

Two-point one-dimensional δ - δ' interactions: non-abelian addition law and decoupling limit

M Gadella¹, J Mateos-Guilarte², J M Muñoz-Castañeda³ and L M Nieto¹

¹Departamento de Física Teórica, Atómica y Óptica, and IMUVA (Instituto de Matemáticas), Universidad de Valladolid, 47011 Valladolid, Spain

²Departamento de Física Fundamental, Universidad de Salamanca, 37008 Salamanca, Spain

³Institut für Theoretische Physik, Universität Leipzig, Germany and Department of Applied Physics, ETSIAE, Universidad Politecnica de Madrid, Spain

E-mail: jose.munoz.castaneda@upm.es

Received 17 June 2015, revised 24 September 2015

Accepted for publication 28 October 2015

Published 4 December 2015



CrossMark

Abstract

In this contribution to the study of one-dimensional point potentials, we prove that if we take the limit $q \rightarrow 0$ on a potential of the type $v_0\delta(y) + 2v_1\delta'(y) + w_0\delta(y - q) + 2w_1\delta'(y - q)$, we obtain a new point potential of the type $u_0\delta(y) + 2u_1\delta'(y)$, when u_0 and u_1 are related to v_0, v_1, w_0 and w_1 by a law with the structure of a group. This is the Borel subgroup of $SL_2(\mathbb{R})$. We also obtain the non-abelian addition law from the scattering data. The spectra of the Hamiltonian in the decoupling cases emerging in the study are also described in full. It is shown that for the $v_1 = \pm 1, w_1 = \pm 1$ values of the δ' couplings the singular Kurasov matrices become equivalent to Dirichlet at one side of the point interaction and Robin boundary conditions at the other side.

Keywords: delta prime, singular potentials, quantum mechanics, group theory, quantum dynamical systems, point interactions

(Some figures may appear in colour only in the online journal)

1. Introduction

One-dimensional models with point interactions [1] have recently received much attention. They are used to model several kinds of extra thin structures [2, 3] or to point to defects in materials, so that effects like tunneling are easily studied. They are also used in the study of

heterostructures, where they may appear in connection to an abrupt effective mass change [4]. The general study of point interactions of the free Hamiltonian $H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$ is mainly due to Kurasov [5, 6] and it is based on the construction of self-adjoint extensions of symmetric operators with identical deficiency indices. More recently, Asorey, Muñoz-Castañeda, and coworkers reformulated the theory of self-adjoint extensions of symmetric operators over bounded domains in terms of meaningful quantities from a quantum field theoretical point of view (see [7–9] and references therein). This new approach allows rigorous study of the theory of quantum fields over bounded domains and the quantum boundary effects. The existence of a four-parameter family of self-adjoint extensions is well known. Some of these extensions are customarily associated with Hamiltonians of the H_0 type plus an interaction of type $a\delta(x-d) + b\delta'(x-d)$, where δ' is the derivative of the Dirac delta while a , b , and d are fixed real numbers. This type of perturbation is of interest in both mathematical and physical quantum mechanics and has been largely discussed [1, 10–17].

Although it is clear which self-adjoint extension corresponds to the 1D-Dirac delta $\delta(x-d)$, there is no consensus on which one should be assigned to its derivative $\delta'(x-d)$ [13–19]. Self-adjoint extensions of H_0 are characterized by matching conditions at $x=d$. The set of self-adjoint extensions of H_0 is given by the unitary group $U(2)$ and therefore is a four-parameter family of operators (see [8, 9]). In previous works, we characterized perturbations of type $a\delta(x-d) + b\delta'(x-d)$ by suitable matching conditions as a two-parameter family of self-adjoint extensions (see [12, 20]). Combining this family of point interactions with some other potentials (or even with mass jumps), we obtained some physical features such as transmission and reflection coefficients, bound and antibound states, and resonance poles [21].

In another physical contexts, scalar quantum field theory (QFT) on a line, point potentials are useful to model impurities and/or provide external singular backgrounds where the bosons move (see, e.g., [22]). The spectra of Hamiltonians with δ and δ' point interactions provide one-particle states in scalar $(1+1)$ -dimensional QFT systems (see, [7–9]). In particular, configurations of two pure delta potentials added to the free Schrödinger Hamiltonian have been used to describe scalar field fluctuations on external backgrounds, (see, e.g., [23–25]), as the corresponding scattering waves. The same configuration of delta interactions is addressed in [18] as a perturbation of the Salpeter Hamiltonian. Moreover, according to the idea proposed by several authors, delta point interactions allow implementing some boundary conditions compatible with a scalar QFT defined on an interval (see [22] and references quoted therein). Additionally, generalized Dirichlet boundary conditions were discussed in [20] where it is shown that the use of δ - δ' potentials provides a much larger set of admissible boundary conditions. The *TGTG*-formula was subsequently applied to compute the corresponding quantum vacuum energies between two plane parallel plates represented by a δ - δ' potential in an arbitrary space-time dimension.

To be more specific, the coupling of a to the δ potential mathematically describes the plasma frequencies in Barton's hydrodynamical model [26], characterizing the electromagnetic properties of the conducting (infinitely thin) plates. The physical meaning of the b coupling to the δ' interaction in the context of Casimir physics has only recently been unveiled in [27]: it describes the response of the orthogonal polarizability of a monatomically thin plate to the electromagnetic field.

Since the zero range potentials mimic the plates in a Casimir effect setup it is interesting to consider a three-plate configuration and investigate the Casimir forces by allowing them to move the plate in the middle freely (see, e.g., [28] for an introduction to Casimir Pistons, and [29–32] for more recent results). In particular, it is meaningful to displace the central plate

toward one of the other two placed in the boundary; thus, we find the main physical motivation to study the particular situation where two δ - δ' interactions are superimposed.

This is the first problem to be analyzed in this work. The outcome is surprising: a non-abelian addition law emerges that corresponds to the Borel subgroup of the $SL_2(\mathbb{R})$ group. The other focus of interest is the case where the δ' -couplings are exceptional, i.e., those couplings for which the transmission coefficients are zero and the plates become completely opaque: the left-right decoupling limit. It happens that the distinguished Dirichlet/Robin boundary conditions are implemented in this case by the δ - δ' -potentials but the superposition law just mentioned becomes singular. We will fully discuss this much more awkward regime in the second part of this paper.

Here, we combine two point potentials of the type $a_d\delta(x-d) + 2b_d\delta'(x-d)$ with $d > 0$, and $a_0\delta(x) + 2b_0\delta'(x)$. In fact, the idea of using double delta potentials ($b_d = b_0 = 0$) has a long tradition. For instance, in condensed matter physics, in the Bardeen-Cooper-Schrieffer (BCS) model [33], in Bose–Einstein condensates [34], or potentials with double pole resonances [35] arise in the description of some unstable states. Finally, in regards to physical contexts where δ - δ' interactions play a rôle, we note that point-supported potentials can be used to model gap-like impurities in graphene layers and nano-ribbons (see [36]). In this physical situation graphene surface plasmons suffer total reflection after collision with a point-supported impurity/edge.

In the present article and after the introduction of our Hamiltonian in section 2, we study bound states, scattering coefficients, and resonances in the so-called regular cases, which are those with $b_d \neq \pm \hbar^2/m \neq b_0$, which is done in section 3. In section 4 we consider the limit $d \rightarrow 0$. In this limit both interactions coincide. The result is again an interaction of the type $a\delta(x) + b\delta'(x)$ and it is interesting to evaluate the composition law for the coefficients, which is not linear as we may expect. Indeed, this composition law establishes the two-dimensional space of couplings $(a, b) \subseteq \mathbb{R}^2$ as the Borel subgroup of $SL_2(\mathbb{R})$, a quite unexpected result. Section 5 is devoted to studying the interesting cases of the left-right decoupling limit, i.e., those with $b_d = \pm \frac{\hbar^2}{m}$ and $b_0 = \pm \frac{\hbar^2}{m}$. The paper closes with some concluding remarks.

2. The Hamiltonian

Let us consider a one-dimensional free Hamiltonian H_0 with a potential of the type $a\delta(x-d) + b\delta'(x-d)$. Its Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + a\delta(x-d)\psi(x) + b\delta'(x-d)\psi(x) = E \psi(x). \quad (2.1)$$

In order to work with dimensionless quantities, we introduce new variables and parameters:

$$x = \frac{\hbar}{mc} y, \quad d = \frac{\hbar}{mc} q, \quad w_0 = \frac{2a}{\hbar c}, \quad w_1 = \frac{mb}{\hbar^2}, \quad \varepsilon = \frac{2E}{mc^2}, \quad \varphi(y) = \psi(x), \quad (2.2)$$

such that (2.1) becomes

$$-\frac{d^2}{dy^2} \varphi(y) + w_0\delta(y-q)\varphi(y) + 2w_1\delta'(y-q)\varphi(y) = \varepsilon \varphi(y). \quad (2.3)$$

From now on, we will consider this version of the Schrödinger equation instead of (2.1).

The point potential we are interested in, $w_0\delta(y-q) + 2w_1\delta'(y-q)$, is usually defined via the theory of self-adjoint extensions of symmetric operators of equal deficiency indices [5, 6], so the total Hamiltonian $H = H_0 + w_0\delta(y-q) + 2w_1\delta'(y-q)$ is self-adjoint. The

crucial point is finding the domain of wave functions $\varphi(y)$ that makes H_0 self-adjoint over the domain $\mathbb{R}/\{q\}$ and characterizes the potential $w_0\delta(y - q) + 2w_1\delta'(y - q)$. As these functions and their derivatives should have a discontinuity at $y = q$, we have to define the products of the form $\delta(y - q)\varphi(y)$ and $\delta'(y - q)\varphi(y)$ in (2.3). These can be done in several ways [15–17], but we choose the following:

$$\delta(y - q)\varphi(y) = \frac{\varphi(q^+) + \varphi(q^-)}{2} \delta(y - q), \tag{2.4}$$

$$\delta'(y - q)\varphi(y) = \frac{\varphi(q^+) + \varphi(q^-)}{2} \delta'(y - q) - \frac{\varphi'(q^+) + \varphi'(q^-)}{2} \delta(y - q), \tag{2.5}$$

where $f(q^+)$ and $f(q^-)$ are the right and left limits of the function $f(y)$ as $y \rightarrow q$, respectively. The Schrödinger equation (2.3) should be viewed as a relation between distributions.

In order to obtain a self-adjoint determination of the Hamiltonian $H = H_0 + w_0\delta(y - q) + 2w_1\delta'(y - q)$, we have to find a self-adjoint extension of H_0 . In order to do that, we have to find a domain on which this extension acts. This domain is given by a space of square integrable functions satisfying certain assumptions including matching conditions at the point q that affects the value of wave functions and their derivatives at q [1, 5]. In particular, this implies that both wave functions and derivatives cannot be continuous at q , so that equations (2.4) and (2.5) make sense.

The functions in the domain of the Hamiltonian H are functions in the Sobolev space¹ $W_2^2(\mathbb{R} \setminus \{q\})$ such that q satisfy the following matching conditions²:

$$\begin{pmatrix} \varphi(q^+) \\ \varphi'(q^+) \end{pmatrix} = \begin{pmatrix} \frac{1 + w_1}{1 - w_1} & 0 \\ \frac{w_0}{1 - w_1^2} & \frac{1 - w_1}{1 + w_1} \end{pmatrix} \begin{pmatrix} \varphi(q^-) \\ \varphi'(q^-) \end{pmatrix}. \tag{2.6}$$

These results could be, in principle, extended to interactions of the type $\sum_{i=1}^N a_i\delta(y - q_i) + 2b_i\delta'(y - q_i)$, where N could be either finite or infinite [5]. In the present paper, we assume that $N = 2$ as above, so the total Hamiltonian takes the form:

$$H = H_0 + V + W, \quad H_0 = -\frac{d^2}{dy^2}, \tag{2.7}$$

with

$$V = v_0\delta(y) + 2v_1\delta'(y), \quad W = w_0\delta(y - q) + 2w_1\delta'(y - q), \tag{2.8}$$

where $v_0 = mA/\hbar c$, $v_1 = mB/\hbar^2$, and A, B are, respectively, the δ and δ' couplings of the pair placed at the origin in the original variables. While V is supported at the origin $y = 0$, W is supported at a point q that we assume positive, $q > 0$. The potential $V + W$ is physically relevant as is related to the Casimir effect [20]. The corresponding dimensionless Schrödinger

¹ This is the space of absolutely continuous functions $f(y)$ with absolutely continuous derivatives $f'(y)$, both having arbitrary discontinuities at q , such that the Lebesgue integral given by

$$\int_{-\infty}^{\infty} \{ |f(y)|^2 + |f'(y)|^2 \} dy$$

converges.

² This is true for $w_1 \neq \pm 1$, while for $w_1 = \pm 1$, we have to define the matching conditions in another way [5]. We will concentrate now on the regular cases, but the decoupling limit will be also considered later on in section 5.

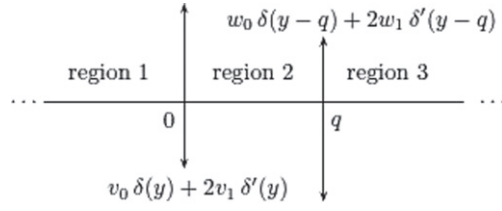


Figure 1. Schematic representation of the physical problem under study, given by (2.9), before considering the limit $q \rightarrow 0$: a potential with a couple of singular point interactions described by a linear combination of Dirac delta distributions and their derivatives.

equation is

$$-\frac{d^2\varphi(y)}{dy^2} + [v_0\delta(y) + 2v_1\delta'(y) + w_0\delta(y - q) + 2w_1\delta'(y - q)]\varphi(y) = \varepsilon \varphi(y). \quad (2.9)$$

One of the aims of the present work is to study the effect resulting of taking the limit in (2.9) as $q \rightarrow 0$, i.e., the effect of the superposition of the two point potentials V and W at the same point. We will see that, as a result of the limit, we obtain a point potential of the type $u_0\delta(y) + 2u_1\delta'(y)$, where u_0 and u_1 are not the sums $v_0 + w_0$ and $v_1 + w_1$, but instead another kind of law, which has a group structure as the Borel subgroup of $SL_2(\mathbb{R})$. Some further results will be discussed.

3. Matching conditions and scattering coefficients

In order to analyze the solution of (2.9), we split the real line into three regions, as shown in figure 1: region 1 is the half line with $y < 0$, region 2 is the segment $0 < y < q$, and region 3 is the half line $y > q$. Supported on $y = 0$ and on $y = q$, we have the point potentials V and W , as in (2.8), respectively. In these regions the values of the solution and their derivatives are those of the free Schrödinger equation, given by

$$\begin{aligned} \varphi_j(y) &= A_j e^{-iky} + B_j e^{iky}, & \varphi'_j(y) &= -ik(A_j e^{-iky} - B_j e^{iky}), \\ j &= 1, 2, 3, & k^2 &= \varepsilon > 0. \end{aligned} \quad (3.1)$$

Since the point potential V is defined by matching conditions like those in (2.6), we have that

$$\begin{pmatrix} A_2 + B_2 \\ -ik(A_2 - B_2) \end{pmatrix} = M_v \begin{pmatrix} A_1 + B_1 \\ -ik(A_1 - B_1) \end{pmatrix}, \quad M_v = \begin{pmatrix} \frac{1 + v_1}{1 - v_1} & 0 \\ \frac{v_0}{1 - v_1^2} & \frac{1 - v_1}{1 + v_1} \end{pmatrix}. \quad (3.2)$$

Defining the following matrix:

$$K = \begin{pmatrix} 1 & 1 \\ -ik & ik \end{pmatrix}, \quad (3.3)$$

Equation (3.2) becomes:

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = K^{-1} M_v K \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \quad (3.4)$$

This expression (3.4) gives the matching conditions at the point $y = 0$. At the point $y = q$ the same procedure works after the appropriate translation:

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = Q^{-1} K^{-1} M_w K Q \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}. \quad (3.5)$$

and we finally obtain:

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = Q^{-1} K^{-1} M_w K Q K^{-1} M_v K \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = T_q \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}, \quad (3.6)$$

where the definition of T_q is obvious, while Q and M_w are, respectively:

$$Q = \begin{pmatrix} e^{-iqk} & 0 \\ 0 & e^{iqk} \end{pmatrix}, \quad M_w = \begin{pmatrix} \frac{1+w_1}{1-w_1} & 0 \\ \frac{w_0}{1-w_1^2} & \frac{1-w_1}{1+w_1} \end{pmatrix}. \quad (3.7)$$

The T_q -matrix in (3.6) relates the asymptotic behavior of the two linearly independent Jost scattering solutions at $x \ll 0$ with their counterparts at $x \gg 0$ (see, e.g., [37]). The T_q -matrix elements satisfy the identities:

$$\det T_q = T_q^{11} T_q^{22} - T_q^{12} T_q^{21} = 1, \quad T_q^{11} = \bar{T}_q^{22}, \quad T_q^{12} = \bar{T}_q^{21},$$

such that T_q is in general an element of the group $SL_2(\mathbb{C})$. From the T_q -matrix one obtains the scattering matrix S_q by a standard procedure. Reshuffling the linear system (3.6) in the form

$$\begin{pmatrix} A_1 \\ B_3 \end{pmatrix} = S_q \begin{pmatrix} A_3 \\ B_1 \end{pmatrix} \quad (3.8)$$

we can easily check that

$$S_q = \frac{1}{T_q^{11}} \begin{pmatrix} 1 & -T_q^{12} \\ T_q^{21} & \det T_q \end{pmatrix}, \quad S_q^\dagger = \frac{1}{\bar{T}_q^{11}} \begin{pmatrix} 1 & \bar{T}_q^{21} \\ -\bar{T}_q^{12} & \det T_q \end{pmatrix}, \quad S_q^\dagger S_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.9)$$

i.e., the scattering S_q -matrix arises as a 2×2 unitary matrix: $S_q^\dagger S_q = S_q S_q^\dagger = 1$. The transmission and reflection coefficients provide the usual form of writing the S_q -matrix elements:

$$S_q = \begin{pmatrix} t(k) & r_R(k) \\ r_L(k) & t(k) \end{pmatrix}. \quad (3.10)$$

$t(k)$ is the amplitude of the transmitted waves coming from the far left or from the far right. Time-reversal invariant potentials give rise to identical transmission coefficients: $t_R(k; q) = t_L(k; q) = t(k; q)$. However, the reflection amplitudes are different for incoming waves either from the far right or the far left such that $r_R(k; q) \neq r_L(k; q)$ because the potential is not parity invariant. Comparison of equations (3.9) and (3.10) shows that:

$$t(k; q) = \frac{1}{T_q^{11}(k)}, \quad r_R(k; q) = -\frac{T_q^{12}(k)}{T_q^{11}(k)}, \quad r_L(k, q) = \frac{T_q^{21}(k)}{T_q^{11}(k)}. \quad (3.11)$$

Transmission and reflection coefficients were calculated in [20] when the two δ - δ' point interactions were symmetrically located with respect to the origin. For the arrangement chosen in this paper we find from formulas (3.11):

$$\begin{aligned}
& r_L(k; q) \\
&= - \frac{e^{-2iqk} \left(2k(v_1^2 + 1) + iv_0 \right) (4kw_1 - iw_0) + (4kv_1 - iv_0) \left(2k(w_1^2 + 1) - iw_0 \right)}{\Delta(k)},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
& r_R(k; q) \\
&= \frac{e^{2iqk} \left(2k(v_1^2 + 1) - iv_0 \right) (4kw_1 + iw_0) + (4kv_1 + iv_0) \left(2k(w_1^2 + 1) + iw_0 \right)}{\Delta(k)},
\end{aligned} \tag{3.13}$$

$$t(k; q) = \frac{4k^2(1 - v_1^2)(1 - w_1^2)}{\Delta(k)}, \tag{3.14}$$

$$\Delta(k) = e^{2ikq} (v_0 + 4ikv_1)(w_0 - 4ikw_1) + (2kv_1^2 + 2k + iv_0)(2kw_1^2 + 2k + iw_0), \tag{3.15}$$

where $\Delta(k)$ is a function of k and the other parameters of the problem.

3.1. Bound/antibound states and resonances

Complex zeroes of

$$T_q^{11}(k) = \frac{\Delta(k)}{4k^2(1 - v_1^2)(1 - w_1^2)}, \tag{3.16}$$

which are poles of the S_q -matrix in the k -complex plane, give rise to bound or antibound states if are located on the purely imaginary, respectively positive or negative half-axis. Complex zeroes coming in pairs having opposite real and identical imaginary parts that correspond to resonances.

The relevant physical information is encoded in the analysis of complex zeroes of $T_q^{11}(k)$, or equivalently of $\Delta(k)$. In order to simplify this analysis, it is useful to make the following changes in the parameters in (3.16):

$$2kq = z, \quad \sigma = \frac{qv_0}{1 + v_1^2}, \quad \tau = \frac{qw_0}{1 + w_1^2}, \quad v = \frac{v_1}{1 + v_1^2}, \quad w = \frac{w_1}{1 + w_1^2}. \tag{3.17}$$

Then, $\Delta(k) = 0$ is equivalent to

$$e^{iz} = - \left(\frac{z + i\sigma}{\sigma + 2ivz} \right) \left(\frac{z + i\tau}{\tau - 2iwz} \right), \quad z = z_r + iz_i, \quad z_r, z_i \in \mathbb{R}. \tag{3.18}$$

Equation (3.18) can be considered as a complex or two-dimensional generalization of the so-called *generalized Lambert equation* (see [38, 39] and references quoted therein). From this complex Lambert equation we can obtain two real equations, corresponding to the real and imaginary parts of (3.18):

$$\begin{aligned}
& \left[4vwz_r^2 - (2vz_i - \sigma)(2wz_i + \tau) \right] \cos z_r - 2z_r(\tau v + 4vwz_i - \sigma w) \sin z_r \\
&= \left[(z_i + \sigma)(z_i + \tau) - z_r^2 \right] e^{z_i}.
\end{aligned} \tag{3.19}$$

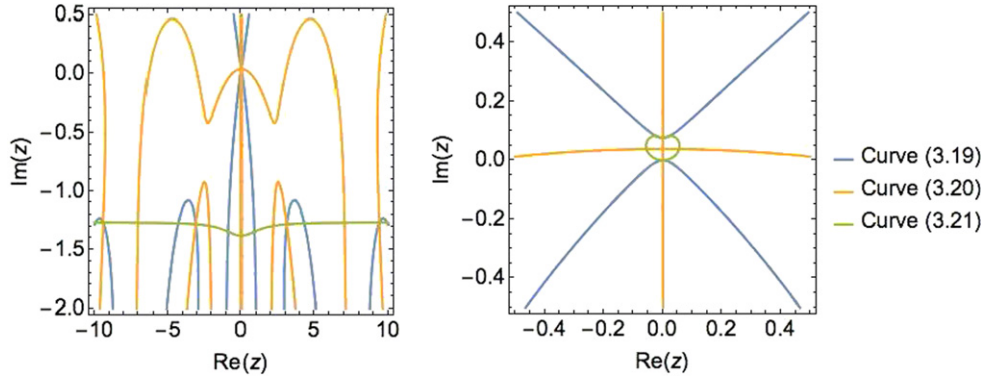


Figure 2. On the left, for some values of the parameters ($v_0 = -2$, $v_1 = 4$, $w_0 = -1$, $w_1 = 3$, and $q = 1/2$), the typical behavior of the curves (3.19)–(3.21) is plotted. The intersections between the blue and orange curves (the values of z such that the real and imaginary parts of $\Delta(z)$ are simultaneously zero) are seen as discrete points, placed on the compatibility condition curve (green). Note that there is an infinite number of such solutions on the lower part of the complex plane, appearing in pairs $z = \pm z_{r,n} - iz_{i,n}$, $z_{r,n}, z_{i,n} > 0$, $n = 1, 2, \dots$, corresponding accordingly to resonances in the spectrum; in addition, there is also one imaginary solution with a positive imaginary part $z_0 = iz_{i,0}$. On the right, a zoom of the same plot near the origin is shown. We may observe a bound state in the upper intersection. The lower intersection, however, happens exactly at the origin and it is not a meaningful physical solution.

$$2z_r(\tau v + 4vwz_i - \sigma w)\cos z_r + [4vwz_r^2 - (2vz_i - \sigma)(2wz_i + \tau)]\sin z_r = -z_r(2z_i + \sigma + \tau)e^{z_i}. \tag{3.20}$$

These expressions for real and imaginary parts of $\Delta(k) = 0$ are rather complicated. A simpler compatibility condition is given by

$$e^{2z_i} = \frac{(4v^2z_r^2 + (2vz_i - \sigma)^2)(4w^2z_r^2 + (2wz_i + \tau)^2)}{(z_r^2 + (z_i + \sigma)^2)(z_r^2 + (z_i + \tau)^2)}, \tag{3.21}$$

which is, indeed, a generalization of the Lambert equation that includes two real variables. The solutions we are looking for must satisfy (3.21), although (3.19) and (3.20) are more restrictive. First of all, we can easily check that a simple solution of the system (3.19) and (3.20) is $z_r = 0$, $z_i = 0$, but this corresponds to $k = 0$, which is a pole of $T_q^{11}(k)$. In addition, it is obvious that the system is symmetric on the variable z_r , although it has no symmetry on z_i . The typical behavior of the solutions of (3.19)–(3.21) is shown in figure 2.

4. Addition law at the $q = 0$ limit

Proposition 1. *The T_q matrix for two-pairs of δ - δ' interactions displaced from each other a distance q is determined in formula (3.6). In the limit $q \rightarrow 0$, T_q becomes:*

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = K^{-1} M_w \cdot M_v K \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = K^{-1} M_u K \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \tag{4.1}$$

i.e., the superposition of these two δ - δ' is equivalent to a single δ - δ' where the matching conditions are characterized by the Kurasov matrix

$$M_u = \begin{pmatrix} \frac{1+u_1}{1-u_1} & 0 \\ \frac{u_0}{1-u_1^2} & \frac{1-u_1}{1+u_1} \end{pmatrix},$$

where $u_1 = \frac{v_1 + w_1}{1 + v_1 w_1}$, $u_0 = \frac{v_0(1-w_1)^2 + w_0(1+v_1)^2}{(1+v_1 w_1)^2}$. (4.2)

Proof. In the $q = 0$ limit, the transfer matrix is $\lim_{q \rightarrow 0} T_q = T_0$, while Q obviously becomes the identity matrix. Since M_w and M_v do not depend on q , the $q = 0$ limit of (3.6) reads:

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = K^{-1} M_w M_v K \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = T_0 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \tag{4.3}$$

We may expect that the product of two Kurasov matrices $M_w M_v$ is another Kurasov matrix. This assumption means that

$$M_w M_v = \begin{pmatrix} \frac{1+w_1}{1-w_1} & 0 \\ \frac{w_0}{1-w_1^2} & \frac{1-w_1}{1+w_1} \end{pmatrix} \begin{pmatrix} \frac{1+v_1}{1-v_1} & 0 \\ \frac{v_0}{1-v_1^2} & \frac{1-v_1}{1+v_1} \end{pmatrix} = \begin{pmatrix} \frac{1+u_1}{1-u_1} & 0 \\ \frac{u_0}{1-u_1^2} & \frac{1-u_1}{1+u_1} \end{pmatrix} = M_u. \tag{4.4}$$

A lengthy but straightforward calculation shows that this is the case with the ‘composite’ couplings:

$$u_1 = \frac{v_1 + w_1}{1 + v_1 w_1}, \quad u_0 = \frac{v_0(1-w_1)^2 + w_0(1+v_1)^2}{(1+v_1 w_1)^2}. \tag{4.5}$$

Thus, the collapsed two δ - δ' pairs are tantamount to a single point potential of the form $u_0 \delta(y) + 2u_1 \delta'(y)$ defined by the transfer matrix $T_0 = K^{-1} M_u K$, □

We would have expected another result such as an additive rule of the type $u_0 = v_0 + w_0$ and $u_1 = v_1 + w_1$, due to the form of point potentials that converge. In fact, for zero δ' couplings we would obtain:

$$\begin{pmatrix} 1 & 0 \\ w_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v_0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w_0 + v_0 & 1 \end{pmatrix}$$

and the fusion of two δ point potentials is a pure abelian process. Nonetheless, the addition law (4.5) is interesting. First of all, the first equation in (4.5) resembles the addition law of velocities in special relativity. However, the second one looks more complicated. In any case, one natural question we may pose is if the composition law (4.5) has a structure of a group and, if this is the case, which group this can be.

Checking the group structure is quite simple. The product of two matrices, M_v and M_w , as in (4.4) gives another matrix with identical structure. This gives the product law. The

associativity comes from the associativity of the product of matrices. The identity of the group is the 2×2 identity matrix \mathbb{I} , which corresponds to both parameters being equal to zero in (2.6), $M_0 = \mathbb{I}$. The inverse of M_v is a matrix M_s so that $M_v M_s = \mathbb{I}$. The calculation is straightforward. For M_v as defined in (3.2) we find

$$M_s = \begin{pmatrix} \frac{1 - v_1}{1 + v_1} & 0 \\ -v_0 & \frac{1 + v_1}{1 - v_1} \end{pmatrix}, \tag{4.6}$$

which shows that taking the inverse in our group of matrices is equivalent to the transformation $(v_0, v_1) \rightarrow (-v_0, -v_1)$. We may denote the inverse of M_v as M_{-v} . Thus, the structure of the group in our set of matrices is confirmed. Furthermore, there are two other remarkable properties of matrices in this group:

(i) Non-commutativity:

$$M_v M_w - M_w M_v = \begin{pmatrix} 0 & 0 \\ \frac{4(v_0 w_1 - w_0 v_1)}{(1 - v_1^2)(1 - w_1^2)} & 0 \end{pmatrix}. \tag{4.7}$$

(ii) The trace of M_v is independent of v_0 :

$$\text{tr } M_v = 2 \frac{1 + v_1^2}{1 - v_1^2}. \tag{4.8}$$

This trace has two singular points $v_1 = \pm 1$, emerging from Kurasov’s analysis, and corresponding to the decoupling limits that are not included in the type of matrices we are considering in the present section (for an analysis of the decoupling or completely opaque limits see section 5). It is positive on the interval $-1 < v_1 < 1$, reaching the minimum value 2 for $v_1 = 0$. On the half lines $-\infty < v_1 < -1$ and $1 < v_1 < \infty$ the trace is negative and its modulus is larger than 2. Its limits when $v_1 \rightarrow \pm\infty$ are equal to -2 . A plot of the function (4.8) can be seen on figure 3.

4.1. Borel subgroup of $SL_2(\mathbb{R})$ and its Lie algebra

Proposition 2. *The Kurasov matrices form a group where the matrix product $M_v M_w = M_u$ is the group law:*

$$u_1 = \frac{v_1 + w_1}{1 + v_1 w_1}, \quad u_0 = \frac{v_0(1 - w_1)^2 + w_0(1 + v_1)^2}{(1 + v_1 w_1)^2}. \tag{4.9}$$

The group of Kurasov matrices is the Borel subgroup of $SL_2(\mathbb{R})$.

Proof. All the previous information shows that the set of M_v matrices is the subgroup of $SL_2(\mathbb{R})$ of all the lower triangular matrices such that $|\text{tr}(g)| \geq 2$. This is the Borel subgroup of $SL_2(\mathbb{R})$.

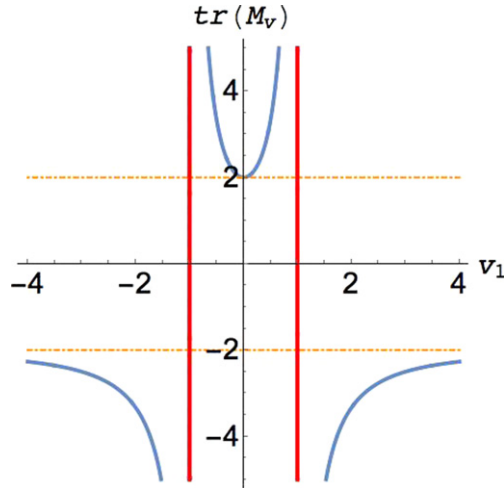


Figure 3. Plot of the trace of a generic matching matrix M_v , given by equation (4.8).

The Lie algebra of the Borel subgroup of $SL_2(\mathbb{R})$ is the minimal parabolic subalgebra of $\mathfrak{sl}_2(\mathbb{R})$. The Cartan basis for $\mathfrak{sl}_2(\mathbb{R})$ is given by the following 2×2 matrices:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.10)$$

Their commutation relations are

$$[e, f] = h, \quad [h, f] = -2f, \quad [h, e] = 2e. \quad (4.11)$$

The Borel subalgebra of a given Lie algebra \mathfrak{g} with Cartan subalgebra \mathfrak{h} is the direct sum of the Cartan subalgebra with a given ordering and the weight spaces of \mathfrak{g} with negative weight (it is equivalent to taking positive weights; this would just change the upper diagonal matrices by lower diagonal matrices. In any case, both algebras, or equivalently both groups, are isomorphic). Therefore, in the case of $\mathfrak{sl}_2(\mathbb{R})$ the Borel subalgebra is

$$\mathfrak{g}_{K(x_0)} = \langle h, f \rangle. \quad (4.12)$$

The elements of the two disconnected components of the group are obtained by exponentiation:

$$e^{\alpha h + \beta f} = \begin{pmatrix} e^\alpha & 0 \\ \beta \frac{\sinh \alpha}{\alpha} & e^{-\alpha} \end{pmatrix}, \quad -e^{\alpha h + \beta f} = \begin{pmatrix} -e^\alpha & 0 \\ -\beta \frac{\sinh \alpha}{\alpha} & -e^{-\alpha} \end{pmatrix}. \quad (4.13)$$

For the component of the elements connected with the identity, the couplings are easily obtained in terms of the group parameters:

$$e^{\alpha h + \beta f} = M_v \implies v_0 = 2 \frac{\beta}{\alpha} \tanh \frac{\alpha}{2}, \quad v_1 = \tanh \frac{\alpha}{2}. \quad (4.14)$$

On the other hand, for the elements in the other connected component, we have:

$$-e^{\alpha h + \beta f} = M_v \implies v_0 = 2 \frac{\beta}{\alpha} \coth \frac{\alpha}{2}, \quad v_1 = \coth \frac{\alpha}{2}. \quad (4.15)$$

The composition law of the group can be expressed in terms of these exponentials. In fact,

$$e^{\alpha_1 h + \beta_1 f} e^{\alpha_2 h + \beta_2 f} = e^{\alpha h + \beta f} \quad (4.16)$$

with

$$\alpha = \alpha_1 + \alpha_2, \quad \beta = \frac{(\alpha_1 + \alpha_2)(e^{\alpha_2} \alpha_2 \beta_1 \sinh(\alpha_1) + e^{-\alpha_1} \alpha_1 \beta_2 \sinh(\alpha_2))}{\alpha_1 \alpha_2 \sinh(\alpha_1 + \alpha_2)}. \quad (4.17)$$

This concludes the discussion on the group structure. Q.E.D.

4.2. Reflection and transmission coefficients due to a single δ - δ' interaction at the origin

Proposition 3. *The $q = 0$ limit of the scattering coefficients of two a priori separated pairs of δ - δ' interactions exactly coincide with the scattering coefficients produced by a single δ - δ' pair with couplings determined from the $M_u = M_v M_w$ Kurasov matrix. In terms of the scattering matrix we can write:*

$$\lim_{q \rightarrow 0} S_q(M_v, M_w) = S(M_v \cdot M_w) = S(M_u). \quad (4.18)$$

Hence, the scattering matrices produced by two superimposed pairs of δ - δ' interactions give rise to a representation of the non-abelian composition law of Kurasov matrices.

Proof. The linear map between asymptotic scattering Jost solutions, respectively, in the far right and the far left:

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = Z \begin{pmatrix} A_1 \\ B_1 \end{pmatrix},$$

produced by a single point interaction $V(x) = u_0 \delta(x) + 2u_1 \delta'(x)$ is provided by a similarity transformation of the Kurasov matrix

$$Z = K^{-1} M_u K, \quad M_u = \begin{pmatrix} \frac{1+u_1}{1-u_1} & 0 \\ u_0 & \frac{1-u_1}{1+u_1} \end{pmatrix}.$$

We obtain for the Z -matrix elements

$$Z^{11}(k) = \frac{2k(1+u_1^2) + iu_0}{2k(1-u_1^2)} = \bar{Z}^{22}(k), \quad Z^{12}(k) = \frac{4ku_1 + iu_0}{2k(1-u_1^2)} = \bar{Z}^{21}, \quad (4.19)$$

where the bar stands for complex conjugation. From this information we read the scattering coefficients

$$r_R(k) = -\frac{Z^{12}(k)}{Z^{11}(k)} = -\frac{4ku_1 + iu_0}{2k(u_1 + 1) + iu_0},$$

$$r_L(k) = \frac{Z^{21}(k)}{Z^{11}(k)} = \frac{4ku_1 - iu_0}{2k(u_1^2 + 1) + iu_0}, \quad (4.20)$$

$$t(k) = \frac{1}{Z^{11}(k)} = -\frac{2k(u_1^2 - 1)}{2k(u_1^2 + 1) + iu_0}, \tag{4.21}$$

in perfect agreement with the results in [12].

Finally, we compare these scattering coefficients with those obtained from the T_0 -matrix that describe the map between asymptotic Jost solutions when the distance q from one δ - δ' point interaction to the other tends to zero. The outcome is:

$$\begin{aligned} r_L(k; 0) &= -\frac{T_0^{12}(k)}{T_0^{11}(k)} \\ &= -\frac{v_1((v_1 + 2)w_0 + 4ik) + 4ikv_1w_1(v_1 + w_1) + 4ikw_1 + v_0(w_1 - 1)^2 + w_0}{-2ikw_1((v_1^2 + 1)w_1 + 4v_1) - 2ik(v_1^2 + 1) + (v_1 + 1)^2w_0 + v_0(w_1 - 1)^2}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} r_R(k; 0) &= \frac{T_0^{21}(k)}{T_0^{11}(k)} \\ &= \frac{v_1(-(v_1 + 2)w_0 + 4ik) + 4ikv_1w_1(v_1 + w_1) + 4ikw_1 - v_0(w_1 - 1)^2 - w_0}{-2ikw_1((v_1^2 + 1)w_1 + 4v_1) - 2ik(v_1^2 + 1) + (v_1 + 1)^2w_0 + v_0(w_1 - 1)^2}, \end{aligned} \tag{4.23}$$

$$\begin{aligned} t(k; 0) &= \frac{1}{T_0^{11}(k)} \\ &= \frac{2k(v_1^2 - 1)(w_1^2 - 1)}{2kw_1((v_1^2 + 1)w_1 + 4v_1) + 2k(v_1^2 + 1) + i(v_1 + 1)^2w_0 + iv_0(w_1 - 1)^2}, \end{aligned} \tag{4.24}$$

to check that the addition law

$$u_1 = \frac{v_1 + w_1}{1 + v_1w_1}, \quad u_0 = \frac{v_0(1 - w_1)^2 + w_0(1 + v_1)^2}{(1 + v_1w_1)^2},$$

also works for the scattering coefficients,

Q.E.D.

5. Left-right decoupling values of the δ' couplings

We have seen that there are singularities of the Kurasov matrix at the critical points, $v_1 = \pm 1$ and $w_1 = \pm 1$. We may expect that these critical cases where the transmission coefficients are zero do not contribute to the group structure we previously analyzed. The completely opaque potentials as given in equation (2.8) have one of the following four forms:

$$V = v_0\delta(x) \pm 2\delta'(x) \quad \text{and/or} \quad W = w_0\delta(x - q) \pm 2\delta'(x - q). \tag{5.1}$$

In order to define these potentials, we obviously cannot use matching conditions of the form (2.6) that are singular. Instead, we will impose [5]:

1. For $v_1 = 1$:

$$\varphi(0^-) = 0, \quad \varphi'(0^+) = \frac{v_0}{4} \varphi(0^+). \quad (5.2)$$

2. For $w_1 = 1$:

$$\varphi(q^-) = 0, \quad \varphi'(q^+) = \frac{w_0}{4} \varphi(q^+). \quad (5.3)$$

Thus, we set Dirichlet boundary conditions on the left of the two points $x = 0$ and $x = q$, whereas Robin boundary conditions are chosen on the right-hand side of $x = 0$ and $x = q$ (see [20]). In the two remaining cases the matching conditions are the mirror images of the previous ones.

3. For $v_1 = -1$:

$$\varphi(0^+) = 0, \quad \varphi'(0^-) = -\frac{v_0}{4} \varphi(0^-). \quad (5.4)$$

4. For $w_1 = -1$:

$$\varphi(q^+) = 0, \quad \varphi'(q^-) = -\frac{w_0}{4} \varphi(q^-). \quad (5.5)$$

We remain, however, in the physical situation depicted in figure 1, where the plane waves and their derivatives are written as in (3.1). There are eight possible configurations involving at least one decoupling configuration of the couplings: either the two δ - δ' interactions build opaque walls both at $x = 0$ and $x = q$, or there is no transmission only at one point. We first discuss the cases when the two δ' couplings take the decoupling limit.

5.1. Two δ' couplings in the decoupling limit

There are four cases in which we have decoupling situations both at $x = 0$ and $x = q$. Let us consider them separately.

Case 1: $v_1 = 1, w_1 = 1$.

The boundary conditions are given by (5.2) and (5.3). Let us consider the situation on the interval $[0, q]$. For $x = 0$, $\varphi'(0^+) = \frac{v_0}{4} \varphi(0^+)$ is written as

$$-ik(A_2 - B_2) = \frac{v_0}{4}(A_2 + B_2) \implies \frac{A_2}{B_2} = -\frac{v_0 - 4ik}{v_0 + 4ik} = -\exp\left(-2i \arctan \frac{4k}{v_0}\right). \quad (5.6)$$

For $x = q$, we write $\varphi(q^-) = 0$ as

$$A_2 e^{-ikq} + B_2 e^{ikq} = 0 \implies \frac{A_2}{B_2} = -e^{2ikq}. \quad (5.7)$$

Then, we have a transcendental equation, which can be written either in the form of a generalized Lambert equation:

$$e^{2ikq} = \frac{v_0 - 4ik}{v_0 + 4ik}, \quad (5.8)$$

or in terms of the function arctan as

$$kq = -\arctan \frac{4k}{v_0} \implies \tan(kq) = -\frac{4k}{v_0}. \quad (5.9)$$

This transcendental equation has a countably infinite number of solutions, k_n , that give the energy levels corresponding to this situation. In the limit $q = 0$, we have only one solution, $k = 0$. Outside the interval $[0, q]$, we have the equations

$$\varphi(0^-) = 0 \implies A_1 + B_1 = 0; \quad \varphi'(q^+) = \frac{w_0}{4} \varphi(q^+) \implies \frac{A_3}{B_3} = -e^{2ikq} \frac{w_0 - 4ik}{w_0 + 4ik}. \quad (5.10)$$

These are relations between coefficients that do not provide of any further information, so that we will not refer to similar situations that will appear for all other cases.

Case 2: $v_1 = 1, w_1 = -1$.

The boundary conditions are (5.2) for $x = 0$ and (5.5) for $x = q$. The condition at $x = 0$ has already been studied, providing equation (5.6), so we consider the new boundary condition at $x = q$. It comes from $\varphi'(q^-) = -\frac{w_0}{4} \varphi(q^-)$:

$$\frac{A_2}{B_2} = -e^{2ikq} \frac{w_0 + 4ik}{w_0 - 4ik} = -e^{2ikq} \exp\left(2i \arctan \frac{4k}{w_0}\right). \quad (5.11)$$

This equation is compared to (5.6). The result given in terms of a generalized Lambert equation is

$$e^{2ikq} = \frac{v_0 - 4ik}{v_0 + 4ik} \frac{w_0 - 4ik}{w_0 + 4ik}. \quad (5.12)$$

This transcendental equation can be written in another way by playing with the formulas for the arctan. Note that from (5.12) we obtain straightforwardly the following expression:

$$-kq = \arctan \frac{4k}{w_0} + \arctan \frac{4k}{v_0} \implies \tan(kq) = -\frac{4k(w_0 + v_0)}{w_0 v_0 - 16k^2}. \quad (5.13)$$

This new transcendental equation gives another set of energy eigenvalues (see the right side of figure 4). In the limit $q = 0$, we have only one solution: $k = 0$.

Case 3: $v_1 = -1, w_1 = 1$.

The boundary conditions are (5.4) and (5.3). This means $\varphi(0^+) = 0$, which implies that $A_2 + B_2 = 0$ or $A_2/B_2 = -1$ and $\varphi(q^-) = 0$, which is

$$A_2 e^{-ikq} + B_2 e^{ikq} = 0 \implies \frac{A_2}{B_2} = -e^{2ikq}, \quad (5.14)$$

so that

$$\frac{A_2}{B_2} = -1 = -e^{-2ikq} \implies k = \frac{\pi n}{q}. \quad (5.15)$$

Again, in the limit $q = 0$, the only solution is $k = 0$.

Case 4: $v_1 = -1, w_1 = -1$.

This correspond to (5.4) and (5.5). The boundary condition at $x = 0$ gives $A_2/B_2 = -1$. The boundary condition at $x = q$ is that given in (5.11). Then, the transcendental equation giving the energy levels is given by

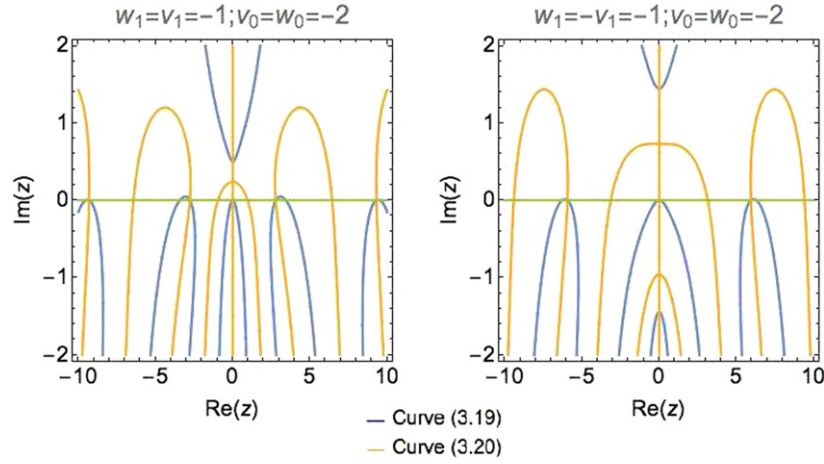


Figure 4. Plots of the equation $\text{Re}(\Delta(z)) = \text{Im}(\Delta(z)) = 0$ over the complex z -plane when both singularities have couplings (as indicated on the top) in the decoupling regime. The green line represents the axis $\text{Im}(z) = 0$.

$$e^{2ikq} = \frac{w_0 - 4ik}{w_0 + 4ik}, \tag{5.16}$$

in the form of a generalized Lambert equation, or

$$\tan(kq) = -\frac{4k}{w_0} \tag{5.17}$$

in the form of a transcendental equation in terms of the tangent. This transcendental equation also has a countably infinite number of solutions, k_n , which gives the energy levels corresponding to this situation (see the left side of figure 4). Once more, in the limit $q = 0$, only the solution $k = 0$ remains.

5.2. Only one δ' coupling in the decoupling limit

Let's now consider the four cases in which we have a decoupling and a regular coupling.

Case 1: $v_1 = 1, w_1 \neq \pm 1$.

From $v_1 = 1$, i.e., from (5.2), we know the conditions $A_1 = -B_1 \neq 0$ and (5.6) must be satisfied, i.e.,

$$\varphi_1(y) \propto \sin(ky), \quad A_2 = -\frac{v_0 - 4ik}{v_0 + 4ik} B_2. \tag{5.18}$$

Note that the system behaves as if there is an impenetrable barrier at $x = 0$. The coefficients A_3, B_3 are obtained from A_2, B_2 as in (3.5):

$$\begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} \frac{2k(1 + w_1^2) + iw_0}{2k(1 - w_1^2)} & e^{2ikq} \frac{4kw_1 + iw_0}{2k(1 - w_1^2)} \\ e^{-2ikq} \frac{4kw_1 - iw_0}{2k(1 - w_1^2)} & \frac{2k(1 + w_1^2) - iw_0}{2k(1 - w_1^2)} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}. \tag{5.19}$$

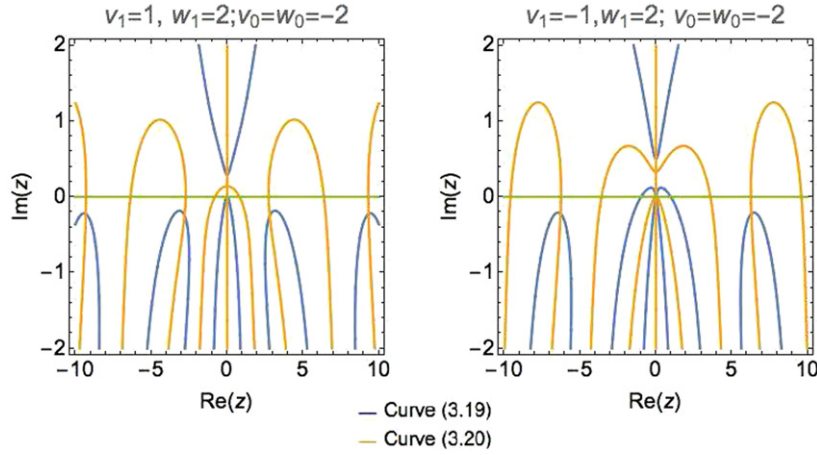


Figure 5. Plots of the equation $\text{Re}(\Delta(z)) = \text{Im}(\Delta(z)) = 0$ over the complex z -plane when one singularity, as indicated on the top, takes the decoupling value for the coupling. The green line represents the axis $\text{Im}(z) = 0$.

All the relevant information about bound states, antibound states, and resonances is obtained by imposing the so-called *purely outgoing boundary condition*, which in our case is $A_3 = 0$. Using (5.18) and (5.19) we get

$$e^{2ikq} = \frac{v_0 - 4ik}{v_0 + 4ik} \frac{2k(1 + w_1^2) + iw_0}{4kw_1 + iw_0}. \tag{5.20}$$

This equation is the equivalent of $\Delta(k) = 0$, where $\Delta(k)$ is given in (3.15), which provides the relevant information in the non-decoupling case. Indeed, (5.20) is obtained making $v_1 = 1$ in $\Delta(k) = 0$. The analysis in this case is similar to the one carried out previously in section 3.1.

We are interested especially in the case where $q \rightarrow 0$. Then, from (5.20) we get

$$(v_0 + 4ik)(4kw_1 + iw_0) = (v_0 - 4ik)(2k(1 + w_1^2) + iw_0). \tag{5.21}$$

The complex solutions of this equation are the following:

$$k_0 = 0, \quad k_1 = -i \frac{4w_0 + v_0(1 - w_1)^2}{4(1 + w_1)^2}. \tag{5.22}$$

The solution $k = 0$ is not relevant, but k_1 produces interesting results: If $4w_0 + v_0(1 - w_1)^2 < 0$, k_1 is on the positive imaginary axis and it corresponds to a bound state, and if $4w_0 + v_0(1 - w_1)^2 > 0$, k_1 is on the negative imaginary axis and it corresponds to an anti bound state. The curves solving the equations $\text{Re}(\Delta) = \text{Im}(\Delta) = 0$ for this case are represented in the left of figure 5.

Case 2: $v_1 = -1, w_1 \neq \pm 1$.

The system also behaves as if there were an impenetrable barrier at $x = 0$. Now, from $v_1 = -1$ (or equivalently from (5.4)), we find the condition $A_2/B_2 = -1$ and consequently (5.19), which must also be satisfied. Then, the *purely outgoing boundary condition* $A_3 = 0$ implies that (see the right side of figure 5)

$$e^{2ikq} = \frac{2k(1 + w_1^2) + iw_0}{4kw_1 + iw_0}. \quad (5.23)$$

In the limit $q \rightarrow 0$ we get the condition $k(1 - w_1)^2 = 0$, which has the unique solution $k = 0$.

Case 3: $v_1 \neq \pm 1, w_1 = 1$.

In this situation we must take into account equations (5.3) and (3.4). The first pair of equations shows the presence of a kind of impenetrable barrier, now in $x = q$, and also the fact that $A_2 = -e^{2ikq}B_2$. Equation (3.4) can be rewritten in the form

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = K^{-1}M_v^{-1}K \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \frac{2k(1 + v_1^2) - iv_0}{2k(1 - v_1^2)} & -\frac{4kv_1 + iv_0}{2k(1 - v_1^2)} \\ \frac{-4kv_1 + iv_0}{2k(1 - v_1^2)} & \frac{2k(1 + v_1^2) + iv_0}{2k(1 - v_1^2)} \end{pmatrix} \begin{pmatrix} A_2 \\ B_2 \end{pmatrix}. \quad (5.24)$$

The purely outgoing boundary condition in the present case is $B_1 = 0$, so that

$$e^{2ikq} = -\frac{2k(1 + v_1^2) + iv_0}{4kv_1 - iv_0}. \quad (5.25)$$

In the limit $q \rightarrow 0$, we get $k(1 + v_1)^2 = 0$, which has the unique solution $k = 0$. Observe that this case is similar to the previous one, Case 2.

Case 4: $v_1 \neq \pm 1, w_1 = -1$.

For $w_1 = -1$, the second equation in (5.5) imposes $\varphi'(q^-) = -\frac{w_0}{4}\varphi(q^-)$. Then

$$A_2 = -e^{2ikq} \frac{w_0 + 4ik}{w_0 - 4ik} B_2. \quad (5.26)$$

Taking this into (5.25), and imposing the purely outgoing boundary condition $B_1 = 0$, we get

$$e^{2ikq} = -\frac{w_0 - 4ik}{w_0 + 4ik} \frac{2k(1 + v_1^2) + iv_0}{4kv_1 - iv_0}. \quad (5.27)$$

In the limit case $q \rightarrow 0$, we have the following complex solutions:

$$k_0 = 0, \quad k_1 = -i \frac{4v_0 + w_0(1 + v_1)^2}{4(1 - v_1)^2}. \quad (5.28)$$

As in Case 1 studied before, the solution $k = 0$ is not relevant, but the other one, k_1 , corresponds either to a bound state if $4v_0 + w_0(1 + v_1)^2 < 0$ or to an anti-bound state if $4v_0 + w_0(1 + v_1)^2 > 0$.

6. Concluding remarks

We studied a one-dimensional two-point perturbation of the free Hamiltonian $H_0 = -d^2/dx^2$ of the form $v_0\delta(x) + 2v_1\delta'(x) + w_0\delta(x - q) + 2w_1\delta'(x - q)$ with $q > 0$. These types of potentials are of interest in one-dimensional quantum field theory and in the study of the Casimir effect as shown by previous work of our group [20, 22, 23]. We have obtained two remarkable and unexpected results.

The former refers to the limit $q \rightarrow 0$. The result is a point potential of the form $u_0\delta(x) + 2u_1\delta'(x)$, where u_0 and u_1 are not the sums $v_0 + w_0$ and $v_1 + w_1$, respectively, but

more complicated functions of these arguments instead. A law of composition for the coefficients of the deltas is produced with the structure of group that coincides with the Borel subgroup of $SL_2(\mathbb{R})$. From the analysis of reflection and transmission coefficients, we obtain the same group law.

The second result comes from the analysis of the decoupling cases that arise by the choices $v_1 = \pm 1$ (decoupling case at $x = 0$) and $w_1 = \pm 1$ (decoupling case at $x = q$). When considering the decoupling cases at $x = 0$ and $x = q$, the barrier seems to be impenetrable at these points so that no scattering states are produced. We find bound or antibound states as solutions of a generalized Lambert equation in terms of the momentum. When the decoupling regime is produced at one point only (either $x = 0$ or $x = q$), we have impenetrability at this point and scattering through the other one, and resonances (as a pair of poles of the analytic continuation of the S -matrix in the momentum representation) can be found. However, it is quite remarkable to note that these poles may lie on the real axis if at least one of the δ' couplings is complex. This situation may violate a widely accepted causality condition [40].

Finally, we mention that in the course of this paper we have clarified the relation between two apparently different ways of characterizing self-adjoint extensions of the H_0 operator. The first one is based on the matching conditions determined from the Kurasov matrices. We have shown that they form the Borel subgroup of $SL_2(\mathbb{R})$. The standard approach to dealing with self-adjoint extensions of symmetric operators, coming back to von Neumann, is through unitary matrices (see, e.g., the previous works [7–9] by Asorey and Muñoz-Castañeda *et al*). The connection from these two points of view starts from the isomorphism between the $SL_2(\mathbb{R})$ and $SU_{1,1}(\mathbb{C})$ groups. In fact, they are conjugate subgroups inside $GL_2(\mathbb{R})$:

$$g SL_2(\mathbb{R})g^{-1} = SU_{1,1}(\mathbb{C}), \quad g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

The matching conditions (2.6) become in the $SU_{1,1}(\mathbb{C})$ framework:

$$\begin{pmatrix} \varphi(q^+) - i\varphi'(q^+) \\ \varphi(q^+) + i\varphi'(q^+) \end{pmatrix} = W_v \begin{pmatrix} \varphi(q^-) - i\varphi'(q^-) \\ \varphi(q^-) + i\varphi'(q^-) \end{pmatrix}, \quad (6.1)$$

where the W_v is conjugated to the Kurasov matrix M_v through the action of g :

$$W_v = g M_v g^{-1} = \frac{1}{2(1 - v_1^2)} \begin{pmatrix} 2(1 + v_1^2) - iv_0 & 4v_1 - iv_0 \\ 4v_1 + iv_0 & 2(1 + v_1^2) + iv_0 \end{pmatrix}. \quad (6.2)$$

In a parallel reshuffling of the linear system (6.1) to that performed to define the S_q scattering matrix from the T_q matrix and passing from the (3.6) equation to (3.8) and (3.9) we rewrite (6.1) in the form

$$\begin{pmatrix} \varphi(q^-) - i\varphi'(q^-) \\ \varphi(q^+) + i\varphi'(q^+) \end{pmatrix} = U_v \begin{pmatrix} \varphi(q^+) - i\varphi'(q^+) \\ \varphi(q^-) + i\varphi'(q^-) \end{pmatrix}, \quad (6.3)$$

where the unitary matrix U_v is obtained from W_v and is

$$U_v = \frac{1}{W_v^{11}} \begin{pmatrix} 1 & -W_v^{12} \\ W_v^{21} & \det W_v \end{pmatrix} = \frac{2(1 - v_1^2)}{2(1 + v_1^2) - iv_0} \begin{pmatrix} 1 & \frac{-4v_1 + iv_0}{2(1 - v_1^2)} \\ \frac{4v_1 + iv_0}{2(1 - v_1^2)} & 1 \end{pmatrix}. \quad (6.4)$$

In this indirect way the Kurasov matrices determining the matching conditions that define the δ - δ' interactions are related to a subset of the $U(2)$ group, which in turn characterizes a variety of self-adjoint extensions of the H_0 operator in the standard manner.

A remarkable fact is that even though M_v and W_v are singular matrices at the decoupling limit $v_1 = \pm 1$ the corresponding unitary matrices are regular:

$$U_v|_{v_1=1} = \begin{pmatrix} 0 & -1 \\ \frac{4+iv_0}{4-iv_0} & 0 \end{pmatrix}, \quad U_v|_{v_1=-1} = \begin{pmatrix} 0 & \frac{4+iv_0}{4-iv_0} \\ -1 & 0 \end{pmatrix}. \quad (6.5)$$

Writing the linear system (6.4) for the decoupling limit of the coupling $v_1 = 1$ we obtain two equations. First,

$$-(\varphi(q^-) + i\varphi'(q^-)) = \varphi(q^-) - i\varphi'(q^-) \Rightarrow \varphi(q^-) = 0, \quad (6.6)$$

i.e., the Dirichlet boundary conditions are satisfied after reaching the point q from the left. Second,

$$\frac{4+iv_0}{4-iv_0}(\varphi(q^+) - i\varphi'(q^+)) = \varphi(q^+) + i\varphi'(q^+) \Rightarrow \varphi(q^+) - \frac{4}{v_0}\varphi'(q^+) = 0 \quad (6.7)$$

sets Robin boundary conditions at q coming from the right. If the other decoupling value for the coupling, $v_1 = -1$, is chosen the situation is identical but the conditions at q from the left or from the right are exchanged. The result above is in perfect agreement with the Kurasov matching conditions at the decoupling limit of the couplings, as expressed in formula (5.2). Finally, we note that in the usual treatment this situation is determined from diagonal rather than anti diagonal matrices. To meet this criterion we merely multiply (6.5) by the σ_1 Pauli matrix.

Finally, we extract some physical consequences from the non-abelian superposition law for the δ - δ' potentials:

1. The quantum self-energy of two δ - δ' configurations due to quantum vacuum scalar fluctuations should inherit some features from the non-abelian composition law. A well known procedure to regularize such divergent quantity is to start from the heat trace of the Hamiltonian operator: $h_H(t) = \text{Tr}_{L^2} \exp[-tH]$, where t is Schwinger proper time. This spectral function is obtained for any Schrödinger Hamiltonian in terms of the bound-state energies and the spectral density, defined in turn from the total phase shift (see the two formulas just above (32) in [22]). From the total phase shift produced by the δ - δ' potential determined by the M_v Kurasov matrix and the bound state energy we find:

$$h_H(t; M_v) = e^{\frac{v_0^2}{4(v_1^2+1)^2}} \left(2i\pi \operatorname{erfc} \left(\frac{\sqrt{t}v_0}{2(v_1^2+1)} \right) + \theta(-v_0) - 4i\pi \right), \quad (6.8)$$

where erfc is the complementary error function and θ the Heaviside step function. Note that the exact heat kernel for a single δ - δ' is well defined in the decoupling limit $v_1 \rightarrow \pm 1$. We conjecture that taking the zero distance limit in the double δ - δ' potential, the heat trace for the superposed potential will be of the form $h_H(t; M_v M_w) = h_H(t; M_u)$. The quantum vacuum energy is essentially the spectral zeta function evaluated at $s = -1/2$. This second spectral function is obtained from the heat trace via Mellin's transform of the heat trace: $\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} h_H(t)$, where $\Gamma(s)$ is the Euler Gamma function and s a complex parameter. $s = -1/2$ is a pole of the meromorphic

function $\zeta_H(s)$ in \mathbb{C} . We regularize the divergent quantum vacuum energy by assigning to ζ_H its value in a regular point. What the non-abelian addition law tell us are the values of the parameters u_0 and u_1 as functions of v_0, w_0, v_1, w_1 entering in this regularized expression after the $q = 0$ limit has been taken.

2. An ideal model of electric conductivity in solids is provided by a δ -Dirac comb where δ -point interactions sit at the ions sites. This model can be enriched in twofold: (1) δ' potentials are added at every site of the lattice. (2) Two species of ions, henceforth two species of δ - δ' interactions, such that the solid is characterized by the periodic potential

$$V(x) = \sum_{n \in \mathbb{Z}} (v_0 \delta(x - nq) + v_1 \delta(x - nq) + u_0 \delta(x - nq + p) + u_1 \delta(x - nq + p)) \quad (6.9)$$

are considered. There are two limits to a single species: $p \rightarrow 0, q$. Unlike in the Dirac comb of two species where the two limits are equivalent, the two merging processes are different in the (6.9) comb due to the non-abelian superposition law. In fact, allowing p to vary we have an infinitely repeated Casimir piston, which can be reduced to one piston by restricting the system to the primitive cell.

Acknowledgments

We acknowledge the financial support of the Spanish MINECO (Project MTM2014-57129-C2-1-P) and Junta de Castilla y León (UIC 011). JMMC would like to acknowledge the fruitful discussions with M Bordag and Klaus Kirsten.

References

- [1] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 *Solvable Models in Quantum Mechanics* 2nd ed (Providence, RI: AMS Chelsea Publishing)
- [2] Zolotaryuk A V and Zolotaryuk Y 2015 A zero-thickness limit of multilayer structures: a resonant-tunnelling δ' -potential *J. Phys. A: Math. Theor.* **48** 035302
- [3] Zolotaryuk A V and Zolotaryuk Y 2015 Controllable resonant tunnelling through single-point potentials: a point triode *Phys. Lett. A* **379** 511–7
- [4] Gadella M, Heras F J H, Negro J and Nieto L M 2009 A delta well with a mass jump *J. Phys. A: Math. Theor.* **42** 465207
- [5] Kurasov P 1996 Distribution theory for discontinuous test functions and differential operators with generalized coefficients *J. Math. Anal. Appl.* **201** 297–323
- [6] Albeverio S and Kurasov P 2000 *Singular Perturbations of Differential Operators* vol 271 (Cambridge University Press)
- [7] Asorey M, Garcia-Alvarez D and Munoz-Castaneda J M 2006 Casimir effect and global theory of boundary conditions *J. Phys. A: Math. Theor.* **39** 6127–36
- [8] Asorey M and Munoz-Castaneda J M 2013 Attractive and repulsive casimir vacuum energy with general boundary conditions *Nucl. Phys. B* **874** 852–76
- [9] Munoz-Castaneda J M, Kirsten K and Bordag M 2015 QFT over the finite line. Heat kernel coefficients, spectral zeta functions and selfadjoint extensions *Lett. Math. Phys.* **105** 523–49
- [10] Golovaty Y 2012 Schrödinger operators with $(\alpha\delta' + \beta\delta)$ -like potentials: norm resolvent convergence and solvable models *Methods Funct. Anal. Topology* **18** 243–55
- [11] Calçada M, Lunardi J T, Manzoni L A and Monteiro W 2014 Distributional approach to point interactions in one-dimensional quantum mechanics *Frontiers in Physics* **2** 2014
- [12] Gadella M, Negro J and Nieto L M 2009 Bound states and scattering coefficients of the $-a\delta(x) + b\delta'(x)$ potential *Phys. Lett. A* **373** 1310–3
- [13] Seba P 1986 Some remarks on the δ' -interaction in one dimension *Rep. Math. Phys.* **24** 111–20

- [14] Albeverio S, Fassari S and Rinaldi F 2013 A remarkable spectral feature of the schrödinger hamiltonian of the harmonic oscillator perturbed by an attractive δ' -interaction centred at the origin: double degeneracy and level crossing *J. Phys. A: Math. Theor.* **46** 385305
- [15] Zolotaryuk A V April 2010 Boundary conditions for the states with resonant tunnelling across the δ' -potential *Phys. Lett. A* **374** 1636–41
- [16] Zolotaryuk A V and Zolotaryuk Y 2011 Controlling a resonant transmission across the δ' -potential: the inverse problem *J. Phys. A: Math. Theor.* **44** 375305
- [17] Zolotaryuk A V and Zolotaryuk Y 2012 Corrigendum: Controlling a resonant transmission across the δ' -potential: the inverse problem *J. Phys. A: Math. Theor.* **45** 119501
- [18] Albeverio S, Fassari S and Rinaldi F 2015 The discrete spectrum of the spinless one-dimensional salpeter hamiltonian perturbed by δ -interactions *J. Phys. A: Math. Theor.* **48** 185301
- [19] Kulinskii V L and Panchenko D Y 2015 Physical structure of point-like interactions for one-dimensional schrödinger operator and the gauge symmetry *Physica B: Condensed Matter* **472** 78–83
- [20] Castañeda J M M and Guilarte J M 2015 $\delta - \delta'$ generalized Robin boundary conditions and quantum vacuum fluctuations *Phys. Rev. D* **91** 025028
- [21] Alvarez J J, Gadella M, Lara L P and Maldonado-Villamizar F H 2013 Unstable quantum oscillator with point interactions: maverick resonances, antibound states and other surprises *Phys. Lett. A* **377** 2510–9
- [22] Munoz-Castaneda J M, Guilarte J M and Mosquera A M 2013 Quantum vacuum energies and Casimir forces between partially transparent δ -function plates *Phys. Rev. D* **87** 105020
- [23] Guilarte J M and Munoz-Castaneda J M 2011 Double-delta potentials: one dimensional scattering. The Casimir effect and kink fluctuations *Int. J. Theor. Phys.* **50** 2227–41
- [24] Bordag M and Munoz-Castaneda J M 2012 Quantum vacuum interaction between two sine-Gordon kinks *J. Phys. A: Math. Theor.* **45** 374012
- [25] Bordag M 1995 Vacuum energy in smooth background fields *J. Phys. A: Math. Theor.* **28** 755–66
- [26] Barton G 2004 Casimir energies of spherical plasma shells *Journal of Physics A: Mathematical and General* **37** 1011
- [27] Bordag M 2014 Monoatomically thin polarizable sheets *Phys. Rev. D* **89** 125015
- [28] Zee A 2010 *Quantum Field Theory in a Nutshell: In a Nutshell Series (Second Edition)* (Princeton, NJ: Princeton University Press)
- [29] Fucci G, Kirsten K and Morales P 2011 Pistons modelled by potentials *Springer Proc. Phys.* **137** 313–22
- [30] Fucci G and Kirsten K 2011 Conical casimir pistons with hybrid boundary conditions *J. Phys. A: Math. Theor.* **44** 295403
- [31] Fucci G and Kirsten K 2012 The casimir effect for generalized piston geometries *Int. J. Mod. Phys. Conf. Ser.* **14** 100–14
- [32] Fucci G 2015 Casimir pistons with general boundary conditions *Nucl. Phys. B* **891** 676–99
- [33] Cooper L N 1956 Bound electron pairs in a degenerate fermi gas *Phys. Rev.* **104** 1189–90
- [34] Pethick C J and Smith H 2008 *Bose-Einstein Condensation in Dilute Gases* 2nd edn (Cambridge University)
- [35] Antoniou I E, Gadella M, Hernández E, Jáuregui A, Melnikov Y, Mondragón A and Pronko G P 2001 Gamow vectors for barrier wells *Chaos Solitons and Fractals* **12** 2719–36
- [36] Garcia-Pomar J L, Nikitin A Y and Martin-Moreno L 2013 Scattering of graphene plasmons by defects in the graphene sheet *ACS Nano* **7** 4988–94
- [37] Boya L J 2008 Quantum mechanical scattering in one dimension *Nuovo Cimento Rivista Serie* **31** 020000–139
- [38] Scott T C, Aubert-Frécon M and Grotendorst J 2006 New approach for the electronic energies of the hydrogen molecular ion *Chem. Phys.* **324** 323–38
- [39] Scott T C, Mann R and Martinez R E II 2006 General relativity and quantum mechanics: towards a generalization of the lambert W function *Appl. Algebra Eng., Commun. Comput.* **17** 41–7
- [40] Nussenzveig H M 1972 *Causality and Dispersion Relations* (New York: Academic Press)