$SU(2)$, Associated Laguerre Polynomials and Rigged Hilbert Spaces

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Abstract We present a family of unitary irreducible representations of $SU(2)$ realized in the plane, in terms of the Laguerre polynomials. These functions are similar to the spherical harmonics defined on the sphere. Relations with an space of square integrable functions defined on the plane, $L^2(\mathbb{R}^2)$, are analyzed. We have also enlarged this study using rigged Hilbert spaces that allow to work with iscrete and continuous bases like is the case here.

1 Introduction

The representations of a Lie algebra are usually considered as ancillary to the algebra and developed starting from the algebra, i.e. from the generators and their commutation relations. The universal enveloping algebra (UEA) is constructed and a complete set of commuting observables selected, choosing between the invariant operators of the algebra and of a chain of its subalgebras. The common eigenvectors of this complete set of operators are a basis of a vector space where the Lie algebra generators are realized as operators.

We propose here an alternative construction that allows to add to the representations obtained following the reported recipe, new ones not achievable following the previous approach. Starting from a concrete vector space

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of functions with discrete labels and continuous variables, we consider the recurrence relations that allow to connect functions with different values of the labels. These recurrence relations are not operators but allow us to introduce, for each label and for each continuous variable, an operator that reads its value. In this way, recurrence relations are rewritten in terms of rising and lowering operators built by means of the above defined operators.These rising and lowering operators are often genuine generators of the Lie algebras considered by Miller [1] and the procedure gives simply the representations of the algebras in a well defined function space [2, 3]. However it can happen that the commutators, besides the values required by the algebra, have additional contributions. The essential point of this paper is that these additional contributions (as exhibited here) can be proportional to the null identity that defines the starting vector space. As this identity is zero on the whole representation, the Lie algebra is well defined and a new representation in a space of functions has been found.

We do not discuss here the general approach, but we limit ourselves to a simple example where all aspects are better understandable. We start thus from the associated Laguerre functions (ALF) and, following the proposed construction, we realize the algebra $su(2)$ in terms of the appropriate rising and lowering operators. The ALF support in reality a larger algebra [4] but we prefer to consider here only the subalgebra $su(2)$. The reasons for this choice are twofold: first in this way the technicalities are reduced at the minimum and second it has been very nice for us to discover that not all representations of a so elementary group like $SU(2)$ where known.

As discussed in [5, 6, 7] the presence of operators with spectrum of different cardinality implies that, as considered for the first time in Lie algebras in [8], the space of the group representation is not a Hilbert space but a rigged Hilbert space (RHS) [9]. Thus, we introduce the above setting within the context of RHS since the RHS is the perfect framework where discrete and continuous bases coexist. In addition, the same RHS serves as a support for a representation on it of a Lie algebra as continuous operators as well as for its UEA. Therefore, the connection between discrete and continuous bases and Lie algebras with RHS is well established.

2 Associated Laguerre polynomials

The ALP [10], $L_n^{(\alpha)}(x)$, depend from a real continuous variable $x \in [0,\infty)$ and from two other real labels $(n,\alpha) : n = 0,1,2,\ldots$ and α (usually assumed as a fixed parameter) continuous and > -1 . They reduce to the Laguerre polynomials for $\alpha = 0$ and are defined by the second order differential equation

$$\left[x\frac{d^2}{dx^2} + (1+\alpha-x)\frac{d}{dx} + n\right] L_n^{(\alpha)}(x) = 0 \, . \tag{1}$$

From the many recurrence relations that can be found in literature [10, 11], we consider the following ones, all first order differential recurrence relations:

$$
\begin{aligned}
&\left[x\frac{d}{dx}+(n+1+a-x)\right] L_n^{(\alpha)}(x)=(n+1)L_{n+1}^{(\alpha)}(x)\,,\\
&\left[-x\frac{d}{dx}+n\right] L_n^{(\alpha)}(x)=(n+\alpha)L_{n-1}^{(\alpha)}(x)\,,\\
&\left[-\frac{d}{dx}+1\right] L_n^{(\alpha)}(x)=L_n^{(\alpha+1)}(x)\,,\\
&\left[x\frac{d}{dx}+\alpha\right] L_n^{(\alpha)}(x)=(n+\alpha)L_n^{(\alpha-1)}(x)\,.
\end{aligned}
\tag{2}
$$

Starting from $L_n^{(\alpha)}(x)$, by means of repeated applications of eqs. (2), $L_{n+k}^{(\alpha+h)}(x)$ –with h and k arbitrary integers– can be obtained through a differential relation of higher order. But, by means of eq. (1), every differential relation of order two or higher can be rewritten as a differential relation of order one. In particular we can obtain

$$
\begin{aligned}
&\left[\frac{d}{dx}+\frac{n}{\alpha+1}\right] L_n^{(\alpha)}(x) \;=\; -\,\frac{\alpha}{\alpha+1}\,L_{n-1}^{(\alpha+2)}(x),\\
&\left[x(\alpha-1)\frac{d}{dx}-x\left(n+3\,\frac{\alpha}{2}\right)+\alpha(\alpha-1)\right] L_n^{(\alpha)}(x)\\
&\qquad\qquad\qquad = (j+\alpha)(\alpha+1)\;L_{n+1}^{(\alpha-2)}(x),
\end{aligned}
\tag{3}
$$

that are the recurrence relations we employ in this paper.

The ALP $L_n^{(\alpha)}(x)$ are –for $\alpha>-1$ and fixed– orthogonal in n with respect the weight measure $d\mu(x)=x^{\alpha}\,e^{-x}\,dx$ [10]:

$$
\begin{aligned}
&\int_0^{\infty} dx\; x^{\alpha}\;e^{-x}\,L_n^{(\alpha)}(x)\;L_{n'}^{(\alpha)}(x) \;=\; \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{nn'}\;,\\
&\sum_{n=0}^{\infty} x^{\alpha}\,e^{-x}\,L_n^{(\alpha)}(x)L_n^{(\alpha)}(x')=\delta(x-x')\,.
\end{aligned}
\tag{4}
$$

The parameter α can be extended to arbitrary complex values [10] and, in particular, for α integer and such that $0\le|\alpha|\le n$, we have the relation

$$
L_n^{(-\alpha)}(x)=(-x)^{\alpha}\,\frac{(n-\alpha)!}{n!}\,L_{n-\alpha}^{(\alpha)}(x)\,. \tag{5}
$$

Here we assume consistently that $n\in\mathbb{N}$, $\alpha\in\mathbb{Z}$ and $n-\alpha\in\mathbb{N}$, and we also consider α as a label, like n, and not a parameter fixed at the beginning. Following the approach of [2], we introduce now a set of alternative variables and include the weight measure inside the functions, in such a way to obtain the bases we are used in quantum mechanics. We define indeed $j:=n+\alpha/2$

and $m := -\alpha/2$ that are such that $j \in \mathbb{N}/2\,,\ j-m \in \mathbb{N}$ and $|m| \leq j$. Note that they look like the parameters j and m used in $SU(2)$. Now we write

$$\mathcal{L}_j^m(x) := \sqrt{\frac{(j+m)!}{(j-m)!}}\; x^{-m}\, e^{-x/2}\, L_{j+m}^{(-2m)}(x)$$

so that, from eq. (5), $\mathcal{L}_j^m(x)$ is symmetric/antisymmetric in the exchange $m \leftrightarrow -m$ since $\mathcal{L}_j^m(x) = (-1)^{2j}\,\mathcal{L}_j^{-m}(x)$. From eqs. (4), we see that the $\mathcal{L}_j^m(x)$ verify, for m fixed, the following orthonormality and completeness relations

$$\int_0^\infty \mathcal{L}_j^m(x)\ \mathcal{L}_{j'}^m(x)\, dx \ =\ \delta_{jj'}\,, \qquad \sum_{j=|m|}^{\infty} \mathcal{L}_j^m(x)\ \mathcal{L}_j^m(x') \ =\ \delta(x-x')\,, \quad (6)$$

and are thus, for any fixed value of m, an orthonormal basis of $L^2(\mathbb{R}^+)$.

Note that, in the algebraic description of the spherical harmonics, the functions $T_j^m(x) = \sqrt{\frac{(j-m)!}{(j+m)!}}\, P_j^m(x)$, related to the associated Legendre functions $P_l^m(x)$ and introduced in [2], satisfy $T_j^m(x) = (-1)^m\, T_j^{-m}(x)$ which is a relation similar to those verified by the $\mathcal{L}_j^m(x)$. Moreover the $T_j^m(x)$, like the $\mathcal{L}_j^m(x)$ on the half-line, are orthogonal –for fixed m– in the interval $(-1,+1) \subset \mathbb{R}$ and a basis for $L^2[-1,1]$.

3 $SU(2)$ representations in the plane

Following now Ref. [2], we define four operators X, D_x, J and M such that

$$\begin{aligned} X\,\mathcal{L}_j^m(x) \ &=\ x\,\mathcal{L}_j^m(x), \qquad & D_x\,\mathcal{L}_j^m(x) \ &=\ \mathcal{L}_j^m(x)'\,, \\ J\,\mathcal{L}_j^m(x) \ &=\ j\,\mathcal{L}_j^m(x), \qquad & M\,\mathcal{L}_j^m(x) \ &=\ m\,\mathcal{L}_j^m(x). \end{aligned} \quad (7)$$

and we can rewrite eq. (1) in terms of the $\mathcal{L}_j^m(x)$ and in operatorial form as

$$E\,\mathcal{L}_j^m(x) \ \equiv\ \left[X\,D_x^2 + D_x - \frac{1}{X}M^2 - \frac{X}{4} + J + \frac{1}{2}\right]\,\mathcal{L}_j^m(x) = 0\,. \quad (8)$$

Thus, the identity $E \equiv 0$ defines $L^2(\mathbb{R}^+)$.

The relations (3) can now be rewritten on terms of the $\mathcal{L}_j^m(x)$ as

$$\begin{aligned} K_+\,\mathcal{L}_j^m(x) &= \sqrt{(j-m)(j+m+1)}\ \ \mathcal{L}_j^{m+1}(x)\,, \\ K_-\,\mathcal{L}_j^m(x) &= \sqrt{(j+m)(j-m+1)}\ \ \mathcal{L}_j^{m-1}(x)\,, \end{aligned} \quad (9)$$

where

$$
\begin{aligned}
K_+ &= -2D_x\left(M+\frac{1}{2}\right)+\frac{2}{X}M\left(M+\frac{1}{2}\right)-\left(J+\frac{1}{2}\right)\,,\\
K_- &= 2D_x\left(M-\frac{1}{2}\right)+\frac{2}{X}M\left(M-\frac{1}{2}\right)-\left(J+\frac{1}{2}\right)\,.
\end{aligned}
\tag{10}
$$

Since, from eqs. (9), we have $[K_+,K_-]\ \mathcal{L}_j^m(x) \;=\; 2\,m\ \mathcal{L}_j^m(x)$ and assuming $K_3 := M$ (i.e. $K_3\ \mathcal{L}_j^m(x) = m\ \mathcal{L}_j^m(x)$) we get the relations

$$
[K_+,K_-]\ \mathcal{L}_j^m(x) \;=\; 2\,K_3\ \mathcal{L}_j^m(x),\quad [K_3,K_\pm]\,\mathcal{L}_j^m(x) \;=\; \pm\,K_\pm\,\mathcal{L}_j^m(x)\,, \tag{11}
$$

that display the fact that, for fixed j, under the action of $K_\pm$ and K_3, the $\mathcal{L}_j^m(x)$ supports the irreducible representation of dimension $2j+1$ of $su(2)$.

However, while as exhibited by (6) the space $\{\mathcal{L}_j^m(x)\}$ has an inner product for m fixed and $j\geq|m|$ (thus supporting a set of UIR of $SU(1,1)$ [4]), the representation (11) of $SU(2)$ is not faithful, since $\mathcal{L}_j^m(x) = (-1)^{2j}\,\mathcal{L}_j^{-m}(x)$, and not unitary. The definition of a scalar product is indeed one of the problems we have in the connection of hypergeometric functions and Lie algebras. Hence, we have two problems: the $\mathcal{L}_j^m(x)$ are not orthonormal for j fixed and functions with opposite m are not independent (as it happens also with the $P_j^m(x)$). Following the same approach of the spherical harmonics to construct the inner product space for j fixed and $|m|\leq j$ we, thus, introduce a new real variable ϕ ($-\pi<\phi\leq\pi$) and the new objects

$$
\mathcal{Z}_j^m(r,\phi) := e^{i\,m\,\phi}\,\mathcal{L}_j^m(r^2),
$$

that verify $\mathcal{Z}_j^m(r,\phi+2\pi) = (-1)^{2j}\,\mathcal{Z}_j^m(r,\phi)$. Under the change of variable $x\to r^2$ equation (8) becomes for $\mathcal{Z}_j^m(r,0)$

$$
\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4m^2}{r}-r^2+4(j+\frac{1}{2})\right]\ \mathcal{Z}_j^m(r,0)=0. \tag{12}
$$

The functions $\mathcal{Z}_j^m(r,\phi)$ are the analogous on the plane of the spherical harmonics $Y_{lm}(\theta,\phi)$ on the sphere. The orthonormality and completeness of the $\mathcal{Z}_j^m(r,\phi)$ is similar to that of $Y_j^m(\theta,\phi)$

$$
\begin{aligned}
&\frac{1}{\pi}\int_{-\pi}^{\pi}d\phi\int_0^\infty\ r\,dr\ \ \mathcal{Z}_j^m(r,\phi)^*\ \mathcal{Z}_{j'}^{m'}(r,\phi) \;=\; \delta_{j,j'}\ \ \delta_{m,m'}\ ,\\
&\sum_{j,m}\ \mathcal{Z}_j^m(r,\phi)^*\ \mathcal{Z}_j^m(r',\phi') \;=\; \frac{\pi}{r}\ \delta(r-r')\,\delta(\phi-\phi')\,.
\end{aligned}
\tag{13}
$$

This means that $\{\mathcal{Z}_j^m(r,\phi)\}$ is a basis of the Hilbert space $L^2(\mathbb{R}^2)$ with measure $d\mu(r,\phi) = r\,dr\,d\phi/\pi$ like $\{Y_j^m(\theta,\phi)\}$ is a basis of $L^2(S^2)$ with $d\Omega$.

Now we consider an abstract Hilbert space $\mathcal{H}$ supporting the $2j+1$ dimensional IR of $su(2)$ spanned by the eigenvectors of J and M (see eq. (7))

$$J\,|j,m\rangle = j\,|j,m\rangle\,,\quad M\,|j,m\rangle = m\,|j,m\rangle\,,\qquad 2j\in\mathbb{N},\ \ |m|\leq j\,.$$

These vectors $|j,m\rangle$ constitute a basis of $\mathcal{H}$ verifying the properties of orthogonality and completeness

$$\langle j,m|j',m'\rangle = \delta_{j,j'}\,\delta_{m,m'}\,,\qquad \sum_{j=0}^{\infty}\sum_{m=-j}^{j}|j,m\rangle\langle j,m| = I$$

Any $|f\rangle\in\mathcal{H}$ may be written as $|f\rangle = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,|j,m\rangle$ if and only if

$$\sum_{j=0}^{\infty}\sum_{m=-j}^{j}|f_{j,m}|^2<\infty\,,\qquad f_{l,m}=\langle l,m|f\rangle\,. \tag{14}$$

A canonical injection $S:\mathcal{H}\to L^2(\mathbb{R}^2)$ can be defined by $|j,m\rangle\to\mathcal{Z}_j^m(r,\phi)$ and extended by linearity and continuity to the whole $\mathcal{H}$. One can easily check that S is unitary. For any $|f\rangle\in\mathcal{H}$ we have the following expression

$$S|f\rangle=\sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,S\,|j,m\rangle=\sum_{j=0}^{\infty}\sum_{m=-j}^{j} f_{j,m}\,\mathcal{Z}_j^m(r,\phi)\,.$$

We now introduce a continuous basis, $\{|r,\phi\rangle\}$, depending on the values of the variables r and ϕ with the help of the discrete basis $\{|j,m\rangle\}$ by

$$\langle r,\phi|j,m\rangle := \mathcal{Z}_j^m(r,\phi)\,. \tag{15}$$

In reality, because of the different cardinality of r and j, we are dealing with a RHS (see next Section). The $\mathcal{Z}_j^m(r,\phi)$ can be seen as the transformation matrices from the irreducible representation states $\{|j,m\rangle\}$ to the localized states in the plane $\{|r,\phi\rangle\}$, like $Y_j^m(\theta,\phi)=\langle j,m|\theta,\phi\rangle$ are the corresponding ones to the localized states $\{|\theta,\phi\rangle\}$ in the sphere [7, 12]. Indeed

$$|j,m\rangle=\frac{1}{\pi}\int_{\mathbb{R}^2}|r,\phi\rangle\mathcal{Z}_j^m(r,\phi)rdrd\phi,\quad |j,m\rangle=\int_{S^2}|\theta,\phi\rangle\sqrt{j+1/2}Y_j^m(\theta,\phi)d\Omega.$$

We continue with the analogy and, from $K_\pm$ and K_3 (10), we define

$$J_\pm\,:=\,e^{\pm i\phi}\,K_\pm,\qquad J_3\,:=\,K_3, \tag{16}$$

with act on the $\mathcal{Z}_j^m(r,\phi)$ as

$$\begin{aligned} J_+\ \mathcal{Z}_j^m(r,\phi) &= \sqrt{(j-m)(j+m+1)}\ \mathcal{Z}_j^{m+1}(r,\phi),\\ J_-\ \mathcal{Z}_j^m(r,\phi) &= \sqrt{(j+m)(j-m+1)}\ \mathcal{Z}_j^{m-1}(r,\phi),\\ J_3\ \mathcal{Z}_j^m(r,\phi)\ &= m\ \mathcal{Z}_j^m(r,\phi)\,. \end{aligned} \tag{17}$$

The functions $\mathcal{Z}_j^m(r,\phi)$ with j fixed and $|m|\leq j$, are orthonormal and determine the representation of dimension $2j+1$ of $su(2)$ as it happens for the $Y_j^m(\theta,\phi)$. However there is a essential difference between the operators $\{J_\pm, J_3\}$ that act on the sphere $\mathbb{S}^2$ that are true generators of $su(2)$ and the $\{J_\pm, J_3\}$ of (16), defined in $\mathbb{R}^2$, that do not close a Lie algebra. Indeed, when we calculate the commutator $[J_+, J_-]$ in terms of the differential operators defined in the eqs. (10) and (16), we obtain $[J_+,J_-] \;=\; 2\,J_3 + \dfrac{8}{R^2}\,J_3\,E$, and only when $E\equiv 0$, i.e. only in the unitary space $L^2(\mathbb{R}^2)$, the $su(2)$ algebra is recovered. On the other hand, E is related to the $su(2)$ Casimir $\mathcal{C}$

$$E \;=\; -\frac{R^2}{4J_3^2+1}\,[\mathcal{C}-J(J+1)] \;\equiv\; -\frac{R^2}{4J_3^2+1}\left[J_3^2+\frac{1}{2}\,\{J_+,J_-\}\; - J(J+1)\right],$$

so equation $E=0$ is equivalent to the $su(2)$ Casimir condition $\mathcal{C}-J(J+1)=0$, that entails the usual Lie algebra in each $su(2)$ representation space.

4 Rigged Hilbert space formulation.

A RHS (or Gelf'and triplet) is a triplet of spaces $\varPhi\subset\mathcal{H}\subset\varPhi^\times$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space, $\varPhi$ is a dense subspace of $\mathcal{H}$ endowed with its own topology, and $\varPhi^\times$ is the dual (or the antidual) space of $\varPhi$ [9, 13, 14]. The topology considered on $\varPhi$ is finer (contains more open sets) than the topology that $\varPhi$ has as subspace of $\mathcal{H}$, and $\varPhi^\times$ is equipped with a topology compatible with the dual pair $(\varPhi,\varPhi^\times)$ [15], usually the weak topology. The topology of $\varPhi$ [16, 17] allows that all sequences which converge on $\varPhi$, also converge on $\mathcal{H}$ but the converse is not true. The difference between topologies gives rise that $\varPhi^\times$ is bigger than $\mathcal{H}$, which is self-dual.

Here, any $F\in\varPhi^\times$ is a continuous linear mapping from $\varPhi$ into $\mathbb{C}$.

An essential property is that if A is a densely defined operator on $\mathcal{H}$, such that $\varPhi$ be a subspace of its domain and that $A\varphi\in\varPhi$ for all $\varphi\in\varPhi$, we say that $\varPhi$ reduces A or that $\varPhi$ is invariant under the action of A, (i.e., $A\varPhi\subset\varPhi$). Then A may be extended unambiguously to $\varPhi^\times$ by the duality formula

$$\langle A^\times F|\varphi\rangle := \langle F|A\varphi\rangle\,,\qquad \forall\,\varphi\in\varPhi\,,\;\; \forall\,F\in\varPhi^\times\,. \tag{18}$$

Moreover if A is continuous on $\varPhi$, then $A^\times$ is continuous on $\varPhi^\times$.

The topology on $\varPhi$ is given by an infinite countable set of norms $\{||-||_{n=1}^\infty\}$. A linear operator A on $\varPhi$ is continuous if and only if for each norm $||-||_n$ there is a $K_n>0$ and a finite sequence of norms $||-||_{p_1}, ||-||_{p_2},\ldots,||-||_{p_r}$ such that for any $\varphi\in\varPhi$, one has [18]

$$||A\varphi||_n \leq K_n\;(||\varphi||_{p_1}+||\varphi||_{p_2}+\cdots+||\varphi||_{p_r})\;. \tag{19}$$

Now let us go to define and use the RHS $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ where discrete and continuous bases coexist and the meaningful operators are well defined and continuous. Since we have a representation in terms of the $\mathcal{Z}_j^m(r,\phi)$, it would be more convenient to start with an equivalent RHS $\mathfrak{D} \subset L^2(\mathbb{R}^2) \subset \mathfrak{D}^\times$, such as $\mathfrak{D}$ is a test functions space with $f(r,\phi) \in L^2(\mathbb{R}^2)$, which therefore admit the span

$$f(r,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m}\, \mathcal{Z}_j^m(r,\phi)\,, \tag{20}$$

where the series converges in the sense of the norm in $L^2(\mathbb{R}^2)$. A necessary and sufficient condition for it is $\sum_{j=0}^{\infty} \sum_{m=-j}^{j} |f_{j,m}|^2 < \infty$. Thus, from (20), we define $\mathfrak{D}$ as the space of functions $f(r,\phi)$ in $L^2(\mathbb{R}^2)$ such that

$$||f(r,\phi)||_n^2 := \sum_{j=0}^{\infty} \sum_{m=-j}^{j} (j+|m|+1)^{2n}\, |f_{j,m}|^2 < \infty\,, \qquad n = 0,1,2,\dots\,. \tag{21}$$

Obviously, all the finite linear combinations of the $\mathcal{Z}_j^m(r,\phi)$ are in $\mathfrak{D}$, hence $\mathfrak{D}$ is dense in $L^2(\mathbb{R}^2)$. Thus, the family of norms $||-||_n$ on $\mathfrak{D}$ (21) gives a topology such that $\mathfrak{D}$ is a Frèchet space (metrizable and complete). Since for $n = 0$ we have the Hilbert space norm, the canonical injection from $\mathfrak{D}$ into $L^2(\mathbb{R}^2)$ is continuous.

Because j goes from 0 to ∞, the operators $J_\pm, J_3$ are all unbounded and, therefore, their respective domains are densely defined on $L^2(\mathbb{R}^2)$, but not on the whole $L^2(\mathbb{R}^2)$. We can prove that all these operators are defined on the whole $\mathfrak{D}$ and are continuous with the topology on $\mathfrak{D}$. The proof is simple and it is essentially the same for all operators. As an example, let us give the proof for J_+. For any function f in $\mathfrak{D}$, we have $J_+ f$, i.e.,

$$J_+ \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m}\, \mathcal{Z}_j^m(r,\phi) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} f_{j,m}\, \sqrt{(j-m)(j+m+1)}\, \mathcal{Z}_j^{m+1}(r,\phi)\,.$$

To show that $J_+ f \in \mathfrak{D}$ we have to prove that for any $n \in \mathbb{N}$, it satisfies (21). So taking into account the shift on the index m (17) we have

$$\sum_{j=0}^{\infty} \sum_{m=-j}^{j} |f_{j,m}|^2\, (j-m)(j+m+1)\, (j+1+|m|+1)^{2n}\,. \tag{22}$$

The following two inequalities are straightforward:

$$(j-m)(j+m+1) \le (j+|m|+1)^2\,, \qquad (j+1+|m|+1)^{2n} \le 2^{2n}\, (j+|m|+1)^{2n}\,.$$

Using these inequalities we see that (22) is bounded by

$$2^{2n} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} |f_{j,m}|^2 \, (j+1+|m|+1)^{2n+2} \,, \tag{23}$$

which converges after (21). Hence, $J_+ f \in \mathfrak{D}$. In order to show the continuity of J_+ on $\mathfrak{D}$, we use (19). Thus, applying J_+ to any $f(r,\phi) \in \mathfrak{D}$ we get

$$||J_+ f(r,\phi)||_n^2 \leq 2^{2n} \, ||f(r,\phi)||_{n+1}^2 \Longrightarrow ||J_+ f(r,\phi)||_n \leq 2^n \, ||f(r,\phi)||_{n+1} \,,$$

which satisfies (19) for all $n = 0, 1, 2, \ldots$. Hence, the continuity of J_+ on $\mathfrak{D}$ has been proved. By means of the duality formula, we extend J_+ to a weakly continuous operator on $\mathfrak{D}^\times$. Same properties can be proved for J_- and J_3.

Now we are able to define the abstract RHS $\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ using the unitary mapping $S : \mathcal{H} \to L^2(\mathbb{R}^2)$ introduced in the previous section. Thus, we define $\mathfrak{G} := S^{-1}\mathfrak{D}$. Hence the topology on $\mathfrak{G}$ is the transported topology from $\mathfrak{D}$ by S, so that if $f \in \mathfrak{G}$, the semi-norms are

$$||f||_n^2 = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} (j+|m|+1)^{2n} \, |f_{j,m}|^2 < \infty \,, \qquad n = 0, 1, 2, \ldots \,.$$

The topology on $\mathfrak{G}$ uniquely defines $\mathfrak{G}^\times$. Moreover there exists a one-to-one continuous mapping from $\mathfrak{G}$ onto $\mathfrak{D}$ with continuous inverse. It is given by an extension, $\widetilde{S}$, of S defined via the duality formula $\langle \widetilde{S} f | \widetilde{S} F \rangle = \langle f | F \rangle$, with $f \in \mathfrak{G}$ and $F \in \mathfrak{G}^\times$.

On the other hand, if an operator O satisfies $O\mathfrak{D} \subset \mathfrak{D}$ with continuity, the same property works for $\widehat{O} = S^{-1} O S$ on $\mathfrak{G}$.

5 Conclusions

Starting from the recurrence relations (3) we obtained the operators $\{J_\pm, J_3\}$ (16). Their general linear algebra is not a Lie algebra. However its representation on $L^2(\mathbb{R}^2)$, characterized by the eigenvalue zero of the operator E, is isomorphic to the regular representation $\{|j,m\rangle\}$ of $su(2)$ and it has therefore a stronger symmetry than the general linear operator structure itself.

We are used in Lie algebra theory to representations that preserve the symmetry of the algebra and to algebras that have the same symmetry of the space where the representation is defined. This is exactly what happens with the spherical harmonics, that are solution of Laplace equation and, thus, have the same intrinsic symmetry of the group $SU(2)$ of which they are representation bases. However, here the situation is different since we represent $SU(2)$ in the plane $\mathbb{R}^2$ which geometry preserves only the subgroup $SO(2)$ of $SU(2)$. Indeed $\{J_\pm, J_3\}$ (16) are defined for arbitrary E, but they generate $su(2)$ only under the assumption $E \equiv 0$, i.e. when we restrict ourselves to functions f verifying the Casimir condition $\mathcal{C}\, f = J(J+1)\, f$,, i.e. that belong to $L^2(\mathbb{R}^2)$.

Reversing the connection, the representations of a Lie algebra have been related not only to the Lie algebra itself but also to a set of operators that do not close a Lie algebra in an universal way but reduce to a Lie algebra only when applied to well defined vector spaces.

This paper offers a method to introduce representations of Lie groups in spaces that are not symmetric under the group action and in situations where the general linear group of operators is not a Lie group in a universal way.

We have also constructed two RHS ($\mathfrak{G} \subset \mathcal{H} \subset \mathfrak{G}^\times$ and $\mathfrak{D} \subset L^2(\mathbb{R}^2) \subset \mathfrak{D}^\times$) supporting two UIR of $SU(2)$, the first one is related with the discrete basis $\{|j,m\rangle\}$ and the other RHS with the continuous one $\{|r,\phi\}$. Both are related by the unitary map $S : |j,m\rangle \to \mathcal{Z}_j^m(r,\phi)$ that also transports the topologies of the first RHS and other properties to the second RHS.

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