A note on Beuker's and related double integrals.

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ABSTRACT

An elementary transformation formula is derived allowing double integrals of the type introduced by F. Beukers to be reduced and new ones to be constructed.

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1 Introduction

In 1979 F. Beukers[1] introduced, as an aid to studying the irrationality of certain mathematical constants, including Apery's constant $\zeta(3)$, introduced a class of double integral representations over the unit square typified by

$$\zeta(2) = \int_0^1 \int_0^1 \frac{dxdy}{1-xy} \tag{1}$$

which the reader will have no difficulty verifying by expanding the denominator as a geometric series and integrating term-wise. Since that time this class of representation has been studied intensively and extendend to nearly all the transcendents important to number theory: Euler's constant, polylogarithms, etc. (See .[2,3,4,5,6] where further references are given.) For example[6]

$$\int_0^1 \int_0^1 \frac{\ln(1+xy)}{1-xy} dxdy = \frac{\pi^2}{4} \ln 2 - \zeta(3) \tag{2}$$

$$\int_0^1 \int_0^1 \frac{\ln(1-xyz)}{1-xy} dxdy = \frac{\pi^2}{6} \ln(1-z) - \sum_{n=1}^{\infty} \frac{H_{n,2}}{n} z^n \tag{3}$$

where $H_{n,m}$ is a Harmonic number.

The aim of this note is to prove the elementary reduction formula:

Theorem

If $f : \mathcal{R} \to \mathcal{R}$ is integrable over the unit interval, then

$$\int_0^1 \int_0^1 f(xy) dxdy = -\int_0^1 \ln(x) f(x) dx. \tag{4}$$

Equation (4) can serve to dispel any mystery one might feel surrounding formulas such as (1); since (1) becomes the familiar representation

$$\zeta(2) = -\int_0^1 \frac{\ln x}{1-x} dx, \tag{5}$$

it provides an easy route to constructing more such representations and might even provide a few new definite integrals.

We also have the obvious:

Corollary

Under the same assumption

$$\int_0^1 \int_0^1 \frac{f(xy)}{\ln(xy)} dxdy = -\int_0^1 f(x) dx. \tag{6}$$

2 Calculation

By symmetry, the left side of (4) is

$$I = 2\int_R f(xy)dxdy \tag{7}$$

where R is the right triangle $(0,0),(1,0),(1,1)$, Under the transformation

$$x = \frac{1}{2}[\sqrt{t^2+4u}+t], \quad y = \frac{1}{2}\sqrt{t^2+4u}-t] \tag{8a}$$

$$u = xy \quad t = x - y \tag{8b}$$

having Jacobian $J = -(t^2+4u)^{-1/2}$, in the $u-t$-plane R becomes the right triangle $(0,0),(0,1),(1,0)$ yielding

$$I = 2\int_0^1 duf(u)\int_0^{1-u}\frac{dt}{\sqrt{t^2+4u}} = 2\int_0^1 duf(u)\sinh^{-1}\left(\frac{1-u}{2\sqrt{u}}\right). \tag{9}$$

Next, let $u = \mathrm{sech}^2 y$, so $(1-u)/2\sqrt{u} = (\cosh t - \mathrm{sech}\, t)/2$ and with $\cosh y = e^x$, $\sinh^{-1}[(1-u)/2\sqrt{u}] = x$. Therefore, with $s = e^{-2x}$

$$I = 4\int_0^\infty dx\, xe^{-2x}f(e^{-2x}) = -\int_0^1 \ln(s)f(s)ds. \tag{10}$$

This completes the proof of (4).

3 Discussion

Let us apply this to the Guillera-Sondow[6] formula (2). Equation (4) immediately gives

$$\int_0^1 \frac{\ln(x)\ln(1+x)}{1-x}dx = \zeta(3) - \frac{\pi^2}{4}\ln 2. \tag{11}$$

Mathematica is able to reproduce this and it can be reproduced using formula (A.3.5) in [7], e.g. However, in the case of (3), Mathematica gives for $0 < z < 1$

$$\int_0^1 \frac{\ln(x)\ln(1-xz)}{1-x}dx = 0 \tag{12}$$

which is clearly incorrect, while Lewin's formula (A.3.5) gives the left hand side of (12) explicitly as an expression, too complicated to present here, containing dilogarithms, trilogarithms and logarithms, meaning that the generating function for the generalised harmonic numbers $H_{n,2}/n$ can be expressed in closed form. In the case $z = 1/2$ the many terms simplify and one finds

$$\sum_{n=1}^\infty \frac{H_{n,2}}{2^n n} = \frac{5}{8}\zeta(3) - \frac{\pi^2}{3}\ln 2. \tag{13}$$

As an example of the corollary we have

$$\int_0^1 \int_0^1 \frac{e^{-xyz}}{\ln(x) + \ln(y)} dxdy = \frac{1 - e^z}{e^z}, \quad Re[z] > 0 \tag{14}$$

which may be new.

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References

[1] F. Beukers, *A note on the irrationality of* $\zeta(2)$ *and* $\zeta(3)$, Bul. Lond. Math. Soc.**11**, 268-272 (1979)

[2] P. Hadjicostas, *Some generalisations of Beuker's integrals*, Kyungpook Math. J. **42**, 399-416 (2003)

[3] P. Hadjicostas, A conjecture-generalisation of Sondow's formula, arXiv:o405428(2004)

[4] J. Sondow, Criteria for the irrationality of Euler's constant, Proc. Amer. Math. Soc.**131**, 3335-3344 (2003)

[5] J, Sondow, Double integrals for Euler's constant and $\ln(4/\pi)$, Amer. Math. Monthly **112**,61-65(2005)

[6] J. Guillera and J. Sondow, Double integrals and infinite products for some classical constants, arXiv:0506319(2006).

[7] L. Lewin, Polylogarithms and Associated Functions, [N. Holland Publishers, NY (1981)]