Delta^{1/2} J$ (Lemma 12.ii.e),

$$\Delta^{-1/4} g_\eta = \Delta^{-1/4} F g_\eta = \Delta^{-1/4} \Delta^{1/2} J g_\eta = \Delta^{1/4} J g_\eta \,,$$

$$[J \Delta^{-1/4} g_\eta](y) = \delta_G^{-1/2}(y) \delta_G^{-1/4}(y^{-1}) \overline{g_\eta(y^{-1})} = [\Delta^{-1/4} g_\eta](y) \,.$$

(iii) Since J is an (antilinear) isometry and $J^2 = I$ (Lemma 12.ii), one has, by Lemma 13 and (20),

$$\begin{aligned} J\langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle &= J\langle \{\lambda(x) \Delta^{-1/4} g_\eta : x \in G\} \rangle = \\ &= \langle \{J\lambda(x) J J \Delta^{-1/4} g_\eta : x \in G\} \rangle = \\ &= \langle \{\rho(x) J \Delta^{-1/4} g_\eta : x \in G\} \rangle = \\ &= \langle \{\rho(x) \Delta^{-1/4} g_\eta : x \in G\} \rangle = \\ &= \langle \{\rho(x) \Delta^{1/4} J g_\eta : x \in G\} \rangle \,, \end{aligned}$$

that is,

$$J\langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle = \langle \mathcal{R}_G \Delta^{1/4} J g_\eta \rangle = \langle \mathcal{R}_G \Delta^{-1/4} g_\eta \rangle \,. \tag{21}$$

On the other hand, by (ii), $g_\eta \in \mathcal{D}(\Delta^{-1/4})$ and, thus, for $x \in G$, the function $y \in G \mapsto \delta_G^{-1/4}(x^{-1}y) g_\eta(x^{-1}y)$ is in $L^2(G)$. But

$$\begin{aligned} \delta_G^{-1/4}(x^{-1}y) g_\eta(x^{-1}y) &= \delta_G^{1/4}(x) \delta_G^{-1/4}(y) g_\eta(x^{-1}y) = \\ &= \delta_G^{1/4}(x) [\Delta^{-1/4} \lambda(x) g_\eta](y) \end{aligned}$$

and this implies that $\lambda(x) g_\eta \in \mathcal{D}(\Delta^{-1/4})$ and

$$\lambda(x) \Delta^{-1/4} g_\eta = \delta_G^{1/4}(x) \Delta^{-1/4} \lambda(x) g_\eta, \quad x \in G \,.$$

Thus,

$$\begin{aligned} \langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle &= \langle \{\lambda(x) \Delta^{-1/4} g_\eta : x \in G\} \rangle = \\ &= \langle \{\Delta^{-1/4} \lambda(x) g_\eta : x \in G\} \rangle = \Delta^{-1/4} \mathcal{H}_\eta \end{aligned}$$

and, then, using again that $g_\eta = g_\eta^\flat$,

$$\begin{aligned} J\langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle &= \langle \{J \Delta^{-1/4} \lambda(x) g_\eta : x \in G\} \rangle = \\ &= \langle \{\Delta^{-1/2} \Delta^{1/4} \lambda(x) g_\eta : x \in G\} \rangle = \\ &= \langle \{\Delta^{-1/4} \lambda(x) g_\eta : x \in G\} \rangle = \\ &= \langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle \,. \end{aligned}$$

This equality, together with (21), lead to the result. $\square$

Thus, if η is an admissible vector for $\{\pi, \mathcal{H}_\pi\}$, according to Lemma 29.iii, we can consider the closed subspace $\hat{\mathcal{H}}_\eta$ of $L^2(G)$ defined by

$$\hat{\mathcal{H}}_\eta := \langle \mathcal{L}_G \Delta^{-1/4} g_\eta \rangle = \langle \mathcal{R}_G \Delta^{-1/4} g_\eta \rangle \,. \tag{22}$$

Obviously, the corresponding orthogonal projection $P_{\hat{\mathcal{H}}_\eta}$ from $L^2(G)$ onto $\hat{\mathcal{H}}_\eta$ belongs to the center $\mathcal{L}_G \cap \mathcal{R}_G$ and the elements of the reduced von Neumann algebras $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ of $\mathcal{L}_G$ and $\mathcal{R}_G$ to $\hat{\mathcal{H}}_\eta$ (see footnote 3) are just the restrictions to $\hat{\mathcal{H}}_\eta$ of elements of $\mathcal{L}_G$ and $\mathcal{R}_G$, i.e.,

$$\begin{aligned} \hat{\mathcal{L}}_\eta &:= \left\{ \lambda(x)_{|\hat{\mathcal{H}}_\eta} : x \in G \right\}'' = [\mathcal{L}_G]_{P_{\hat{\mathcal{H}}_\eta}} = \mathcal{L}_G P_{\hat{\mathcal{H}}_\eta} \,, \\ \hat{\mathcal{R}}_\eta &:= \left\{ \rho(x)_{|\hat{\mathcal{H}}_\eta} : x \in G \right\}'' = [\mathcal{R}_G]_{P_{\hat{\mathcal{H}}_\eta}} = \mathcal{R}_G P_{\hat{\mathcal{H}}_\eta} \,. \end{aligned} \tag{23}$$

Both (reduced) von Neumann algebras $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ act on $\hat{\mathcal{H}}_\eta$ and, by definition, $[\hat{\mathcal{L}}_\eta]' = \hat{\mathcal{R}}_\eta$.

Moreover, the definition of $\hat{\mathcal{H}}_\eta$ also implies that $\Delta^{-1/4} g_\eta$ is a cyclic vector for both $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$. Recall that, given a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, an element $\xi \in \mathcal{H}$ is called a **separating vector** for $\mathcal{M}$ if, for any $A \in \mathcal{M}$, $A\xi = 0$ implies $A = 0$. It is well-known that ξ is a separating vector for $\mathcal{M}$ if and only if ξ is a cyclic vector for $\mathcal{M}'$; see e.g. [30, Prop.2.5.3]. Thus, $\Delta^{-1/4} g_\eta$ is a cyclic and separating vector of $\hat{\mathcal{H}}_\eta$ for both $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$.

These results and additional ones are included in the next theorem.

Theorem 30 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Let $\hat{\mathcal{H}}_\eta$ be the closed subspace of $L^2(G)$ defined by (22) and let $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ be the reduced von Neumann algebras of $\mathcal{L}_G$ and $\mathcal{R}_G$ to $\hat{\mathcal{H}}_\eta$ given in (23). Then:*

(i) $P_{\hat{\mathcal{H}}_\eta} \in \mathcal{L}_G \cap \mathcal{R}_G$.

(ii) $[\hat{\mathcal{L}}_\eta]' = \hat{\mathcal{R}}_\eta$.

(iii) $\Delta^{-1/4} g_\eta$ *is a cyclic and separating vector of* $\hat{\mathcal{H}}_\eta$ *for both* $\hat{\mathcal{L}}_\eta$ *and* $\hat{\mathcal{R}}_\eta$.

(iv) $[\pi_r(\Delta^{-1/4} g_\eta)]_{|\hat{\mathcal{H}}_\eta} = I_{|\hat{\mathcal{H}}_\eta}$. *If, in addition,* $[\Delta^{-1/4} g_\eta]^\flat = \Delta^{1/4} g_\eta$ *belongs to* $\hat{\mathcal{H}}_\eta$, *then* $\pi_r(\Delta^{-1/4} g_\eta) = P_{\hat{\mathcal{H}}_\eta}$.

(v) Let J be the modular conjugation in $L^2(G)$ defined by (17). Then J and $P_{\hat{\mathcal{H}}_\eta}$ commute. Let us consider

$$\hat{J}_\eta := J_{|\hat{\mathcal{H}}_\eta} \,.$$

Then $\hat{J}_\eta$ is an antilinear isometry of $\hat{\mathcal{H}}_\eta$ onto itself such that $\hat{J}_\eta^2 = I_{|\hat{\mathcal{H}}_\eta}$.

(vi) Let $\mathfrak{P}$ be the self-dual closed convex cone of $L^2(G)$ given by (19). Let us define

$$\hat{\mathfrak{P}}_\eta := \mathfrak{P} \cap \hat{\mathcal{H}}_\eta .$$

Then $\hat{\mathfrak{P}}_\eta$ is a self-dual closed convex cone of $\hat{\mathcal{H}}_\eta$ and

$$\hat{\mathfrak{P}}_\eta = \left\{ [\mathbb{R}^+ g_\eta - \mathfrak{P}] \cap \mathfrak{P} \right\}^- .$$

(vii) $\left\{ \hat{\mathcal{L}}_\eta, \hat{\mathcal{H}}_\eta, \hat{J}_\eta, \hat{\mathfrak{P}}_\eta \right\}$ is a standard form of the von Neumann algebra $\hat{\mathcal{L}}_\eta$.

Proof: (i), (ii) and (iii) have been proved in the preceding comments.
(iv) Since $\pi_r(g_\eta) = P_{\mathcal{H}_\eta}$, one has, for $x, z \in G$,

$$\begin{aligned}
&[\lambda(z)\Delta^{-1/4} g_\eta] * [\Delta^{-1/4} g_\eta](x) = \\
&= \int_G [\lambda(z)\Delta^{-1/4} g_\eta](y)[\Delta^{-1/4} g_\eta](y^{-1}x)\, dy = \\
&= \int_G \delta_G^{-1/4}(z^{-1}y) g_\eta(z^{-1}y) \delta_G^{-1/4}(y^{-1}x) g_\eta(y^{-1}x)\, dy = \\
&= \delta_G^{-1/4}(z^{-1}x)[\lambda(z) g_\eta] * g_\eta(x) = \\
&= \delta_G^{-1/4}(z^{-1}x)[\lambda(z) g_\eta](x) = [\lambda(z)\Delta^{-1/4} g_\eta](x) .
\end{aligned}$$

Thus, $[\pi_r(\Delta^{-1/4} g_\eta)]_{|\hat{\mathcal{H}}_\eta} = I_{|\hat{\mathcal{H}}_\eta}$.

Since $g_\eta = g_\eta^\flat$, a simple calculation leads to $[\Delta^{-1/4} g_\eta]^\flat = \Delta^{1/4} g_\eta$. If, in addition, $[\Delta^{-1/4} g_\eta]^\flat \in \hat{\mathcal{H}}_\eta$, then $\pi_r(\Delta^{-1/4} g_\eta) = P_{\hat{\mathcal{H}}_\eta}$. Indeed, in such case, for $f \in \hat{\mathcal{H}}_\eta^\perp$ and $x \in G$,

$$\begin{aligned}
f * [\Delta^{-1/4} g_\eta](x) &= \int_G f(y)[\Delta^{-1/4} g_\eta](y^{-1}x)\, dy = \\
&= \int_G f(y) \delta_G^{-1/4}(y^{-1}x) g_\eta(y^{-1}x)\, dy = \\
&= \int_G f(y) \delta_G^{1/4}(x^{-1}y) \overline{g_\eta^\flat(x^{-1}y)}\, dy = \\
= \int_G f(xy) \delta_G^{1/4}(y) \overline{g_\eta^\flat(y)}\, dy &= \int_G f(xy) \overline{[\Delta^{-1/4} g_\eta]^\flat(y)}\, dy = \\
&= (\lambda(x^{-1}) f | [\Delta^{-1/4} g_\eta]^\flat) = (f | \lambda(x)[\Delta^{-1/4} g_\eta]^\flat) = 0 .
\end{aligned}$$

(v) By Lemma 12.ii.b, Lemma 13 and (20), for $x \in G$,

$$J\lambda(x)\Delta^{-1/4} g_\eta = J\lambda(x) J J \Delta^{-1/4} g_\eta = \rho(x) J \Delta^{-1/4} g_\eta = \rho(x)\Delta^{-1/4} g_\eta ,$$

so that J and $P_{\hat{\mathcal{H}}_\eta}$ commute and $\hat{J}_\eta = J_{|\hat{\mathcal{H}}_\eta}$ is an antilinear isometry of $\hat{\mathcal{H}}_\eta$ onto itself such that $\hat{J}_\eta^2 = I_{|\hat{\mathcal{H}}_\eta}$.

Finally, (vi) and (vii) follow from [18, IX.1.8] and [18, IX.1.11]. $\square$

Let $\mathcal{M}$ be a von Neumann algebra of operators acting on a Hilbert space $\mathcal{H}$. Recall that $\mathcal{M}$ is said to be **σ-finite** if all collections of mutually orthogonal projections in $\mathcal{M}$ have at most a countable cardinality. It is well-known [30, Prop.2.5.6] that $\mathcal{M}$ admits a cyclic and separating vector if and only if $\mathcal{M}$ is σ-finite. Thus, Theorem 30.iii implies the following result.

Corollary 31 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Then the reduced von Neumann algebras $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ are both σ-finite.*

Now, let us recall the notion of *weight* on a von Neumann algebra.

Definition 32 *A* **weight** *on a von Neumann algebra $\mathcal{M}$ is a map $\varphi : \mathcal{M}^+ \to [0,\infty]$, the extended positive real numbers, satisfying the following conditions:*

$$\begin{aligned} \varphi(A+B) &= \varphi(A) + \varphi(B), \quad A, B \in \mathcal{M}^+ , \\ \varphi(sA) &= s\varphi(A), \quad A \in \mathcal{M}^+,\, s \geq 0 , \end{aligned}$$

where we use the convention $0(+\infty) = 0$. It is said to be **semi-finite** *if*

$$\mathfrak{P}_\varphi := \{A \in \mathcal{M}^+ : \varphi(A) < \infty\}$$

generates $\mathcal{M}$; **faithful** *if $\varphi(A) \neq 0$ for every non-zero $A \in \mathcal{M}^+$;* **normal** *if $\varphi(\sup A_i) = \sup \varphi(A_i)$ for every bounded increasing net $\{A_i\}$ in $\mathcal{M}^+$.*

Given a weight φ on $\mathcal{M}$, $\mathfrak{P}_\varphi$ is a convex cone of $\mathcal{M}^+$,

$$\mathfrak{n}_\varphi := \{A \in \mathcal{M} : A^*A \in \mathfrak{P}_\varphi\}$$

is a left ideal of $\mathcal{M}$ and

$$\mathfrak{m}_\varphi := \Big\{ \sum_{i=1}^n B_i^* A_i : A_i, B_i \in \mathfrak{n}_\varphi,\, 1 \leq i \leq n \Big\}$$

is an $*$-subalgebra such that $\mathfrak{m}_\varphi \cap \mathcal{M}^+ = \mathfrak{P}_\varphi$ and every element of $\mathfrak{m}_\varphi$ is a linear combination of four elements of $\mathfrak{P}_\varphi$. The $*$-subalgebra $\mathfrak{m}_\varphi$ is called the *definition domain* of the weight φ or the *definition subalgebra* of φ. See e.g. [18, VII.1.2].

There is one to one correspondence between full right and left Hilbert algebras and faithful semi-finite normal weights [18, Sect.VII.2]. Here we pay attention to the full convolution right and left Hilbert algebras $\mathcal{U}'$ and $\mathcal{U}''$ of a lc group G defined in (14) and (15). Recall that

$$\pi_r(\mathcal{U}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*, \quad \pi_l(\mathcal{U}'') = \mathfrak{n}_l \cap \mathfrak{n}_l^* .$$

Set

$$\begin{aligned} \mathfrak{m}_r &:= \mathfrak{n}_r^* \mathfrak{n}_r = \Big\{ \sum_{i=1}^n B_i^* A_i : A_i, B_i \in \mathfrak{n}_r,\, 1 \leq i \leq n \Big\} , \\ \mathfrak{m}_l &:= \mathfrak{n}_l^* \mathfrak{n}_l = \Big\{ \sum_{i=1}^n B_i^* A_i : A_i, B_i \in \mathfrak{n}_l,\, 1 \leq i \leq n \Big\} . \end{aligned}$$

We define an extended positive real valued function φ_r on $\mathcal{R}_G^+$ as follows:

$$\varphi_r(A) := \begin{cases} ||g||, & \text{if } A^{1/2} = \pi_r(g),\ g \in \mathcal{U}'\,, \\ +\infty, & \text{otherwise}\,, \end{cases} \tag{24}$$

Similarly, we define φ_l on $\mathcal{L}_G^+$:

$$\varphi_l(A) := \begin{cases} ||g||, & \text{if } A^{1/2} = \pi_l(g),\ g \in \mathcal{U}''\,, \\ +\infty, & \text{otherwise}\,. \end{cases} \tag{25}$$

Then the formulas (24) and (25) give opposite faithful semifinite normal weights φ_r on $\mathcal{R}_G$ and φ_l on $\mathcal{L}_G$ such that

$$\mathfrak{m}_{\varphi_r} = \mathfrak{m}_r, \quad \mathfrak{n}_{\varphi_r} = \mathfrak{n}_r, \quad \mathfrak{m}_{\varphi_l} = \mathfrak{m}_l, \quad \mathfrak{n}_{\varphi_l} = \mathfrak{n}_l$$

and

$$\begin{aligned} \varphi_r(\pi_r(g)^*\pi_r(f)) = (f|g), \qquad f, g \in \mathcal{B}'\,, \\ \varphi_l(\pi_l(g)^*\pi_l(f)) = (f|g), \qquad f, g \in \mathcal{B}\,. \end{aligned}$$

See e.g. [18, VII.2.5]. The weight φ_l on $\mathcal{L}_G$ is called the **Plancherel weight**; see e.g. [31, 32], [18, Sect.VII.3], [24, Chapter 3].[4]

The associated modular operator Δ and modular conjugation J are defined, respectively, in (16) and (17). The modular operator Δ induces a one parameter unitary group $\{\Delta^{it} : t \in \mathbb{R}\}$ acting on $\mathcal{U}''$ and $\mathcal{U}'$ as automorphisms, i.e.,

$$\begin{aligned} \pi_r(\Delta^{it}g) &= \Delta^{it}\pi_r(g)\Delta^{-it}, \quad g \in \mathcal{U}'\,, \\ \pi_l(\Delta^{it}g) &= \Delta^{it}\pi_l(g)\Delta^{-it}, \quad g \in \mathcal{U}''\,. \end{aligned}$$

Since the modular conjugation J maps $\mathcal{U}''$ (resp. $\mathcal{U}'$) onto $\mathcal{U}'$ (resp. $\mathcal{U}''$) anti-isomorphically (Lemma 12.iii),

$$\begin{aligned} \pi_r(J\Delta^{it}g) &= J\Delta^{it}\pi_l(g)\Delta^{-it}J, \quad g \in \mathcal{U}''\,, \\ \pi_l(J\Delta^{it}g) &= J\Delta^{it}\pi_r(g)\Delta^{-it}J, \quad g \in \mathcal{U}'\,. \end{aligned}$$

See [18, VI.1.19]. Thus, Δ gives rise to a (unique) one parameter automorphism group $\{\sigma_t^{\varphi_l}\}_{t\in\mathbb{R}}$ of $\mathcal{L}_G$ given by

$$\sigma_t^{\varphi_l}(A) := \Delta^{it}A\Delta^{-it}, \quad A \in \mathcal{L}_G\,,$$

which satisfies the *modular condition* for φ_l:

(i) $\varphi_l = \varphi_l \circ \sigma_t^{\varphi_l}$, for $t \in \mathbb{R}$;

(ii) For every pair $A, B \in \pi_l(\mathcal{U}'')$, there exists a bounded continuous function $F_{A,B}$ on the closed horizontal strip $\overline{D}$ bounded by $\mathbb{R}$ and $\mathbb{R} + i$ which is holomorphic on the open strip D such that[5]

$$F_{A,B}(t) = \varphi_l(\sigma_t^{\varphi_l}(A)B), \quad F_{A,B}(t+i) = \varphi_l(B\sigma_t^{\varphi_l}(A)), \quad t \in \mathbb{R}.$$

[4] Really, one should say that φ_l on $\mathcal{L}_G$ (resp. φ_r on $\mathcal{R}_G$) is the left (resp. right) Plancherel weight.

[5] For the weight φ_r on $\mathcal{R}_G$ and the one parameter automorphism group $\{\sigma_t^{\varphi_r}\}_{t\in\mathbb{R}}$ of $\mathcal{R}_G$ given by $\sigma_t^{\varphi_r}(A) := \Delta^{it}A\Delta^{-it}$, $A \in \mathcal{R}_G$, one must consider the closed horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} - i$.

See [18, VIII.1.2]. $\{\sigma_t^{\varphi_l}\}_{t\in\mathbb{R}}$ is called the **modular automorphism group** associated with φ_l.

Now, let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$. Since $\Delta^{-1/4}g_\eta$ is a cyclic and separating vector of $\hat{\mathcal{H}}_\eta$ for both $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$ (Theorem 30.iii), the expression

$$\hat{\omega}_\eta(A) := \omega_{\Delta^{-1/4}g_\eta}(A) = (A\Delta^{-1/4}g_\eta|\Delta^{-1/4}g_\eta), \quad A \in \hat{\mathcal{L}}_\eta\,, \tag{26}$$

defines a faithful finite normal weight $\hat{\omega}_\eta$ on $\hat{\mathcal{L}}_\eta$. The next result determines the associated modular operator $\hat{\Delta}_\eta$, modular conjugation $\hat{J}_\eta$ and involutions $\hat{F}_\eta$ and $\hat{S}_\eta$ in $\hat{\mathcal{H}}_\eta$.

Proposition 33 *Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$ and let $\hat{\omega}_\eta$ be the faithful finite normal weight on $\hat{\mathcal{L}}_\eta$ defined by (26). Then:*

(i) The modular conjugation is $\hat{J}_\eta = J_{|\hat{\mathcal{H}}_\eta}$.

(ii) The involutions $\hat{F}_\eta$ and $\hat{S}_\eta$ in $\hat{\mathcal{H}}_\eta$ are determined by

$$\hat{F}_\eta[\rho(x)\Delta^{-1/4}g_\eta] = \rho(x^{-1})\Delta^{-1/4}g_\eta, \quad x \in G\,,$$

$$\hat{S}_\eta[\lambda(x)\Delta^{-1/4}g_\eta] = \lambda(x^{-1})\Delta^{-1/4}g_\eta, \quad x \in G\,.$$

(iii) The modular operator $\hat{\Delta}_\eta$ satisfies

$$\hat{\Delta}_\eta[\lambda(x)\Delta^{-1/4}g_\eta] = \rho(x)\Delta^{-1/4}g_\eta, \quad x \in G\,.$$

Proof: (i) The result is already proved in Theorem 30.v.

(ii) This is the standard definition for the involutions $\hat{F}_\eta$ and $\hat{S}_\eta$ when $\Delta^{-1/4}g_\eta$ is a cyclic and separating vector of $\hat{\mathcal{H}}_\eta$ for $\hat{\mathcal{L}}_\eta$ and $\hat{\mathcal{R}}_\eta$; see e.g. [30, Sect.2.5.2].

(iii) The modular operator $\hat{\Delta}_\eta := \hat{F}_\eta\hat{S}_\eta$ satisfies, for $x \in G$,

$$\begin{aligned}\hat{\Delta}_\eta[\lambda(x)\Delta^{-1/4}g_\eta] &= \hat{F}_\eta\hat{S}_\eta[\lambda(x)\Delta^{-1/4}g_\eta] = \hat{F}_\eta[\lambda(x^{-1})\Delta^{-1/4}g_\eta] = \\ &= \hat{F}_\eta[JJ\lambda(x^{-1})JJ\Delta^{-1/4}g_\eta] = \hat{F}_\eta[J\rho(x^{-1})J\Delta^{-1/4}g_\eta] = \\ &= \hat{F}_\eta[J\rho(x^{-1})\Delta^{-1/4}g_\eta] = \hat{F}_\eta[\rho(x^{-1})\Delta^{-1/4}g_\eta] = \rho(x)\Delta^{-1/4}g_\eta\,,\end{aligned}$$

since, for $x, y \in G$,

$$\begin{aligned}[J\rho(x^{-1})\Delta^{-1/4}g_\eta](y) &= J\delta_G^{1/2}(x^{-1})\delta_G^{-1/4}(yx^{-1})g_\eta(yx^{-1}) = \\ &= \delta_G^{-1/4}(x)J\delta_G^{-1/4}(y)g_\eta(yx^{-1}) = \delta_G^{-1/4}(x)\delta_G^{-1/4}(y)g_\eta(yx^{-1}) = \\ &= \delta_G^{1/2}(x^{-1})\delta_G^{-1/4}(yx^{-1})g_\eta(yx^{-1}) = [\rho(x^{-1})\Delta^{-1/4}g_\eta](y)\,,\end{aligned}$$

where we use $\hat{J}_\eta = J_{|\hat{\mathcal{H}}_\eta}$, $\hat{J}_\eta^2 = I_{|\hat{\mathcal{H}}_\eta}$, Lemma 13, (20) and $g_\eta = g_\eta^\flat$. $\square$

As before, the modular operator $\hat{\Delta}_\eta$ leads to a one parameter automorphism group $\{\sigma_t^{\hat{\omega}_\eta}\}_{t\in\mathbb{R}}$ of $\hat{\mathcal{L}}_\eta$ given by

$$\sigma_t^{\hat{\omega}_\eta}(A) := \hat{\Delta}_\eta^{it} A \hat{\Delta}_\eta^{-it}, \quad A \in \hat{\mathcal{L}}_\eta\,,$$

which satisfies the modular condition for $\hat{\omega}_\eta$.

The weights φ_l and $\hat{\omega}_\eta$ and the corresponding one parameter automorphism groups $\{\sigma_t^{\varphi_l}\}$ and $\{\sigma_t^{\hat{\omega}_\eta}\}$ are related by the so-called *cocycle derivative*, a one parameter family $\{U_t\}$ of partial isometries on $L^2(G)$ with initial and final space $\hat{\mathcal{H}}_\eta$ in this case.

Theorem 34 *Let φ_l be the Plancherel weight on $\mathcal{L}_G$. Let η be an admissible vector for $\{\pi, \mathcal{H}_\pi\}$ and let $\hat{\omega}_\eta$ be the faithful finite normal weight on $\hat{\mathcal{L}}_\eta$ defined by (26). Then there exists a unique one parameter family $\{U_t\}$ of partial isometries such that*

(i) $t \in \mathbb{R} \mapsto U_t \in \mathcal{L}_G$ *is σ-strong* continuous.*

(ii) $U_{s+t} = U_s \sigma_s^{\varphi_l}(U_t)$, $s, t \in \mathbb{R}$.

(iii) $U_t U_t^* = U_t^* U_t = P_{\hat{\mathcal{H}}_\eta}$, $t \in \mathbb{R}$.

(iv) $U_t \sigma_t^{\varphi_l}(\mathfrak{n}_{\hat{\omega}_\eta}^* \cap \mathfrak{n}_{\varphi_l}) \subset \mathfrak{n}_{\hat{\omega}_\eta}^* \cap \mathfrak{n}_{\varphi_l}$, $t \in \mathbb{R}$.

(v) For each $A \in \mathfrak{n}_{\hat{\omega}_\eta} \cap \mathfrak{n}_{\varphi_l}^*$ and $B \in \mathfrak{n}_{\varphi_l} \cap \mathfrak{n}_{\hat{\omega}_\eta}^*$, there exists there exists a bounded continuous function F on the closed horizontal strip $\overline{D}$ bounded by $\mathbb{R}$ and $\mathbb{R} + i$ which is holomorphic on the open strip D such that, for $t \in \mathbb{R}$,

$$F(t) = \hat{\omega}_\eta(U_t \sigma_t^{\varphi_l}(B) A), \quad F(t+i) = \varphi_l(A U_t \sigma_t^{\varphi_l}(B))\,.$$

(vi) $\sigma_t^{\hat{\omega}_\eta}(A) = U_t \sigma_t^{\varphi_l}(A) U_t^*$, $A \in \hat{\mathcal{L}}_\eta$, $t \in \mathbb{R}$.

Furthermore, the above property (v) of $\{U_t\}$ determines the cocycle uniquely.

Proof: Since $P_{\hat{\mathcal{H}}_\eta}$ belongs to the center $\mathcal{L}_G \cap \mathcal{R}_G$, by [18, VI.1.23], one has $\sigma_t^{\varphi_l}(P_{\hat{\mathcal{H}}_\eta}) = P_{\hat{\mathcal{H}}_\eta}$, $t \in \mathbb{R}$. The rest of the Theorem adapts [18, VIII.3.19] to this context. □

The one parameter family $\{U_t\}$ of the above theorem is called the **cocycle derivative** of $\hat{\omega}_\eta$ relative to φ_l and denoted by $(D\hat{\omega}_\eta : D\varphi_l)_t$, $t \in \mathbb{R}$.

Finally, we include a sort of **orthogonality relations** in this context.

Theorem 35 *Let η_1 and η_2 be admissible vectors for $\{\pi, \mathcal{H}_\pi\}$. The following conditions are equivalent:*

(i) $\Delta^{-1/4} g_{\eta_1} \perp \Delta^{-1/4} g_{\eta_2}$.

(ii) $\hat{\mathfrak{P}}_{\eta_1} \perp \hat{\mathfrak{P}}_{\eta_2}$.

(iii) $\hat{\mathcal{H}}_{\eta_1} \perp \hat{\mathcal{H}}_{\eta_2}$.

Proof: See [18, IX.1.12]. $\square$

Remark 36 Phillips [14] studies orthogonality relations for irreducible square-integrable representations of left Hilbert algebras similar to those given by Duflo and Moore [22] for irreducible unitary representations of nonunimodular groups. See, in particular, [14, Th.2.4] and [14, Cor.2.5]. See also Grossmann, Morlet and Paul [3]. See also the reproducing kernel Hilbert space approach given by Carey [33]. Additional comments on orthogonality relations can be found in Rieffel [16, Sect.8].

6 Final remarks

Some final remarks for further development:

1. The unimodular case deserves particular attention to exploit the very special character of *traces* and the measurable structure around them [18, Sect.IX.2]. This study must connect with the work of Barbieri, Hernández and Parcet [7] on Riesz and frame systems and generalized Zak transforms. The integral and L^1-norm introduced by Phillips [13, Sect.2] in the general case should be taken into account too.

2. The subspaces $\mathcal{H}_\eta$ and $J\mathcal{H}_\eta$ are reproducing kernel Hilbert spaces. See Remarks 8.d, 11.b and 17. We do not use this property here, but it deserves attention. See Carey [33, 34, 25]. Reproducing kernel Hilbert spaces are closely related to the theory of *coherent states*, see e.g. Perelomov [35].

3. An analysis of the subspace $\hat{\mathcal{H}}_\eta$ could be done in the light of Rieffel and van Daele's approach to Tomita-Takesaki theory [36].

4. A lc group gives rise to two dual structures: one is associated with the Haar measure and multiplication operators and the other is related with the Plancherel weight and convolution operators. It is clear that, as presented here, the theory of admissible vectors is developed in the second structure. Deep understanding of this duality involves *crossed products* [31, 32] and *Kac algebras* [24].

5. Examples of application shall be given elsewhere.

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