Dynamics of algebras in quantum unstable systems

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Abstract

We introduce a dynamical evolution operator for dealing with unstable physical process, such as scattering resonances, photon emission, decoherence and particle decay. With that aim, we use the formalism of rigged Hilbert space and represent the time evolution of quantum observables in the Heisenberg picture, in such a way that time evolution is non-unitary. This allows to describe observables that are initially non-commutative, but become commutative after time evolution. In other words, a non-abelian algebra of relevant observables becomes abelian at $t \to \infty$. We finally present some relevant examples.

1 Introduction

In previous papers [1, 2, 3], we considered the quantum-to-classical transition from the point of view of the algebra of quantum observables. If a quantum system undergoes a physical process such that its behavior becomes classical, then its algebra of observables should undergo a transition from a non-Abelian algebra to an Abelian one. In order to describe this kind of time evolutions, we have proposed to use the Heisenberg picture, so we can consider the time evolution of the whole algebra of observables. It is important to remark that, in the standard formalism of quantum mechanics, a closed system always evolves unitarily. So, even in the Heisenberg picture, if two observables are incompatible at one time, they will remain incompatible for every time. Therefore, with the aim of describing the quantum-to-classical transition of the algebra of observables, it is necessary to go beyond unitary time evolutions.

In this paper, we continue with this approach by studying more general models. We introduce a dynamical evolution operator for dealing with unstable physical process (such as scattering resonances, photon emission, decoherence, relaxations and particle decay). In order to study the time evolution of their algebras of observables, we use the formalism of rigged Hilbert space (RHS), which

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is a natural choice for describing these kind of systems. The RGS description of quantum mechanics is an alternative formalism to that of von Neumann. It has several applications, particularly in particle physics and in the study of scattering processes. It also provides a rigorous description of eigenstates of the position and momentum operators, in fact, it serves as a rigorous mathematical basis for the Dirac formulation of quantum mechanics [4, 5, 6, 7, 8, 9].

As mentioned above, the use of the Heisenberg picture allows to study the classical limit from a different point of view. We show that an initially non-abelian algebra of relevant observables, becomes an abelian one when times goes to infinity. We refer to this non-abelian/abelian transition as *commutation process*. The study of this process focuses on the dynamics of the algebras of observables. In this work we provide an explicit representation of the time evolution operator for an extensive family of models described by the RHS formalism. We show that, under certain conditions, a commutation process (of the form described in [1, 2, 3]) is obtained for them. This phenomenon could be of interest for the study of quantum scattering resonances. It consists in a scattering process in which the scattered particle ends up in a quasi-stationary state. As a result of our work, it turns out that the use of non-Hermitian Hamiltonians of the form $H + \lambda V$, introduces a natural ground for the study of the commutation process of algebras.

The paper is organized as follows. In Section 2, we introduce the problem of the dynamical evolution of algebras and the logical quantum-to-classical transition. We illustrate our ideas by discussing a simple case: quantum operations and the quantum damping channel. Next, in Section 3, we discuss the fundamental aspects of the RHS formalism. In Section 4, we introduce a time evolution operator for observables in the RHS formalism. This allows us to describe the commutation process for a family of models of unstable systems in Section 5.

2 Logical quantum-to-classical transition

The sets of properties of classical and quantum systems have a logical structure, given by their orthocomplemented lattice structure [10] (see also [11, 12, 13] for a recent discussion on the subject). The propositional approach to quantum systems has been used in diverse areas of the foundations of quantum mechanics, as for example, in the study of quantum histories [14, 15, 16, 17, 18, 19, 20, 21]. Due to this structure, logical operations and logical relations between properties can be defined, such as disjunction ($\vee$), conjunction ($\wedge$), negation ($\neg$) and implication ($\leq$). All orthocomplemented lattices satisfy certain relations, called *distributive inequalities* [22]:

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c), \qquad a \vee (b \wedge c) \geq (a \vee b) \wedge (a \vee c), \tag{1}$$

where a, b, and c are arbitrary properties of the system. When the equalities hold, the lattice is called *distributive*. An orthocomplemented and distributive

lattice is called a *Boolean lattice*. The distributive property is an essential feature which differentiates classical and quantum lattices.

In classical mechanics a physical system is represented by a phase space and the properties of the system are represented by measurable subsets of its phase space. Therefore, the logical structure of classical systems is given by the algebraic structure of sets [10]. The resulting lattice is not only an orthocomplemented lattice, but also a distributive one, i.e., it is a Boolean lattice. This logical structure is naturally related with *classical logic*.

The quantum case is very different. In quantum mechanics a physical system is represented by a Hilbert space, observables are represented by self-adjoint operators on the Hilbert space and physical properties are represented by orthogonal projectors [23]. The logical structure of quantum systems is the algebraic structure of orthogonal projectors, and it is known as *quantum logic* [24, 25].

Unlike classical logic, quantum logic is a non-distributive orthocomplemented lattice. While in the classic lattice, all properties satisfy the distributive equalities, in the quantum lattice, only distributive inequalities hold in general [10, 22]. However, for some subsets of quantum properties the equalities hold. When a subset of properties satisfies the distributive equalities, they are called *compatible properties*. It can be proved that properties associated with different observables are compatible if the observables commute. If, on the contrary, two observables do not commute, some of the properties associated with them are not compatible [10, 25]. Therefore, by extension, commuting observables are called *compatible observables*.

The differences between classical and quantum logic are of fundamental importance for describing the quantum-to-classical transition. If a quantum system undergoes a physical process and as a consequence of this its behaviour becomes classic, then the logical structure of its properties should undergo a transition from quantum logic to classical logic, i.e. its lattice structure should become distributive. In order to give an adequate description of the logical structure transition, we have proposed to describe the classical limit in terms of the Heisenberg picture [1, 2, 3]. This perspective allows to consider the time evolution of the whole lattice of properties, and to study the transition from classical to quantum logic.

It is important to remark that, when governed by the Schrödinger equation, the time evolution of a closed system is always unitary. Even in the Heisenberg picture, if two observables are incompatible at one time, they will remain incompatible at any time [3]. Therefore, for describing the logical quantum-to-classical transition, it is necessary to consider more general time evolutions.

In order to describe adequately the logical quantum-to-classical transition, let us consider a quantum system with a general time evolution, and a time-dependent set of relevant observables, $\mathcal{O}(t) = \{\hat{O}_i(t)\}_{i\in I}$ (I an arbitrary set of indexes). Each set $\mathcal{O}(t)$ generates an algebra of observables $\mathcal{V}(t)$, and each algebra has associated an orthocomplemented lattice $\mathcal{L}_{\mathcal{V}(t)}$. We assume that initially some observables are incompatible, i.e., there are $i, j \in I$ such that $\left[\hat{O}_i(0), \hat{O}_j(0)\right] \neq 0$. Therefore, the lattice $\mathcal{L}_{\mathcal{V}(0)}$ is a non-distributive lattice.

For quantum systems with only one characteristic time t_c, the quantum-to-classical transition is given by the following process:

$$\left[\hat{O}_i(0), \hat{O}_j(0)\right] \neq 0 \longrightarrow \left[\hat{O}_i(t_c), \hat{O}_j(t_c)\right] = 0, \quad \forall i, j. \tag{2}$$

The logical classical limit is expressed by the fact that, while $\mathcal{L}_{\mathcal{V}(0)}$ is a non-distributive lattice, $\mathcal{L}_{\mathcal{V}(t_c)}$ is a Boolean one, i.e., it is a classical logic. In this way, we obtain an adequate description of the logical evolution of a quantum system.

In order to illustrate the general idea of the logical classical limit, we are going to show the logical transition of a physical with a quantum evolution given by a quantum channel.

2.1 A simple case: quantum operations

We consider a time evolution given by a quantum operation, and we define the corresponding Heisenberg representation. Once we have defined the quantum operations on the space of quantum observables, we study the logical quantum-to-classical transition of one relevant example: the amplitude damping channel. We show that, when time goes to infinity, the logical structure of the system becomes classical.

A quantum operation is a linear and completely positive map from the set of density operators into itself [26]. For each time t, the quantum operation $\mathcal{E}_t$ maps the initial state $\hat{\rho}_0$ to the state at time t, i.e.,

$$\mathcal{E}_t(\hat{\rho}_0) = \hat{\rho}(t), \tag{3}$$

In terms of the sum representation, we can express $\mathcal{E}_t$ as follows [26],

$$\mathcal{E}_t(\hat{\rho}_0) = \sum_{\mu} \hat{E}_\mu(t)\hat{\rho}_0\hat{E}_\mu^\dagger(t), \tag{4}$$

where $\hat{E}_\mu(t)$ are the Kraus operators [27] associated with the map $\mathcal{E}_t$.

Now, we define the Heisenberg representation of a quantum operation $\mathcal{E}_t$ as an operator $\tilde{\mathcal{E}}_t$ which maps each observable $\hat{O}$ to another observable $\hat{O}(t) = \tilde{\mathcal{E}}_t(\hat{O})$. We interpret $\hat{O}(t)$ as the time evolved observable of $\hat{O}$ under the quantum operation. The map $\tilde{\mathcal{E}}_t$ must preserve the mean values of all the observables, i.e.,

$$\begin{aligned} \mathrm{Tr}\left(\hat{\rho}(t)\hat{O}\right) &= \mathrm{Tr}\left(\sum_{\mu} \hat{E}_\mu(t)\hat{\rho}_0\hat{E}_\mu^\dagger(t)\hat{O}\right) = \\ &= \mathrm{Tr}\left(\hat{\rho}_0 \sum_{\mu} \hat{E}_\mu^\dagger(t)\hat{O}\hat{E}_\mu(t)\right) = \mathrm{Tr}\left(\hat{\rho}_0\hat{O}(t)\right). \end{aligned} \tag{5}$$

Therefore, the map $\tilde{\mathcal{E}}_t$ is given by

$$\tilde{\mathcal{E}}_t(\hat{O}) = \hat{O}(t) = \sum_{\mu} \hat{E}_\mu^\dagger(t)\hat{O}\hat{E}_\mu(t). \tag{6}$$

It easy to check that, if $\tilde{\mathcal{E}}_t(\hat{O})$ is a self-adjoint operator, then $\tilde{\mathcal{E}}_t$ maps the space of observables into itself.

Once we have defined quantum operations in the Heisenberg picture, we can study the logical classical limit of the system. Let us illustrate this process with a simple example: the amplitude damping channel [26]. The amplitude damping channel is useful for describing the energy dissipation of a quantum system due to the effects of an environment. It has many applications in quantum information processing, because it is appropriate for modelling the effects of quantum noise. This quantum map can be used to describe the decay of an excited state of a two-level atom due to the spontaneous emission of a photon. If the atom is in the ground state there is no photon emission, and the atom continues in the ground state. But, if the atom is in the excited state, after an interval of time τ, there is a probability p that the state has decayed to the ground state and a photon has been emitted [26].

The quantum map which represents the amplitude damping channel can be expressed in term of two Kraus operators [26],

$$\mathcal{E}_\tau(\hat{\rho}_0) = \hat{E}_0\hat{\rho}_0\hat{E}_0^\dagger + \hat{E}_1\hat{\rho}_0\hat{E}_1^\dagger, \tag{7}$$

with the Kraus operators given by

$$\hat{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{(1-p)} \end{pmatrix}, \quad \hat{E}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. \tag{8}$$

In the Heisenberg picture, we have an associated quantum map $\tilde{\mathcal{E}}_\tau$ acting on the space of observables, given by $\tilde{\mathcal{E}}_\tau(\hat{O}) = \hat{E}_0^\dagger\hat{O}\hat{E}_0 + \hat{E}_1^\dagger\hat{O}\hat{E}_1$. In matrix form, we have the following expression,

$$\tilde{\mathcal{E}}_\tau(\hat{O}) = \begin{pmatrix} O_{00} & \sqrt{1-p}\,O_{01} \\ \sqrt{1-p}\,O_{10} & pO_{00} + (1-p)O_{11} \end{pmatrix}. \tag{9}$$

If we apply the amplitude damping channel n times, we obtain the quantum map $\tilde{\mathcal{E}}_{n\tau}(\hat{O})$ given by

$$\tilde{\mathcal{E}}_{n\tau}(\hat{O}) = \begin{pmatrix} O_{00} & \sqrt{(1-p)^n}\,O_{01} \\ \sqrt{(1-p)^n}\,O_{10} & \sum_{i=0}^{n-1} p(1-p)^i O_{00} + (1-p)^n O_{11} \end{pmatrix}. \tag{10}$$

This can be reduced to

$$\tilde{\mathcal{E}}_{n\tau}(\hat{O}) = \begin{pmatrix} O_{00} & \sqrt{(1-p)^n}\,O_{01} \\ \sqrt{(1-p)^n}\,O_{10} & (1-p)^n O_{11} + O_{00}\left[1-(1-p)^n\right] \end{pmatrix}. \tag{11}$$

Considering the limit $n \longrightarrow \infty$, we obtain

$$\tilde{\mathcal{E}}_\infty(\hat{O}) = \begin{pmatrix} O_{00} & 0 \\ 0 & O_{00} \end{pmatrix}. \tag{12}$$

Thus, when $t \longrightarrow \infty$, all observables become multiples of the identity. This means that the whole algebra of observables becomes trivially commutative, and therefore, the associated lattice becomes Boolean.

The quantum-to-classical transition was extensively studied in the physics literature from the point of view of the quantum state evolution. However, from this perspective observables do not evolve on time. In previous papers, we argued that this kind of descriptions of the classical limit based on the Schrödinger picture is not adequate for explaining the quantum-to-classical transition of the logical structure of a system. Instead, the description in terms of the Heisenberg picture allows to describe how the quantum structure of properties becomes a Boolean.

3 Unstable systems and rigged Hilbert space

In the previous section we have studied the commutation process for a simple case. From a more general perspective, this phenomenon can appear when the evolution of the system is non-unitary [28, 29, 30]. A natural way of describing this kind of processes has been largely studied in the literature of resonances and unstable quantum systems [31, 32, 33, 34, 35, 36, 37]. In most of these models, resonances appear associated with poles of the scattering matrix and give place to decay times, which can be related with relaxation and decoherence processes [38, 39, 40, 41, 42]. The formalism of rigged Hilbert space is a natural choice for describing these kind of physical processes. In what follows, we will study the commutation process in the context of this formalism.

The study of unstable physical systems usually appeals to a master equation or a non-Hermitian Hamiltonian, giving place to a non-unitary evolution in the Hilbert space [36, 37, 43, 44, 45]. In this paper we explore a different approach: we change the Hilbert space for a rigged Hilbert space, in which we obtain an evolution that is suitable for describing unstable systems, and we construct a time evolution operator which is formally Hermitian although not unitary. This non-unitarity will allow the evolution from a non-commutative algebra of observables to a commutative one.

Resonance scattering is produced by a Hamiltonian pair $\{H_0, H\}$ with $H = H_0 + V$. Here, H_0 is the so called *free Hamiltonian* and V is an interaction. If we consider a three dimensional system, usually $H_0 = \mathbf{p}^2/2m$ and V is given by a spherically symmetric function of the position $\mathbf{r}$, $V(\mathbf{r})$. For simplicity, we also assume that $V(\mathbf{r})$ is short range (vanishes at the infinity faster than the Coulomb interaction) or even of compact support (it is zero outside a finite region).

Quasi stationary states are produced when an incoming particle enters into the interacting region, where the potential V is non-zero, and stays in this region for a much longer period of time than it would have been if the interaction were absent.

Quasi stationary states are usually identified with resonances [46]. Resonances are conceptually defined in two ways. We may always assume that the continuous spectrum of both H_0 and H is given by $\mathbb{R}^+ \equiv [0, \infty)$. For simplicity, we also may assume that both Hamiltonians do not have bound states (which implies a restriction to the continuous subspace), singular spectrum or even

that the absolutely continuous spectrum is not degenerate (which in the case of three dimensional spherically symmetric potentials is equivalent to choose the subspace with $\ell = 0$). Although none of these simplifications is essential, we will restrict our considerations to Hamiltonians with a non-singular continuous spectrum.

Definition 1.- For any pure state $\psi \in \mathcal{H}$ in the Hilbert space $\mathcal{H}$, let us consider the following pair of complex functions

$$F_0(z) := \langle\psi|(H_0 - z)^{-1}|\psi\rangle, \quad F(z) := \langle\psi|(H - z)^{-1}|\psi\rangle. \tag{13}$$

These functions are meromorphic having the positive semi-axis $\mathbb{R}^+$ as branch cut. Then, if for some $\psi \in \mathcal{H}$, $F(z)$ has a pole at Z_R and $F_0(z)$ does not, then we say that the Hamiltonian pair $\{H_0, H\}$ has a resonance at z_R [47].

Definition 2.- Take the S matrix in the momentum representation, so that S is a function of the modulus, $p := |\mathbf{p}|$ of the momentum $\mathbf{p}$, so that $S \equiv S(p)$. Under some hypothesis related with causality, $S(p)$ is analytically continuable to the complex plane as a meromorphic function (that may have additionally branch cuts). The isolated singularities of this extension are poles (never essential singularities). Poles on the imaginary axis are always simple. Poles on the positive imaginary axis represent bound states, poles on the negative imaginary axis are linked to the existence of antibound or virtual states. Finally, *resonances are given by pair of poles on the lower half plane, equidistant with respect to the imaginary axis. Each of these poles represent a single resonance.* In principle, there is no restriction with respect to the order of resonance poles.

It is customary to represent the S matrix in terms of the energy under the transformation $p = \sqrt{2mE}$. Since the square root is a multiform function supported on a two sheeted Riemann surface, the same property is shared by the function $S(E)$ [48]. On this Riemann surface, resonance poles appear in complex conjugate pairs and lie on the second sheet.

The equivalence of both definitions has not been thoroughly investigated, although it goes well for some simple models. In addition, there are some other definitions based on physical notions, which are only equivalent under additional assumptions [48, 49]. We may add that, although the first definition we give here is widely accepted by mathematicians, the second one is more popular among physicists. We are using this definition in the sequel.

In a high number of previous articles, we have discussed the construction of Gamow vectors in an abstract setting when the potential satisfies the above mentioned conditions. Let us summarize the main properties of these Gamow vectors.

- Let us consider a resonance defined as a pair of complex conjugate poles of the analytic continuation of the $S(E)$ matrix in the energy representation. These poles are located at the points $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$. Let us assume that these resonance poles are simple, otherwise unnecessary complications will emerge in the model. The general theory shows that one

may define two vectors, $|\psi^G\rangle$ and $|\psi^D\rangle$, related to z_R^* and z_R, respectively, with some properties that we mention in the sequel.

- Both Gamow vectors, $|\psi^G\rangle$ and $|\psi^D\rangle$, are eigenvectors of the total Hamiltonian H with respective eigenvalues given by z_R^* and z_R, so that

$$H|\psi^G\rangle = z_R^* \, |\psi^G\rangle \,, \qquad H|\psi^D\rangle = z_R \, |\psi^D\rangle. \tag{14}$$

  These relations define both Gamow vectors.

- However, H is a self adjoint Hamiltonian and a self adjoint Hamiltonian cannot have complex eigenvectors. The situation is saved if we extend H to the anti-dual space $\Phi^\times$ of a rigged Hilbert space (RHS in the sequel) $\Phi \subset \mathcal{H} \subset \Phi^\times$. In general, one may construct two RHS $\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^\times$ such that $H|\psi^G\rangle = z_R^* \, |\psi^G\rangle$ is valid in $\Phi_-^\times$ and $H|\psi^D\rangle = z_R \, |\psi^D\rangle$ is valid in $\Phi_+^\times$.

- The spaces Φ_+ and Φ_- are unitarily equivalent to spaces of complex analytic functions on the upper and lower half planes, respectively. This construction permits the use of complex analytic function techniques to obtain our results. In particular, the use of Hardy functions on a half plane permits a formulation for time asymmetric quantum mechanics valid for scattering processes.

- We may extend the evolution operator to the antidual spaces $\Phi_-^\times$ and $\Phi_+^\times$, so that this operator may be applied to the Gamow vectors. The result is given by the following pair of relations

$$e^{-itH}|\psi^G\rangle = e^{-itE_R} \, e^{t\Gamma/2} \, |\psi^G\rangle, \quad e^{-itH}|\psi^D\rangle = e^{-itE_R} \, e^{-t\Gamma/2} \, |\psi^D\rangle. \tag{15}$$

  Note that $|\psi^G\rangle$ grows and $|\psi^D\rangle$ decays as time increases in the positive direction. Consequently, $|\psi^G\rangle$ and $|\psi^D\rangle$ are named the *growing* and the *decaying* Gamow vector, respectively.

- When the spaces $\Phi_\pm$ are constructed using Hardy functions, the first relation in (15) is valid for $t \leq 0$ only. Analogously, the second relation in (15) is valid for $t \geq 0$ only. In this formalism, the unitary group of time evolution splits into two semigroups, one for $t \leq 0$ and the other for $t \geq 0$. Thus, these RHS supports a semigroup representation of time evolution.

- Thus, we have two apparently different processes, one for $t \leq 0$ and the other for $t \geq 0$. However, the time reversal operator T transforms a process into the other, so that both are essentially equivalent. In particular,

$$T|\psi^G\rangle = |\psi^D\rangle \,, \qquad T|\psi^D\rangle = |\psi^G\rangle. \tag{16}$$

- Nevertheless, the basis for time asymmetric quantum mechanics consists in giving a completely different interpretation to both processes. Roughly speaking, the RHS $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ contains the system observables, while $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ the states. Then, both are different and, thus, time asymmetry acquires a sense.

In this s ection we have presented the standard formalism of Rigged Hilbert spaces. Our aim is to introduce a time evolution operator in order to describe the evolution of operator algebras in this setting. Thus, we need to write first the non-Hermitian Hamiltonian in a spectral decomposition-like expression. This is the subject of the next section.

4 Generalized time evolution operator in the rigged Hilbert space formalism

In the usual approach to rigged Hilbert space, the dynamical description is focused on the time evolution of mean values of observables. However, the expression of a time evolution operator for states (or operators, as seen from the perspective of the Heisenberg picture) was not present in the literature. Here we introduce such a time evolution operator. This will allow us to map non-Abelian algebras into Abelian ones.

4.1 Non-Hermitian Hamiltonian

It was shown that any vector $\varphi_+ \in \Phi_+$ can be expanded as

$$\varphi_+ = \sum_i \alpha_i |\psi_i^D\rangle + |\psi_B\rangle, \tag{17}$$

where the sum extends to all resonances, α_i are complex numbers and $|\psi_B\rangle$ is the background term. This term is added in order to avoid a purely exponential decay of normalizable vectors in Hilbert space. Since $\Phi_+ \subset \Phi_+^\times$, equation (17) is valid in $\Phi_+^\times$. Since φ_+ is normalizable and the Gamow vectors are not, we conclude that the background term $|\psi_B\rangle$ cannot be normalizable either.

Analogously, for any $\varphi_- \in \Phi_-$, we have the following expansion:

$$\varphi_- = \sum_i \beta_i |\psi_i^G\rangle + |\phi_B\rangle. \tag{18}$$

Correspondingly, in [50], we have shown that there are two possible expansions for the total Hamiltonian H given by the following expressions,

$$H = \sum_i z_{R_i} |\psi_i^D\rangle\langle\psi_i^G| + BGR \tag{19}$$

$$H^\dagger = \sum_i z_{R_i}^* |\psi_i^G\rangle\langle\psi_i^D| + BGR^* \,. \tag{20}$$

Both expressions are the formal adjoint of each other. This is why we add the dagger in the second expansion. They act on different spaces:

$$H \in \mathcal{L}(\Phi_-, \Phi_+^\times), \qquad H^\dagger \in \mathcal{L}(\Phi_+, \Phi_-^\times), \tag{21}$$

where $\mathcal{L}(\Phi_\pm, \Phi_\mp^\times)$ is the space of continuous linear mappings from $\Phi_\pm$ to $\Phi_\mp^\times$. Expressions like BGR or BGR^* denote the projection onto the background subspace.

In general, the leading term corresponds to the resonance contribution and the background in (19) is usually small. As a matter of fact, deviations of the exponential law occur for very small and for very large times and are difficult to be observed. This is why we may omit the background term for most of observational times. In consequence, a good approximation for expansions (19) and (20) is given if we omit the background, so that

$$H = \sum_i z_{R_i} |\psi_i^D\rangle\langle\psi_i^G| \,, \qquad H^\dagger = \sum_i z_{R_i}^* |\psi_i^G\rangle\langle\psi_i^D|. \tag{22}$$

Let us insist that the distinction between H and $H^\dagger$ is purely conventional so that we could have called H or $H^\dagger$ to any of them.

Now, we have the mathematical tools to analyze decoherence produced by resonances. We show that the above described time evolution gives place to a commutation process.

4.2 Time evolution operator

In order to avoid possible convergence problems, we may assume that the number of resonances is finite. From the purely mathematical point of view, this assumption is not fulfilled for most quantum models, but it is still valid in some useful toy models, such as Friedrichs's one. It is nevertheless true that resonances with large imaginary terms are not observable, as the inverse of the imaginary part is related with the mean life. Also, in the context of a non relativistic theory, large values for the resonance energy E_R are meaningless. In this way, the approximation having a finite number of resonances is a reasonable one.

In [50], we have defined a pseudometrics for Gamow vectors. The idea of using pseudometrics was discussed heuristically in previous articles [51, 52]. As mentioned before, let us assume that the number of resonance poles is finite $\{z_1, z_1^*, z_2, z_2^*, \ldots, z_N, z_N^*\}$. Let us consider the $2N$ dimensional space, $\mathcal{H}^G$, spanned by the Gamow vectors corresponding to these resonances,

$$\{|\psi_1^D\rangle, |\psi_1^G\rangle, |\psi_2^D\rangle, |\psi_2^G\rangle, \ldots, |\psi_N^D\rangle, |\psi_N^G\rangle\}. \tag{23}$$

Notice that $\mathcal{H}^G \subset \Phi_\mp^\times$. We define a pseudometrics in $\mathcal{H}^G$ by appealing to a

matrix:

$$A := \begin{pmatrix} 0 & 1 & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & 1 & \dots & \dots & \dots \\ \dots & \dots & 1 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & 1 & 0 \end{pmatrix}. \tag{24}$$

All entries of A which are not explicitly given are zero. Then, the pseudoscalar product of two vectors $|\psi\rangle, |\varphi\rangle \in \mathcal{H}^G$, $(\psi|\varphi)$, is $(\psi|\varphi) = \langle\psi|A|\varphi\rangle$. For the basis (23), the pseudoscalar products are given by

$$(\psi_i^D|\psi_j^D) = (\psi_i^G|\psi_j^G) = 0, \quad (\psi_i^D|\psi_j^G) = (\psi_i^G|\psi_j^D) = \delta_{ij}, \tag{25}$$

where δ_{ij} is the Kronecker delta.

In order to define a sort of time evolution on the space $\mathcal{H}^G$, we need to use this pseudometrics. First, let us replace the Hamiltonian (22) by

$$H = \sum_{i=1}^{N} z_i \, |\psi_i^D)(\psi_i^G|. \tag{26}$$

More details are discussed in the Appendix. Using the pseudometrics, the square of H should be

$$H^2 = \sum_{i=1}^{N} z_i \, |\psi_i^D)(\psi_i^G| \sum_{j=1}^{N} z_j \, |\psi_j^D)(\psi_j^G| = \sum_{i=1}^{N}\sum_{j=1}^{N} z_i \, z_j \, |\psi_i^D)(\psi_i^G|\psi_j^D)(\psi_j^G|$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} z_i \, z_j \, |\psi_i^D)\delta_{ij}(\psi_j^G| = \sum_{i=1}^{N} z_i^2 \, |\psi_i^D)(\psi_i^G|, \tag{27}$$

so that

$$H^n = \sum_{i=1}^{N} z_i^n \, |\psi_i^D)(\psi_i^G|. \tag{28}$$

This allows us to define an expression of the type e^{-itH} as follows

$$U := e^{-itH} = \sum_{j=1}^{N} e^{-itz_j} \, |\psi_j^D)(\psi_j^G|. \tag{29}$$

The above expression can be used as time evolution operator on the space of Gamow vectors. Notice that, in principle, the expression is valid for any value of t. Furthermore, we have $e^{-itH} \in \mathcal{L}(\Phi_-, \Phi_+^\times)$.

5 Commutators

In this section, we finally deal with the Fortín, Holik, Vanni approach for the decoherence of observables [2]. In the first term, we discuss the simplest case in which a single resonance is present, like in the basic Friedrichs model. We show that the difference between the existence of one or more resonances goes beyond the actual complication on the notation.

We recall that we are studying a resonance scattering process in which the background term is neglected. Then, our construction is restricted to the space spanned by the Gamow vectors. In addition, we use the approximation of having a finite number of resonances (this approximation was motivated in the previous sections). In this way, the observables under our consideration are operators acting on the finite dimensional space spanned by the Gamow vectors.

5.1 One resonance

Let O be an observable on the space of Gamow states. Assume that O evolves with time. In this case, we have to define what we understand by time evolution of an observable. In the case of having just one resonance, we propose to use the "complete" Hermitian Hamiltonian defined as

$$H = z_R|\psi^D)(\psi^G| + z_R^*|\psi^G)(\psi^D|. \tag{30}$$

With the aid of the pseudo-metrics (25), we obtain the following expressions

$$H^n = z_R^n|\psi^D)(\psi^G| + (z_R^*)^n|\psi^G)(\psi^D| \tag{31}$$

and

$$U(t) := e^{-itH} = e^{-itz_R}|\psi^D)(\psi^G| + e^{-itz_R^*}|\psi^G)(\psi^D|. \tag{32}$$

To find the formal inverse of e^{-itH}, we need to know an expression for the identity. Since $\{|\psi^D), |\psi^G)\}$ is a basis in $\mathcal{H}^G$, let us write

$$I := |\psi^D)(\psi^G| + |\psi^G)(\psi^D|. \tag{33}$$

This is the identity on $\mathcal{H}^G$. Indeed,

$$I|\psi^D) = |\psi^D)(\psi^G|\psi^D) + |\psi^G)(\psi^D|\psi^D) = |\psi^D). \tag{34}$$

Analogously, $I|\psi^G) = |\psi^G)$. Then, the linearity of I shows that this is indeed the identity on $\mathcal{H}^G$. Then, the inverse of $U(t)$ is

$$U(t)^{-1} = e^{itz_R}|\psi^D)(\psi^G| + e^{itz_R^*}|\psi^G)(\psi^D| = U(-t), \tag{35}$$

since, using the pseudometric relations (25), we have that $U(t) = U(-t) = I$.

Next, the time evolution of the operator, which at time $t = 0$ is O is now

$$O(t) = U(t)OU(-t). \tag{36}$$

We have just one resonance with resonance pole at $z_R = E_R - i\Gamma/2$, so that the above equation reads

$$O(t) = \left[e^{-itz_R}|\psi^D)(\psi^G| + e^{-itz_R^*}|\psi^G)(\psi^D|\right] O \left[e^{itz_R}|\psi^D)(\psi^G| + e^{itz_R^*}|\psi^G)(\psi^D|\right] =$$
$$= |\psi^D)(\psi^G|O|\psi^D)(\psi^G| + e^{-t\Gamma}|\psi^D)(\psi^G|O|\psi^G)(\psi^D| +$$
$$+ e^{t\Gamma}|\psi^G)(\psi^D|O|\psi^D)(\psi^G| + |\psi^G)(\psi^D|O|\psi^G)(\psi^D|, \tag{37}$$

where the "averages" $(\psi^G|O|\psi^G)$, etc are in principle well defined, since we work on a finite dimensional space. In fact, the dimension is 2 in this case.

This provides undesirable terms in $e^{t\Gamma}$ with $t > 0$, so that we have to give up the condition $U(t)U(-t) = I$. A second choice for the time evolution of the observables will include the following ingredients:

$$U(t) = e^{-itz_R}|\psi^D)(\psi^G| + e^{itz_R^*}|\psi^G)(\psi^D|. \tag{38}$$

The point is that this operator is formally Hermitic. Clearly, its square is not the identity, instead

$$U(t)U^\dagger(t) = U^2(t) = e^{-t\Gamma} I. \tag{39}$$

This choice provides more desirable results. The standard definition of the time evolution for an observable states that the value of the observable O after a time t is given by $O(t) := U^\dagger(t)OU(t)$. In our case, $U^\dagger(t) = U(t)$. Note that the commutator

$$[O_1(t), O_2(t)] = U(t)O_1U(t)U(t)O_2U(t) - U(t)O_2U(t)U(t)O_1U(t) =$$
$$= e^{-t\Gamma}[U(t)O_1O_2U(t) - U(t)O_2O_1U(t)] =$$
$$= e^{-t\Gamma}U(t)[O_1, O_2]U(t). \tag{40}$$

Using the above machinery, we may calculate $U(t)[O_1, O_2]U(t)$. It is a sum of four terms:

$$e^{-2iE_R}\, e^{-t\Gamma}\, |\psi^D)(\psi^G|[O_1, O_2]|\psi^D)(\psi^G|,$$
$$e^{2iE_R}\, e^{-t\Gamma}\, |\psi^G)(\psi^D|[O_1, O_2]|\psi^G)(\psi^D|,$$
$$e^{-t\Gamma}\, |\psi^D)(\psi^G|[O_1, O_2]|\psi^D)(\psi^G|,$$
$$e^{-t\Gamma}\, |\psi^G)(\psi^D|[O_1, O_2]|\psi^G)(\psi^D|.$$

The terms of the form $(\psi^G|[O_1, O_2]|\psi^D)$ and $(\psi^D|[O_1, O_2]|\psi^G)$ should be well defined as they are "averages" of linear operators on finite dimensional spaces. We realize that (40) is of the form:

$$[O_1(t), O_2(t)] = e^{-2t\Gamma}\, \{\alpha(t)|\psi^D)(\psi^G| + \beta(t)|\psi^G)(\psi^D|\}, \tag{41}$$

where $\alpha(t)$ and $\beta(t)$ are constants for which the dependence on t is just a phase of the form $e^{\pm 2itE_R}$ with E_R real. We have obtained the result given in [2].

5.2 More than one resonance

Here, the procedure is the same, although calculations are more cumbersome. In general, one should obtain

$$[O_1(t), O_2(t)] = \sum_{j=1}^{N} e^{-2t\Gamma_j} \{\alpha_j(t)|\psi_j^D)(\psi_j^G| + \beta_j(t)|\psi_j^G)(\psi_j^D|\}, \tag{42}$$

where the resonance poles are located at the points $z_j = E_j - i\Gamma_j/2$, $j = 1, 2, \ldots, N$.

5.3 Compatibility with TAQM

It is important to remark that the formalism presented in this work can be applied to a wide family of physical models of interest. As an example, we show in this section the compatibility with the TAQM formalism [53, 54], which finds applications in scattering resonances and more recently in classical and quantum optics [55, 56].

Let us go to (38) and observe the coefficients e^{-itz_R} for $|\psi^G)(\psi^D|$ and $e^{itz_R^*}$ for $|\psi^D)(\psi^G|$. Also note that, in the standard formulation of TAQM using RHS of Hardy functions on a half plane, we have the following evolution rules:

$$e^{-itH}|\psi^D) = e^{-itz_R}|\psi^D)\,, \qquad t \geq 0, \tag{43}$$

and

$$e^{-itH}|\psi^G) = e^{-itz_R^*}|\psi^G) = e^{i(-t)z_R^*}|\psi^G)\,, \qquad t \leq 0. \tag{44}$$

If we want to have a forward time evolution, we use the conversion $-t \longmapsto t$ in (44). Thus, for $t \geq 0$, the expression $e^{itz_R}|\psi^G)$ has full sense. Contrary, for $t \leq 0$ time evolution for these Gamow vectors does not exist in the context of TAQM. Therefore, equations (43,44) are valid for $t \geq 0$ only from this point of view.

6 Conclusions

In previous works [1, 2, 3], we have studied the quantum-to-classical transition from the point of view of the algebra of observables of the system. If a quantum system undergoes a physical process such that its behavior becomes classical, then its algebra of observables should undergo a process from a non-Abelian algebra to an Abelian one.

In this paper, we continue this approach. We introduce a dynamical evolution operator for dealing with unstable physical process (such as scattering resonances, photon emission, decoherence and particle decay). In order to do this, we use the formalism of rigged Hilbert space and we represent the time evolution of quantum observables in the Heisenberg picture, in such a way that time evolution is non-unitary. This allows us to describe observables that are initially non-commutative, but become commutative after time evolution. Therefore, we

show that the quantum-to-classical transition based in dynamical algebras, occurs in a rich family of models of unstable systems.

Appendix

The replacement of (22) by (26) is the change of H given by (22) by BHB, where B is a square root of A. As a matter of fact, this means the use of a new operator H of the form:

$$H = \sum_{i=1}^{N} z_i \, B|\psi_j^D\rangle\langle\psi_j^G|B, \tag{45}$$

where B is not uniquely defined. We may choose the following definition for B: replace the 2×2 dimensional nonvanishing boxes in A by

$$(-i)^{1/2} \begin{pmatrix} i\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & i\sqrt{2}/2 \end{pmatrix}. \tag{46}$$

Note that $B|\psi_i^D\rangle = |\psi_i^D)$ and $\langle\psi_i^G|B = (\psi_i^G|$. The square of H as in (45) is given by

$$\begin{aligned} H^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} z_i z_j B|\psi_i^D\rangle\langle\psi_i^G|BB|\psi_j^D\rangle\langle\psi_j^G|B = \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} z_i z_j B|\psi_i^D\rangle\langle\psi_i^G|A|\psi_j^D\rangle\langle\psi_j^G|B = \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} z_i z_j B|\psi_i^D\rangle \, \delta_{ij} \, \langle\psi_j^G|B = B\left[\sum_{i=1}^{N} z_i^2 |\psi_i^D\rangle\langle\psi_j^G|\right] B. \end{aligned} \tag{47}$$

Thus,

$$e^{-itH} = B\left[\sum_{i=j}^{N} e^{-itz_j}|\psi_j^D\rangle\langle\psi_j^G|\right] B. \tag{48}$$

In relation with the formal adjoint $H^\dagger$, it seems convenient to use another square root of A, that we call C. Formally, C is the adjoint of B, $C := B^\dagger$. It is a square root of A since $B^\dagger B^\dagger = (BB)^\dagger = A^\dagger = A$. Then, the new $H^\dagger$ would be

$$H^\dagger = C\left[\sum_{i=1}^{N} z_i^* |\psi_i^G\rangle\langle\psi_i^D|\right] C = \sum_{i=1}^{N} z_i^* |\psi_i^G)(\psi_i^D|. \tag{49}$$

Then, $|\psi_i^G) = C|\psi_i^G\rangle$ and $(\psi_i^D|C = \langle\psi_i^D|$.

This choice has an interest by its own. In fact, note that the following expression is formally Hermitian:

$$H = \sum_{i=1}^{N} z_i |\psi_i^D\rangle\langle\psi_i^G| + \sum_{j=1}^{N} z_j^* |\psi_j^G\rangle\langle\psi_j^D|. \tag{50}$$

Then, the formal hermiticity of

$$H = \sum_{i=1}^{N} z_i |\psi_i^D)(\psi_i^G| + \sum_{j=1}^{N} z_j^* |\psi_j^G)(\psi_j^D| \tag{51}$$

requires that (51) be equal to

$$H = B \left[\sum_{i=1}^{N} z_i |\psi_i^D\rangle\langle\psi_i^G| \right] B + C \left[\sum_{j=1}^{N} z_j^* |\psi_j^G\rangle\langle\psi_j^D| \right] C. \tag{52}$$

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