Superpositions of bright and dark solitons supporting the creation of balanced gain and loss optical potentials

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Abstract

We address the construction of analytically integrable complex-valued potentials by linear superpositions of fundamental bright and dark optical solitons that solve cubic nonlinear Schrödinger equations. The real part of the potentials coincides with the bright soliton intensity. The imaginary part results from the convolution of the bright soliton with its concomitant, a localized dark excitation that arises from repulsive nonlinearities in the media. In general, the method leads to the Gross-Pitaevskii nonlinear differential equation, so the above results correspond to the absence of external interactions. The potentials presented here may find applications in the study of self-focussing signals that propagate in nonlinear media with balanced gain and loss since they are parity-time symmetrical.

1 Introduction

Optical solitons are localized pulses that do not change shape as they propagate in nonlinear media [1–3]. Dispersion and nonlinearity conspire to cancel the spatial dependence in the dynamics, which is usually described by nonlinear differential equations [4, 5], so the solitons are not only shape invariant, they are also very stable when their area is a constant of motion. This last defines the so-called bright solitons and shows how practical is their presence in optics [2]. Being bound states of the cubic nonlinear Schrödinger equation, bright optical solitons exist because attractive nonlinearities are originated in the media by the Kerr effect [6, 7]. Solutions for repulsive nonlinearities, known as dark

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optical solitons, are also available and useful [6,7]. An interesting application of the optical soliton properties concerns the recent practical validation of the parity-time symmetry in optics [8]. Such a symmetry means invariance under parity and time-reversal transformations in quantum mechanics [9], and expresses that self-adjointness is not a necessary condition to have physical observables with real spectrum. The experimental proof proportioned in [8] is based on the formal equivalence between some dynamical equations in optics and the Schrödinger equation in quantum mechanics. A complex refractive–index $n(x) = n_R(x)+in_I(x)$ serves as an 'optical potential' that can be realized in the laboratory. The gain and loss regions of the material are associated with the imaginary distribution $n_I(x)$, which may be chosen odd $n_I(-x) = n(x)$ to balance the gain–loss rates. An even distribution $n_R(-x) = n_R(x)$ would guide the signal along the propagation direction that is transversal to x. The above provides a balanced gain-loss optical potential that can be used to propagate self-focussing electromagnetic signals if, in addition, the media is nonlinear [1–3, 6, 7].

Quite recently, we have developed a formalism that permits the construction of analytically solvable complex-valued potentials with real spectrum [10–13]. The model is based on the properties of the Riccati equation in the complex domain [14,15], and the Darboux method [16]. The transformation theory introduced by Darboux in 1882 is useful to intertwine the energies of two different spectral problems in contemporary physics [17], and find immediate applications in soliton theory [18] as well as in supersymmetric quantum mechanics [17,19]. The present work is motivated by the usefulness of the complex-valued function $V = u^2 + iu_x$ as the seed of solutions u for nonlinear equations like the modified Korteweg–de Vries, sine–Gordon and cubic nonlinear Schrödinger ones [4]. The judicious selection of u may provide a parity-time symmetric potential V addressed to generate the nonlinearities that are necessary to control the propagation of light in optical media. Indeed, we find that combining the model introduced in [10] with the above expression of V leads automatically to the Gross–Pitaevskii nonlinear equation [20, 21], which offers a natural arena to study Bose-Einstein condensates [22, 23] and is reduced to the cubic nonlinear Schrödinger equation in the absence of external interactions [6, 7]. The latter is exactly solvable by using the inverse scattering method [5, 24] but the former is very restrictive in the search for integrable models.

The organization of the paper is as follows. In Section 2 we revisit the main ideas and results introduced in [10]. Then, conditions are imposed to obtain a balanced gain–loss optical potential and the Gross–Pitaevskii equation is derived in Section 2.1. We specialize the model to the free-particle potential and show that the nonlinear Schrödinger equation defines the profile of u in $V = u^2 + iu_x$. Then, we find that u^2 coincides with the intensity of a bright optical soliton while u_x is defined by the convolution of two optical solitons, one obeying attractive nonlinearities and the other responding to repulsive nonlinear interactions. That is, potential V is generated from the linear superposition of bright and dark optical solitons, both of them in either the stationary regime or in a flat configuration. Finally, in Section 3 we give some conclusions of our work.

2 Model and results

Using the Darboux approach [16], stationary one-dimensional Schrödinger equations,

$$-\psi_{xx} + V\psi = k^2\psi \tag{1}$$

and

$$-\varphi_{xx} + V_0\varphi = k^2\varphi, \tag{2}$$

can be intertwined through the relationship

$$V = V_0 + 2\beta_x, \quad \psi = \varphi_x + \beta\varphi, \tag{3}$$

with β a solution of the nonlinear Riccati equation

$$-\beta_x + \beta^2 = V_0 - \epsilon. \tag{4}$$

Assuming that V_0 is a real-valued measurable function such that Eq. (2) is integrable in $\text{Dom}V_0 = (a_1, a_2) \subseteq \mathbb{R}$, with the real eigenvalues $E = k^2$, one can construct a complex-valued function V such that Eq. (1) is integrable with the same energies $E = k^2$ plus an additional real eigenvalue ϵ [10]. Indeed, for any $\epsilon \in \mathbb{R}$, a complex-valued solution $\beta = \beta_R + i\beta_I$ of (4) must satisfy the coupled system

$$-\beta_{Rx} + \beta_R^2 - \beta_I^2 + \epsilon - V_0 = 0, \tag{5}$$

$$-\beta_{Ix} + 2\beta_I\beta_R = 0. \tag{6}$$

Once the solutions of (5)-(6) have been supplied, the real and imaginary parts of the complex-valued potential V are

$$V_R = V_0 + 2\beta_{Rx}, \quad V_I = 2\beta_{Ix}. \tag{7}$$

Given a bound state ψ_n of such potential, the conventional notions of probability density $\rho_n = |\psi_n|^2$ and probability current $\mathcal{J}_n = i(\psi_n\, \psi_{nx}^* - \psi_{nx}\psi_n^*)$ apply [11], the asterisk stands for complex conjugation, and they are such that the *condition of zero total area* [12],

$$\int_{DomV_0} \text{Im}V_\lambda(x)dx = 2\beta_I(x)|_{a_1}^{a_2} = 0, \tag{8}$$

ensures conservation of total probability.

The new potential V may feature the parity-time (PT) symmetry, defined as the invariance under parity (P) and time-reversal (T) transformations. In quantum mechanics the former corresponds to spatial reflection $p \to -p$, $x \to -x$, and the latter to $p \to -p$, $x \to x$, together with complex conjugation $i \to -i$ [9]. Thus, a necessary condition for PT-symmetry is that the complex-valued potential $V(x)$ should satisfy $V(x) = V^*(-x)$. In our case this last requires initial potentials represented by even functions $V_0(x) = V_0(-x)$ in $\text{Dom}V_0$. Then, it is sufficient to take β_R even and β_I odd in $\text{Dom}V_0$ to get parity-time symmetric potentials V.

The straightforward calculation shows that β is parameterized by a real number λ as follows

$$\beta = -\frac{\alpha_x}{\alpha} + i\frac{\lambda}{\alpha^2}, \tag{9}$$

where the function

$$\alpha(x) = \left[av^2(x) + bv(x)u(x) + cu^2(x)\right]^{1/2} \tag{10}$$

is real-valued and free of zeros in $\text{Dom}V_0$ when the parameters $\{a, b, c\}$ are real and satisfy $4ac - b^2 = 4(\lambda/\omega_0)^2$ [10]. Here u and v are two linearly independent solutions of Eq. (2) for $k^2 = \epsilon$, and $w_0 = W(u, v)$ is their Wronskian.

Of particular interest, the complex-valued potentials (7) may be constructed with the profile

$$V = -(\vartheta^2 + i\vartheta_x), \tag{11}$$

where the function ϑ is (at least) twice differentiable with respect to x and should contain a parameter, say z, so that $\vartheta = \vartheta(x; z)$. Potentials satisfying (11) are very important in soliton theory since ϑ can be used to solve the three nonlinear evolution equations known as the modified Korteweg–de Vries equation, the sine–Gordon equation, and the cubic nonlinear Schrödinger equation, all of them defining the propagation of waves in dispersive media [4].

In the following we show that complex-valued potentials (7) and (11) are compatible for the appropriate solutions of the system (5)-(6). Such relationship supplies a meaning for the real and imaginary parts of the β-function that generates potential (7) via the Darboux transformation (3).

2.1 The Gross–Pitaevskii equation

From (7) and (11) we obtain the system

$$V_0 + 2\beta_{Rx} = -\vartheta^2, \qquad 2\beta_{Ix} = -\vartheta_x. \tag{12}$$

After integrating, the last of the above equations leads to

$$\vartheta = -2\beta_I + \vartheta_0, \tag{13}$$

with ϑ_0 an integration constant. The combination of (13) with (6) produces

$$\vartheta_x = 2(\vartheta - \vartheta_0)\beta_R. \tag{14}$$

Then, the real and imaginary parts of β are respectively given by

$$\beta_R = \frac{\vartheta_x}{2(\vartheta - \vartheta_0)}, \qquad \beta_I = -\left(\frac{\vartheta - \vartheta_0}{2}\right). \tag{15}$$

Using these results in (9) yields the expression

$$\vartheta = -2\frac{\lambda}{\alpha^2} + \vartheta_0, \tag{16}$$

where the constant arising from the integration of β_R has been fixed as -2λ, for consistency. Now, to find a mechanism to determine ϑ, let us introduce (13) into the equation for β_R in (12). After using Eq. (5) we obtain

$$2(\beta_R^2 + \beta_I^2 + \epsilon) - V_0 = \vartheta_0(4\beta_I - \vartheta_0). \tag{17}$$

Without loss of generality we make $\vartheta_0 = 0$. Then, the above equation is reduced to the constraint

$$|\beta|^2 = \tfrac{1}{2}V_0 - \epsilon. \tag{18}$$

As $|\beta| \geq 0$ we immediately have $V_0 \geq 2\epsilon$. Besides, from (15) we realize that (18) produces the nonlinear differential equation

$$\vartheta_x^2 + (4\epsilon - 2V_0)\vartheta^2 + \vartheta^4 = 0, \tag{19}$$

which defines the analytic form of ϑ.

The next step is to determine whether or not the function ϑ features a soliton profile. With this aim notice that the derivative of (14), after using (12) and condition (18), gives

$$-\vartheta_{xx} + (V_0 - 2\vartheta^2)\vartheta = 4\epsilon\vartheta. \tag{20}$$

Now, we introduce a real parameter z via the equation

$$i\vartheta_z = 4\epsilon\vartheta, \tag{21}$$

with solution

$$\vartheta(x;z) = \vartheta(x)\exp(-i4\epsilon z + \xi_0), \tag{22}$$

where ξ_0 is an integration constant. Considering this new form of ϑ, to avoid dependence on $\arg(\vartheta) = -i4\epsilon z + \xi_0$, let us change ϑ^3 by $|\vartheta|^2\vartheta$ in (20). We obtain the spectral problem

$$-\vartheta_{xx} + \left(V_0 - 2|\vartheta|^2\right)\vartheta = 4\epsilon\vartheta, \tag{23}$$

which is named after Gross [20] and Pitaevskii [21], and currently known as the time-independent Gross-Pitaevskii (GP) equation. Of course, (20) and (23) coincide for real ϑ. Combining (21) and (23) one has

$$-\vartheta_{xx} + \left(V_0 - 2|\vartheta|^2\right)\vartheta = i\vartheta_z. \tag{24}$$

The latter is called time-dependent GP equation (or simply GPE), mainly when the propagation parameter z is treated as the evolution variable. In analogy with the Schrödinger equation, V_0 is an external potential and the nonlinear term $-2|\vartheta|^2$ represents an attractive interaction that is proportional to the local density $|\vartheta|^2$. The GPE is a powerful tool to study Bose-Einstein condensates (BEC) in the mean-field approximation [23], where the nonlinearity represents an effective potential to which is subjected each atom because its interaction with all other particles, and $|\vartheta|^2$ stands for the atomic density. In such approach the external potential V_0 produces the BEC confinement and may adopt different forms. The trapping in 3D models can be either magnetic or optical, the latter with the advantage that optical traps are extremely flexible and controllable in shape [22]. Lower dimensional BECs are possible at temperatures close to zero when phase fluctuations are negligible. For instance, magnetic traps include external harmonic potentials that can be produced with highly anisotropic profiles. If the longitudinal frequency ω_z is such that $\omega_z << \omega_\perp \equiv \omega_x = \omega_y$, then the fully 3D GPE can be reduced to an effectively 1D model described by the GPE (24), where V_0 is an oscillator of frequency ω_z [22].

In general, the GPE (24) cannot be solved analytically for arbitrary V_0. Particular examples include periodic potentials $V_0(x+L) = V_0(x)$ with period L for which the Bloch theory [25] gives rise to discrete solitons [26]. However, the simplest exactly solvable case is given by the free–particle potential $V_0 = 0$, which reduces (24) to the cubic nonlinear Schrödinger equation (NLSE),

$$-\vartheta_{xx} - 2|\vartheta|^2\vartheta = i\vartheta_z. \tag{25}$$

Eq. (25) is useful to describe the dynamics of complex field envelopes in nonlinear dispersive media [7], as well as the paraxial approximation of the light propagation in Kerr media [3]. In the last case, the propagation parameter z refers to the distance along the beam and the variable x stands for the direction transverse to the propagation. Therefore, ϑ is the normalized amplitude of the electric field envelope describing the pulse. The nonlinearity $-2|\vartheta|^2$ is due to the Kerr effect and represents the refractive index, its effect on the light rays increases with the light intensity $|\vartheta|^2$ and leads to the self-focussing of the beam [2], Ch.1 (see also [3] and [7]). In counterposition to the GPE (24), the NLSE (25) is exactly integrable in the inverse scattering approach [24] for the boundary condition $|\vartheta| \to 0$ at $x \to \pm\infty$. It possesses localized solutions representing 'bright' solitons while its counterpart, constructed with repulsive nonlinearity $+2|\vartheta|^2$, includes localized 'dark' pulses [6].

Some remarks are necessary. First, constraint (18) delimitates the class of real-valued functions V_0 that are useful to construct complex-valued potentials V featuring the special form (11). Usually β_R and β_I are finite in DomV_0 and go to zero as $x \to a_{1,2}$. Thus, the above approach applies specially for functions V_0 that are finite in their respective domains and vanish asymptotically. As we are going to see, the free–particle potential is an immediate example. The family of transparent potentials produced via supersymmetry [19, 27, 28] and shape invariance [19] can be useful as well. Second, the phase of the polar form (22) cannot be included in the identification (15) since it produces complex-valued functions β_I. Although $\arg(\vartheta)$ allows the propagation of ϑ along z, as it is determined by the linear derivative $i\vartheta_z$ in either (24) or (25), the relationship between $\beta = \beta_R + i\beta_I$ and ϑ is clearly valid at the stationary case ($z = 0$). Third, potentials V_0 fulfilling (18) provide a Darboux profile (16) for $\beta_I = -\frac{1}{2}\vartheta$ that can be applied in the systematic search for analytically solvable GPEs (23).

On the other hand, for the sake of completeness, we may remove β_I from (5) by using (18). The result is the nonlinear Riccati equation

$$-\beta_{Rx} + 2\beta_R^2 - \tfrac{3}{2}V_0 + 2\epsilon = 0, \tag{26}$$

which is reduced to (19) after using (15).

2.2 Optical soliton engineering

Consider the free–particle potential $V_0 = 0$, then $\text{Dom}V_0 = \mathbb{R}$. To find an expression for ϑ let us divide the nonlinear equation (19) by ϑ^4. After introducing $y = -\vartheta^{-1}$ we have

$$y_x^2 + 4\epsilon y^2 + 1 = 0. \tag{27}$$

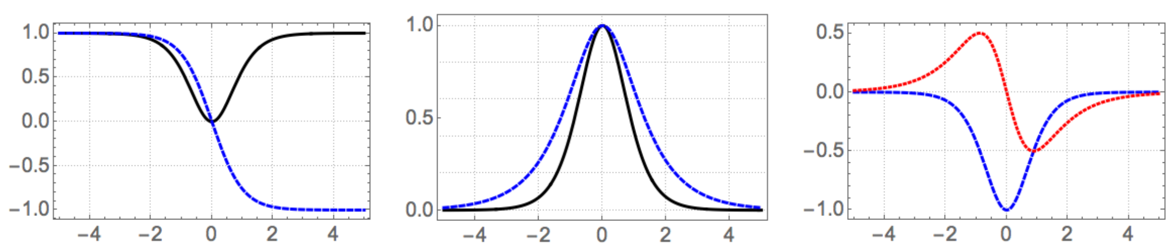
From (18) we know that this case requires $\epsilon \leq 0$. Making $k = i\frac{\kappa}{2}$ with $\kappa \geq 0$, the eigenvalue $\epsilon = -\kappa^2/4$ gives the negative coefficient $-\kappa^2$ for y^2 in (27). Then $y = \kappa^{-1}\cosh[\kappa(x + x_0)]$, with x_0 an integration constant, and

$$\vartheta(x; z) = -\frac{\kappa e^{(i\kappa^2 z + \xi_0)}}{\cosh\left[\kappa(x + x_0)\right]}, \tag{28}$$

where we have used (22). Without loss of generality we make $x_0 = \xi_0 = 0$ to reduce (28) to the conventional form of the fundamental bright soliton

$$\vartheta(x; z) = -\frac{\kappa e^{i\kappa^2 z}}{\cosh(\kappa x)}, \tag{29}$$

which does not change shape as it propagates along the z-axis. The latter because the two left–terms of (25) conspire to cancel the dependence on x, as it is expected from the balanced relationship between nonlinearity and dispersion in soliton profiles [4]. Indeed, the area $A_b = \int_{\mathbb{R}} \vartheta(x)dx = \pi$ does not depend on κ, so it is a constant of motion for the bright soliton [2], Ch.2. In Fig. 1(b) we show the behavior of $\vartheta(x; z)$ at $z = 0$.



(a) $4|\beta_R(x)|^2$ (b) $4|\beta_I(x)|^2$ (c) $V(x)$

Figure 1: (Color online) Excitations of the nonlinear Schrödinger equation (25) used to construct a complex-valued potential (7) with balanced gain and loss. (**a**) Stationary dark soliton (31) associated with (25) for the repulsive nonlinearity $+2|\vartheta|^2$. (**b**) Stationary bright soliton (30) associated to the NLSE (25). In both cases $\kappa = 1$. (**c**) Potential (33) with V_R and V_I even and odd, respectively. V_R is defined by the bright soliton intensity profile and V_I by the product of the bright and dark solitons described above. In all cases $|\vartheta|^2$, ϑ and V_I are in black-solid, blue-dashed and red-dotted lines respectively.

The imaginary part of β can be now expressed in terms of the stationary profile of the above bright soliton solution

$$2\beta_I(x) = -\left.\vartheta(x; z)\right|_{z=0} = \frac{\kappa}{\cosh(\kappa x)}. \tag{30}$$

In turn, the real part of β can be obtained from either (15), no matter the phase $e^{i\kappa^2 z}$, or the constraint (18), by avoiding such phase. The latter produces

$$2\beta_R(x) = \pm\kappa \left[1 - \frac{1}{\cosh^2(\kappa x)}\right]^{1/2} = \pm\kappa \tanh(\kappa x). \tag{31}$$

For consistency with (15) we shall preserve the minus sign. Thus, writing $2\beta_R(x) = -\theta(x)$, we immediately recognize $\theta(x) = \kappa \tanh(\kappa x)$ as the fundamental dark soliton solution of (25), where the attractive nonlinearity $-2|\vartheta|^2$ is replaced by the repulsive one $+2|\vartheta|^2$. Including the z-dependence we have

$$\theta(x; z) = \kappa e^{i\kappa^2 z} \tanh(\kappa x). \tag{32}$$

In contrast with the bright soliton (29), the area defined by $\theta(x)$ is not finite. Besides, although the area $A_d = 2\kappa\pi$ described by the 'hole' $\kappa^2 - \theta^2$ is finite, this is not a constant of motion since it depends on κ. Fig. 1(a) illustrates the 'hole' pulse described by the density profile of the dark soliton (32).

Using the stationary versions of the optical solitons (29) and (32), potential (11) becomes

$$V(x) = -\vartheta^2(x) - i\vartheta(x)\theta(x). \tag{33}$$

That is, V_R is defined by the bright soliton intensity while V_I results from the convolution of the bright and dark solitons, both cases in the stationary regime, see Fig. 1. However, to elucidate the meaning of expressions (29) and (32) in our model, let us rewrite (33) in a more convenient form

$$V(x) = -|\vartheta(x; z)|^2 - i\vartheta(x; z)\theta^*(x; z). \tag{34}$$

Notice that $V(x)$ does not depend on the propagation parameter z, despite it is explicitly included in the soliton solutions. The situation changes for the β-function since it becomes the following linear superposition of bright and dark solitons

$$\beta_0(x; z) = -\tfrac{1}{2}\left[\theta(x; z) + i\vartheta(x; z)\right] = \beta(x) e^{i\kappa^2 z}. \tag{35}$$

The constraint (18) is not affected by the z-dependence since $|\beta_0(x; z)|^2 = |\beta(x)|^2 = \frac{\kappa^2}{4}$. Then, the superposition (35) does not change shape as it propagates along the z-axis. Nevertheless, a striking expression for $V(x)$ and $\beta(x)$ is still available. Considering that only the stationary version of $\theta(x; z)$ is involved in the definition of β, while the phase of $\vartheta(x; z)$ is permitted, we would write

$$V(x) = -|\vartheta(x; z)|^2 - i\vartheta(x)\theta(x). \tag{36}$$

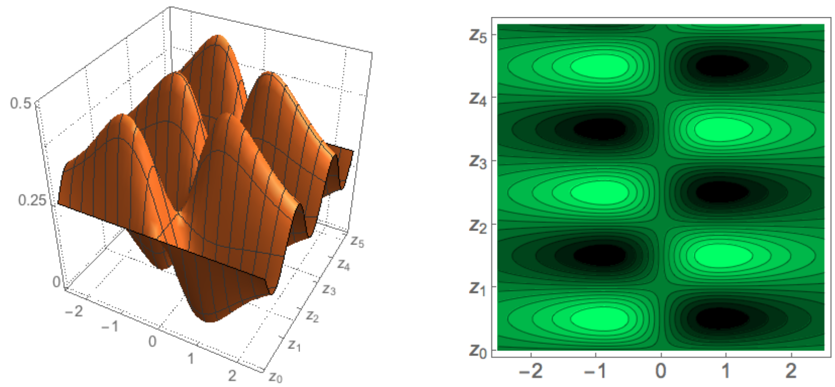
Therefore

$$\beta_1(x; z) = -\tfrac{1}{2}\left[\theta(x) + ie^{i\kappa^2 z}\vartheta(x)\right] = -\tfrac{1}{2}\left[\theta(x) - \sin(\kappa^2 z)\vartheta(x) + i\cos(\kappa^2 z)\vartheta(x)\right], \tag{37}$$

and the pulse

$$|\beta_1(x; z)|^2 = \tfrac{\kappa^2}{4} - \tfrac{1}{2}\sin(\kappa^2 z)\theta(x)\vartheta(x) \tag{38}$$

oscillates with period $\frac{2\pi}{\kappa^2}$ as it propagates along z. Clearly, constraint (18) is satisfied at $\pm z_n = \pm\left(\frac{\pi}{\kappa^2}\right)n$, with $n = 0, 1, \ldots$ In Fig. 2 we can appreciate that excitation (38) is indeed a pair 'hole–hill' that borns shyly at $z = 0$, maturates up to a robust configuration at $z = \frac{\pi}{2\kappa^2}$, and decays slowly up to its annihilation at $z_1 = \frac{\pi}{\kappa^2}$. Then the configuration twirls to provide a pair 'hill–hole', and the process initiates again to finish at $z_2 = \frac{2\pi}{\kappa^2}$. The entire cycle $z_0 \to z_2$ is repeated over and over as z grows up. The annihilation positions z_n define a flat configuration of the excitation that serves to construct potential (36).



(a) (b)

Figure 2: Pulse (38) generated by the linear superposition (37) that includes the bright soliton $\vartheta(x; z)$ and the stationary dark soliton $\theta(x)$, Eqs. (29) and (31) respectively. The excitation oscillates as z increases and constraint $|\beta_1(x; z)|^2 = \kappa^2/4$ is satisfied at the points $z_n = (\pi/\kappa^2)n$, where the pulse becomes flat. At $z = z_1/2$, the configuration involves a hole (dark soliton) in $x > 0$, and a hill (bright soliton) in $x < 0$, which acquires a new shape at $z = z_3/2$ since it includes a hill in $x > 0$, and a hole in $x < 0$. (**a**) The pulse propagates from z_0 to z_5. (**b**) Distribution of holes and hills along the z-axis.

On the other hand, from (16) we have $\lambda = \frac{\kappa}{2}$ and $\alpha^2(x) = \cosh(\kappa x)$. To verify that these results are recoverable from the Darboux expressions of Section 2 let us take $v = e^{-ikx}$ and $u = e^{ikx}$, with $w_0 = -2ik$, in (10). The simple choice $a = c = 1/2$ gives

$$\alpha(x) = [\cos(2kx) + b]^{1/2}, \qquad b^2 = 1 + \tfrac{\lambda^2}{k^2}. \tag{39}$$

We have already taken $k = i\frac{\kappa}{2}$, so that $\alpha(x) = [\cosh(\kappa x) + b]^{1/2}$ is reduced to the expression we are looking for when $b^2 = 1 - (\frac{2\lambda}{\kappa})^2$ is cancelled. Thus $\lambda = \frac{\kappa}{2}$, as expected. Indeed, potential (33) has been already reported in the context of the Darboux transformations [10]. There, it is shown that only the real energy $\epsilon = -\frac{\kappa^2}{4}$ permits a normalizable solution of the Schrödinger equation (1). Such eigenfunction is of the form

$$\psi_\epsilon(x) = \frac{\vartheta(x)}{\sqrt{\kappa\pi}}\left[\cosh\left(\frac{\kappa x}{2}\right) + i\sinh\left(\frac{\kappa x}{2}\right)\right], \tag{40}$$

and satisfies $|\psi_\epsilon|^2 = \frac{1}{\pi}\vartheta$. That is, the density of the ground state (40) has the bright soliton profile, see Fig. 3.

Potential (36) may be classified in the Scarf I-hyperbolic type (in notation of [29], Table 1, use $\alpha = \pm\frac{3}{2}$, $\beta = \pm\frac{1}{2}$). This is a family member of PT-invariant potentials

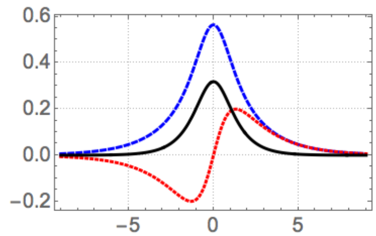


Figure 3: (Color online) Real (blue-dashed) and imaginary (red-dotted) parts of the single bound state (40) associated with the potential shown in Fig. 1(c). Up to the factor $1/\pi$, the corresponding pulse (black-solid) has the bright soliton profile shown in Fig. 1(b).

studied in [30] to show that non-Hermitian Hamiltonians have both real and complex discrete spectrum, and fully analytically. The model includes different global factors for V_R and V_I and investigates whether the eigenvalues are real or complex in terms of such parameters. It is conjectured that "when the real part of the PT-invariant potential is stronger than its imaginary part, the eigenspectrum will be real, and they will be mixed (real and complex) otherwise" [30]. As our model considers the same global factor for V_R and V_I, namely κ^2, the above conjecture is automatically verified (see Fig. 4), so that no complex eigenvalues are expected. Interestingly, potential (36) has been implemented, with the global factors modified as in [30], as the external field in the GPE [31]. When V_I is weighted by a factor $1/2$, it is found an exact solution for $\epsilon = 0.98$ which acquires the analytic form given in Eq. (40). Besides, the existence and stability of solitons in these potentials, with self-focusing and self-defocusing nonlinear cases, has been recently investigated in e.g. [32–34]. The above results open the possibility of scaling our model to the more general case where the global factors of V_R and V_I are different.

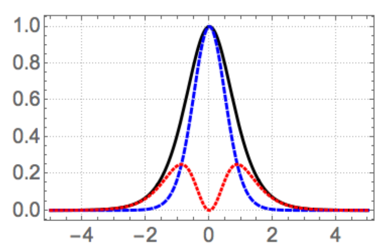


Figure 4: (Color online) Total intensity (black-solid) of potential (36), see also Fig. 1(c). The real part contribution (blue-dashed) is stronger than the imaginary one (red-dotted).

3 Conclusion

In conclusion, we have demonstrated how the superpositions of nonlinear localized modes lead to complex-valued potentials with real energy spectrum and balanced gain–loss profile. In particular, we found a potential that is defined by the intensity of the fundamental bright optical soliton in its real part, and by the convolution of this soliton with the fundamental dark mode in the imaginary branch. Although the analytic expression of this potential has been already studied in different approaches, as far as we know, previous to the present work, there is no information about the origin of such interaction. Indeed, we have shown that the superposition leading to the optical potential defines also a 'breathing' pulse with striking properties. The pulse is composited by a hill (bright soliton intensity) and a hole (dark soliton intensity) that propagate while they interchange roles: the hole becomes a hill and vice versa. The entire process starts with a flat signal that grows up shyly, maturates up to a robust configuration, and decays slowly up to its annihilation. In a second part of the evolution, the hole and hill interchange roles and the signal grows up and then decays again to finish in flat configuration. The definition of the optical potential occurs when the superposition is in flat configuration.

The model can be scaled in different directions. For instance, fundamental solitons may be replaced by excited modes in the definition of β, so it becomes a superposition of excited localized modes of the cubic nonlinear Schrödinger equation. Remarkably, the difficulty of using excited physical energies in the Darboux transformation is not present in the construction of complex-valued potentials since the conventional oscillation theorems do not operate in such a case [12]. Then, it is expected the same situation for the excited soliton modes. Another option trends towards the Gross–Pitaevskii equation where the external potential is not trivially zero. Namely, to satisfy the constraint (18) that delimits the class of external potentials V_0 that are useful in our model, periodic potentials might be investigated. The same holds for the family of transparent potentials that either vanish or become finite asymptotically. In any case, the complex-valued potential $V = V_0 + 2\beta_x$ will be expressed as $V = u^2 + iu_x$, with u a localized mode of either the Gross–Pitaevskii equation or the cubic nonlinear Schrödinger equation.

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