



# Bosonic $D = 11$ supergravity from a generalized Chern–Simons action

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## Abstract

It is shown that the action of the bosonic sector of  $D = 11$  supergravity may be obtained by means of a suitable scaling of the originally dimensionless fields of a generalized Chern–Simons action. This follows from the eleven-form CS-potential of the most general linear combination of closed, gauge invariant twelve-forms involving the  $sp(32)$ -valued two-form curvatures supplemented by a three-form field. In this construction, the role of the skewsymmetric four-index auxiliary function needed for the first order formulation of  $D = 11$  supergravity is played by the gauge field associated with the five Lorentz indices generator of the bosonic  $sp(32)$  subalgebra of  $osp(1|32)$ .

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## 1. Introduction

It is known [1–4] that various  $D = 3$  (super)gravities are actually Chern–Simons (CS) theories based on Lie superalgebras. Although supergravities in  $D > 3$ ,  $D$  odd, do not have a true CS nature, it has been argued that certain CS theories may be related to supergravities for odd  $D > 3$  dimensions. These CS theories have been generically called ‘CS supergravities’ [5–7] (see [8] for further references).

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CS actions are constructed (see *e.g.* [9]) as follows. Let  $A^i, F^i$  ( $i = 1, \dots, \dim \mathcal{G}$ ) be the Maurer–Cartan (MC) gauge fields and curvatures associated with a Lie algebra  $\mathcal{G}$  in a certain basis. Then, the  $2\ell$ -form (the exterior product symbol  $\wedge$  will be omitted throughout)

$$H = k_{i_1 \dots i_\ell} F^{i_1} \dots F^{i_\ell}, \tag{1.1}$$

where  $k_{i_1 \dots i_\ell}$  are the coordinates of a symmetric invariant tensor of order  $\ell$ , is closed and gauge invariant. Since a gauge free differential algebra is contractible,  $H$  is also exact,  $H = dB$ , and the potential  $B$  defines a Chern–Simons  $(2\ell - 1)$ -form, which is gauge invariant up to an exterior differential. Then, the CS action is given by the integral

$$I_{CS} = \int_{\mathcal{M}^{2\ell-1}} B \tag{1.2}$$

over a  $(2\ell - 1)$ -dimensional manifold  $\mathcal{M}^{2\ell-1}$ ; it is gauge invariant up to non-trivial topological situations ignored in this paper.

The possible connection between CS supergravity and the actual supergravities for  $D > 3$  suggested in refs. [10–13] (see [14] for another connection in  $D = 11$  based on the comparison of the linearized models) is best analyzed by expressing the gauge fields and curvatures associated with the superalgebra  $\mathcal{G}$  in terms of supermatrices  $\mathbb{A}$  and  $\mathbb{F}$ , with one- and two-form fields entries respectively. This is the case for  $D = 3$  and  $\mathcal{G} = osp(p|2) \oplus osp(q|2)$ , for  $D = 5$  and  $\mathcal{G} = su(1|2, 2)$  and for  $D = 11$  and  $\mathcal{G} = osp(1|32)$  (or  $osp(1|32) \oplus osp(1|32)$ ).  $H$  is typically of the form  $H = \text{Tr}(\mathbb{F}^\ell)$  where  $\text{Tr}$  denotes the supertrace, although other non-primitive, closed gauge invariant forms will be considered below. Depending on the case, the MC one-form gauge fields of the superalgebras may, or may not, correspond to the fields of  $D$ -dimensional supergravities. In the second and almost general case, the association between ‘CS supergravities’ and the standard supergravities in  $D$  dimensions fails. Let us show this by summarizing the  $D = 3, 5$  and  $11$  cases.

We use mostly plus metric throughout.

### 1.1. The $D = 3$ case

Let us first consider the simplest algebra  $\mathcal{G} = osp(1|2) \oplus sp(2)$  (*i.e.*  $p = 1, q = 0$  above). The  $osp(1|2)$  and  $sp(2)$  gauge fields, denoted  $\mathbb{A}$  and  $\tilde{\mathbb{A}}$  respectively, can be written in matrix form as

$$\mathbb{A} = \begin{pmatrix} f & \xi \\ \tilde{\xi} & 0 \end{pmatrix}, \quad f = f_a \gamma^a; \quad \tilde{\mathbb{A}} = \tilde{f}, \quad \tilde{f} = \tilde{f}_a \gamma^a, \tag{1.3}$$

where  $\xi$  is a two-component Grassmann odd Majorana spinor form and  $\gamma^a$  are the  $2 \times 2$   $D = 2$  gamma matrices. Note that  $osp(1|2)$  alone would not provide enough fields for  $D = 3$  supergravity and that all fields  $f_a, \xi$  and  $\tilde{f}_a$  in (1.3) are necessarily dimensionless; to define ‘physical’ one-form fields, we introduce a scale parameter  $\lambda, [\lambda] = L^{-1}$ . We use geometrized units for which  $c = 1 = G$ , so that all the quantities have physical dimensions in terms of powers of length; with them, the dimensions of an action in  $D$ -dimensional spacetime is  $L^{(D-2)}$ . The new fields  $\omega_a, e_a$ , and  $\psi$  obtained from  $f, \xi$  and  $\tilde{f}$  are then defined by

$$f_a = \omega_a + \lambda e_a, \quad \tilde{f}_a = \omega_a, \quad \xi = \lambda^{\frac{1}{2}} \psi, \tag{1.4}$$

so that they have the right dimensions  $[\omega_a] = L^0, [e_a] = L^1$  and  $[\psi] = L^{\frac{1}{2}}$  to be identified with the fields of  $D = 3, N = 1$  supergravity in the first order formulation.

The action is constructed starting from the closed, invariant polynomial four-form

$$H(f, \tilde{f}, \xi; \alpha) = \text{Tr}(\mathbb{F}^2) + \alpha \text{Tr}(\tilde{\mathbb{F}}^2), \tag{1.5}$$

where  $\alpha$  is a dimensionless constant and

$$\mathbb{F} = d\mathbb{A} + \mathbb{A}^2 = \begin{pmatrix} df + f^2 + \xi \tilde{\xi} & d\xi + f\xi \\ d\tilde{\xi} + \tilde{\xi} f & 0 \end{pmatrix}, \quad \tilde{\mathbb{F}} = d\tilde{f} + \tilde{f}^2. \tag{1.6}$$

Inserting (1.4) into (1.5) and collecting the terms in equal powers of  $\lambda$  gives

$$H(\omega, e, \psi; \lambda, \alpha) = H_0 + \lambda H_1 + \lambda^2 H_2 + \lambda^3 H_3, \tag{1.7}$$

where  $H_0 = H_0(\omega, \alpha)$  only since  $H(\omega, e, \psi; \lambda, \alpha)$  is dimensionless and  $H_{1,2,3} \neq H_{1,2,3}(\alpha)$ . We note in passing that this re-scaling in  $\lambda$  is the starting point of the (super)Lie algebra expansions procedure, introduced in [15] and considered in general in [16], by which new (super)algebras may be obtained from a given one. Note that, unlike in the contraction of algebras, where the dimensions of the original algebra and that of the contracted one are necessarily equal, the dimension of the expanded algebra is usually higher since the expansion process is not dimension-preserving in general<sup>1</sup> (see [16,17] for details).

By construction, the above two-form  $H$  and the associated CS action are  $osp(1|2) \oplus sp(2)$  gauge-invariant. In particular, the local supersymmetry transformations under the odd dimensionless gauge parameter  $\eta$  that corresponds to the gauge field  $\xi$  are, written in terms of  $\epsilon = \lambda^{-\frac{1}{2}}\eta$ ,  $[\epsilon] = L^{1/2}$ ,

$$\begin{aligned} \delta_\epsilon e^a &= \bar{\psi} \gamma^a \epsilon, \\ \delta_\epsilon \psi &= D\epsilon + \lambda e_a \gamma^a \epsilon, \\ \delta_\epsilon \omega^a &= 0, \end{aligned} \tag{1.8}$$

where  $D = d - \omega_a \gamma^a$  is the Lorentz covariant derivative. Since  $\omega^a$  is supersymmetry invariant, so is  $H_0$  which only contains this field. Thus, the action obtained from  $H(\omega, e, \psi; \lambda, \alpha) - H_0(\omega, \alpha)$  is invariant under the local supersymmetry transformations (1.8), and provides the first order formulation of (1, 0)  $D = 3$  AdS supergravity. Moreover, the leading  $\lambda$  term in  $H - H_0$ ,  $H_1$ , is also invariant under the transformations (1.8) for  $\lambda = 0$ , and hence provides the action for  $D = 3$  Poincaré supergravity; this will not be the case for higher  $D$ . Also, as noted in [3], in the general  $(p, q)$  case the action contains a term that comes from  $H_0$  which is not invariant under the  $\epsilon$  gauge transformation that cannot be ignored and the linear term in  $\lambda$  does not yield Poincaré supergravity. In this case, a proper Poincaré limit may still be taken by enlarging  $\mathcal{G}$  as  $\mathcal{G}$  to  $osp_+(p|2) \oplus osp_-(q|2) \oplus so(p) \oplus so(q)$ , and adding to  $H$  the two invariant  $so(p)$  and  $so(q)$ -valued four-forms [4,18].

### 1.2. The $D = 5$ case

The next simplest case is  $D = 5$ . The smallest real superalgebra that contains the  $AdS_5$  one  $so(4, 2) \sim su(2, 2)$  is the 24-dimensional  $\mathcal{G} = su(1|2, 2)$ . A  $su(1|2, 2)$ -valued form can be written in the form

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<sup>1</sup> It is terminologically unfortunate that algebras of different dimensions are sometimes said to be related by so-called ‘generalized’. Inönü–Wigner contractions. There are, of course, generalizations of the original I–W contraction procedure with respect to a subalgebra, but these are also dimension-preserving, as it corresponds to the mathematical idea of contraction (see e.g. [17]).

$$\mathbb{A} = \begin{pmatrix} f & \xi \\ i\bar{\xi} & 4if_0 \end{pmatrix}, \quad f = if_0 + f_a\gamma^a + \frac{1}{4}f_{ab}\gamma^{ab}; \quad \mathbb{F} = d\mathbb{A} + \mathbb{A}^2, \tag{1.9}$$

where  $\gamma^a$ ,  $a = 0, \dots, 4$  are  $4 \times 4$  gamma matrices,  $\xi$  is a four-component spinor form and  $\bar{\xi}$  its adjoint. Let us introduce again  $\lambda$ ,  $[\lambda] = L^{-1}$ , and new fields  $e_a$ ,  $\phi$ ,  $\omega_{ab}$  and  $\psi$ , with dimensions 1, 1, 0 and 1/2 respectively, through the scalings  $f_0 = \lambda\phi$ ,  $f_a = \lambda e_a$ ,  $f_{ab} = \omega_{ab}$ ,  $\xi = \lambda^{\frac{1}{2}}\psi$ . We now express the 16 real bosonic fields  $1(\phi) + 5(e) + 10(\omega)$  and the 4 complex fermionic ones  $\psi$  (8 real) associated with the supergroup parameters in the form

$$f = i\lambda\phi + \lambda e_a\gamma^a + \frac{1}{4}\omega_{ab}\gamma^{ab}, \quad \xi = \lambda^{\frac{1}{2}}\psi. \tag{1.10}$$

Using these expressions in  $\mathbb{F}$  and  $H = \text{Tr}(\mathbb{F}^3)$  and collecting the different powers in  $\lambda$  we obtain

$$H(\phi, e, \omega, \psi) = H_0 + H_1\lambda + H_2\lambda^2 + H_3\lambda^3 + H_4\lambda^4 + H_5\lambda^5, \tag{1.11}$$

where  $H_0 = H_0(\omega)$  and  $H_i$ ,  $i = 1, \dots, 5$ , depend on the gauge fields  $e_a$ ,  $\phi$ ,  $\omega_{ab}$  and  $\psi$ .

The term  $H_3$  in  $\lambda^3$  has the right dimension  $[H_3] = L^{D-2} = L^3$  for a  $D = 5$  action. Therefore, it makes sense comparing the CS action obtained from  $H_3$  with that of simple  $D = 5$  supergravity which, in the first order formulation, has the same spacetime fields content; including also the terms proportional to  $\lambda^4$  and  $\lambda^5$  and retaining only the last three terms would lead (removing a common  $\lambda^3$  factor) to an action with a ‘cosmological constant’ term in  $\lambda^2$  coming from  $H_5$  (as it would be similarly the case taking the higher order terms in  $D = 3$  [1]). However, here there is no reason why local supersymmetry should be preserved by selecting any group of terms in (1.11): since the  $su(1|4) \in$  gauge transformations in terms of the rescaled fields depend on  $\lambda$ ,

$$\begin{aligned} \delta_\epsilon \phi &= -\frac{1}{4}(\bar{\epsilon}\psi - \bar{\psi}\epsilon) \\ \delta_\epsilon e^a &= -\frac{i}{4}(\bar{\epsilon}\gamma^a\psi - \bar{\psi}\gamma^a\epsilon) \\ \delta_\epsilon \omega^{ab} &= \frac{i\lambda}{2}(\bar{\epsilon}\gamma^{ab}\psi - \bar{\psi}\gamma^{ab}\epsilon) \\ \delta_\epsilon \psi &= d\epsilon + \frac{1}{4}\omega_{ab}\gamma^{ab}\epsilon + \lambda(-3i\phi + e_a\gamma^a)\epsilon, \end{aligned} \tag{1.12}$$

the individual terms are not invariant separately. The leading  $H_0$  term will be invariant under the above gauge algebra for  $\lambda = 0$ , but this will not be the case for the other terms including the one with the correct dimension  $H_3$ . In fact, it is easily seen that the  $H_3$  term in (1.11) does not lead to  $D = 5$  supergravity. The quickest way to see it is by noticing that this  $H_3$  term coming from the  $su(1|4)$  based CS action is not gauge invariant under the one-dimensional subgroup of transformations  $\varphi$  corresponding to the field  $\phi$ ,  $\delta_\varphi\phi = d\varphi$ , in contrast with the action of the  $D = 5$  supergravity.

### 1.3. The $D = 11$ case: preliminary considerations

The  $D = 11$  AdS algebra  $so(2, 10)$  is contained in  $sp(32)$ , which is of dimension  $(32 + 1) \cdot 16$ . The relevant superalgebra in this case would be, in principle, the smallest one that contains  $sp(32)$ , namely  $osp(1|32)$ , of dimension  $528 + 32 = 560$ . A convenient way of describing its elements is provided by the  $osp(1|32)$ -valued one-form gauge field supermatrix  $\mathbb{A}$  given by

$$\mathbb{A} = \begin{pmatrix} f & \xi \\ \bar{\xi} & 0 \end{pmatrix}, \quad f = f^a\gamma_a + \frac{1}{4}f^{ab}\gamma_{ab} + f^{a_1\dots a_5}\gamma_{a_1\dots a_5}, \tag{1.13}$$

where  $\gamma_a$  are the  $32 \times 32$  gamma matrices and  $\xi$  is a 32-component Majorana spinor one-form. Clearly, the  $osp(1|32)$ -valued one-forms in (1.13) cannot be identified with the one-form fields  $e_a, \omega_{ab}, \psi^\alpha$  and the three-form field  $A$  of Cremmer–Julia–Scherk (CJS)  $D = 11$  supergravity [19].

One could think of using two copies [10]  $osp(1|32), \widetilde{osp}(1|32)$ , to write the gauge fields  $f^a, \tilde{f}^a, f^{ab}, \tilde{f}^{ab}, f^{a_1 \dots a_5}, \tilde{f}^{a_1 \dots a_5}, \xi^\alpha, \tilde{\xi}^\alpha$ , as linear combinations of new fields  $e^a, B^a, \omega^{ab}, B^{ab}, B^{a_1 \dots a_5}, B'^{a_1 \dots a_5}, \psi^\alpha, \psi'^\alpha$ , with dimension  $L$  except for  $[\psi^\alpha] = L^{\frac{1}{2}}, [\psi'^\alpha] = L^{\frac{3}{2}}, [\omega^{ab}] = L^0$  (and perhaps  $B'^{a_1 \dots a_5}$  and  $B^a$ ) using the scale factor  $\lambda, [\lambda] = L^{-1}$ . It was conjectured in [10] that the three-form field  $A$  could be a composite of  $e^a, B^{ab}, B^{a_1 \dots a_5}, \psi^\alpha, \psi'^\alpha$  as explicitly considered in [20]. A closed,  $osp(1|32)$  gauge invariant twelve-form  $H$  has the general expression

$$H = \text{Tr}(\mathbb{F}^6) + \alpha \text{Tr}(\mathbb{F}^2)\text{Tr}(\mathbb{F}^4) + \beta \left( \text{Tr}(\mathbb{F}^2) \right)^3, \tag{1.14}$$

where  $\mathbb{F} = d\mathbb{A} + \mathbb{A}^2$ . The corresponding form  $\tilde{H}$  for  $\widetilde{osp}(1|32)$  is expressed similarly in terms of  $\tilde{\mathbb{F}} = d\tilde{\mathbb{A}} + \tilde{\mathbb{A}}^2$ . Then, introducing  $H'(\lambda) = H(\lambda) + \tilde{H}(\lambda)$  and collecting the different powers of  $\lambda$  we can write

$$H'(\lambda) = H(\lambda) + \tilde{H}(\lambda) = H'_0 + \dots + H'_9 \lambda^9 + H'_{10} \lambda^{10} + H'_{11} \lambda^{11}. \tag{1.15}$$

It was conjectured [10] that the  $H'_0$  term would depend on  $\omega^{ab}, e^a$  and  $\psi^\alpha$ , with the remaining fields either included in  $A = A(e^a, B^{ab}, B^{a_1 \dots a_5}, \psi^\alpha, \psi'^\alpha)$  or absent, and that it would also be invariant under local supersymmetry. However, this has not been verified, and there are arguments against this being the case. First, the bosonic and fermionic on-shell degrees of freedom do not match unless there is a large hidden extra gauge symmetry. To be more precise, let us consider Horava’s choice of  $osp(1|32) \oplus \widetilde{osp}(1|32)$  and possible gauge action depending on  $e^a, B^a, \omega^{ab}, B^{ab}, B^{a_1 \dots a_5}, B'^{a_1 \dots a_5}, \psi^\alpha$  and  $\psi'^\alpha$  with the following assumptions: (a) the action corresponding to  $H'_0$  has the gauge symmetries of the above fields realized in the generic form  $\delta A^i = d\alpha^i + \dots$ ; (b) the fields  $B^a$  and  $B'^{a_1 \dots a_5}$ , which do not enter in  $A$ , are also absent in  $H'_0$ , so that we can ignore them; (c) the field equations of  $\omega^{ab}$  can be used to eliminate the  $\omega^{ab}$ ; (d) the linearized field equations for the elfbein  $e^a$  and the gauge one-form fields  $B^{ab}$  and  $B^{a_1 \dots a_5}$  have a structure similar to the  $e^a$  equation of  $D = 11$  supergravity and (e) the linearized field equations for  $\psi^\alpha$  and  $\psi'^\alpha$  are linearized Rarita–Schwinger equations. With these assumptions, the counting of on-shell degrees of freedom goes as follows:

$$\begin{pmatrix} e^a_\mu & (\psi^\alpha_\mu, \psi'^\alpha_\mu) & B^{ab}_\mu & B^{a_1 \dots a_5}_\mu \\ 9 \cdot 11 - 55 & \frac{32 \cdot 8}{2} \text{ each} & 9 \cdot \binom{11}{2} & 9 \cdot \binom{11}{5} \end{pmatrix}, \tag{1.16}$$

*i.e.* there are 4697 bosonic and 256 fermionic degrees of freedom.<sup>2</sup> But  $D = 11$  supergravity has  $44 + 84 = 128$  bosonic and 128 fermionic degrees of freedom, so that for  $H'_0$  to lead to CJS supergravity there should be 128 fermionic and 4569 bosonic extra hidden gauge symmetries.

Secondly, there is no reason why the  $H'_0$  term in the expansion (1.15) of the right dimension  $L^9$  should correspond to a locally supersymmetric action. Besides, the local supersymmetry

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<sup>2</sup> The vielbein  $e^a_\mu$  and Rarita–Schwinger  $\psi^\alpha_\mu$  fields in  $D$  dimensions have, respectively,  $(D - 2)D - \binom{D}{2} = \frac{1}{2}(D - 1)(D - 2) - 1$  (after using local Lorentz invariance) and  $\frac{1}{2}(D - 3)2^{[D/2]}$  on-shell degrees of freedom. Similarly, a  $p$ -form gauge field  $A_{\mu_1 \dots \mu_p}$  has  $\binom{D-2}{p}$  on-shell *d.o.f.*; the  $B$ ’s above are one-form gauge fields with additional antisymmetric  $a$  indices.

transformations of  $D = 11$  supergravity are not  $osp(1|32)$  gauge transformations, but rather local superspace transformations of the component fields the commutators of which close on-shell only (see, for instance, [21]). It is thus unclear how the  $osp(1|32)$  gauge transformations could lead to these local superspace transformations after selecting the  $H'_9$  term in (1.15).

A second problem is the three-form field  $A$  in the action of CJS supergravity. For  $A$  to be a composite field,  $A = A(e^a, B^{ab}, B^{a_1 \dots a_5}, \psi^\alpha, \psi'^\alpha)$ , the supersymmetry algebra of the  $H'_9$  term in (1.15) would have to be related with the algebra defined by the MC equations including the one-form gauge fields appearing in the expression of a composite  $A$ . A natural candidate for a supersymmetry algebra would be a contraction of  $osp(1|32) \oplus \overline{osp}(1|32)$  but, as shown in [22], there is no way of obtaining by contraction the algebras given in [20,23] that allow for a one-forms decomposition of the CJS supergravity three-form field  $A$ .

As we have seen, already in the  $D = 5$  case where there is no  $A$  complicating matters, the CS action does not lead to  $D = 5$  supergravity. So it is hard to imagine why moving to  $D = 11$  would improve the situation so that supersymmetry is preserved after selecting the proper  $H'_9$  term in the expansion (1.15). Further, if there were such a mechanism, working only in  $D = 11$  and ensuring local supersymmetry after taking a non-leading term, it would presumably also apply to the  $H'_{10}$  and  $H'_{11}$  terms in (1.15); again, this would yield a  $D = 11$  supergravity with a cosmological constant, which has been shown not to exist [24].

The  $D = 11$  case is more convoluted than the  $D = 5$  one not only due to the three-form field  $A$ , but also because of the auxiliary zero-form field  $F_{a_1 \dots a_4}$  which has to be added in the first order formulation of  $D = 11$  supergravity, which is the one that would naturally appear from a CS action. But even if these difficulties were overcome, the  $D = 5$  case already tells us that the resulting action would not be locally supersymmetric. In fact, an attempt made in [13] using just one  $osp(1|32)$  algebra, ignoring  $A$  and  $F_{a_1 \dots a_4}$  and keeping only  $e_a$ ,  $\omega_{ab}$  and  $\psi^\alpha$ , supports this conclusion.

One may consider adding separately an  $osp(1|32)$ -gauge invariant dimensionless three-form field  $\mathcal{A}$  to look for an action involving the fields of a single  $osp(1|32)$ . The additional  $\mathcal{A}$  is inert under  $osp(1|32)$  gauge transformations and, under two-form gauge transformations  $\Lambda$ ,  $\mathcal{A}$  transforms as  $\delta_\Lambda \mathcal{A} = d\Lambda$ ; thus, the four-form  $d\mathcal{A}$  is  $\delta_\Lambda$ -gauge invariant. Then, the general gauge invariant twelve-form  $H(\mathbb{F}, \mathcal{A})$  (cf. (1.14)) is given by

$$\begin{aligned}
 H = & Tr(\mathbb{F}^6) + \alpha Tr(\mathbb{F}^4)Tr(\mathbb{F}^2) + \beta \left( Tr(\mathbb{F}^2) \right)^3 + \nu Tr(\mathbb{F}^4)d\mathcal{A} \\
 & + \delta \left( Tr(\mathbb{F}^2) \right)^2 d\mathcal{A} + \rho Tr(\mathbb{F}^2)(d\mathcal{A})^2 + \sigma (d\mathcal{A})^3, \tag{1.17}
 \end{aligned}$$

where  $\alpha, \dots, \sigma$  are dimensionless constants.

An action with the right dimensions would correspond to the  $H_9$  term in the expression above with  $\mathcal{A} = \lambda^3 A$ ,  $[A] = L^3$ . However, this construction still would not explain the need for the auxiliary  $F_{a_1 \dots a_4}$  fields. In fact, one of the results of this paper is that, since contractions do not appear to play a role in the present problem, the field re-scalings need not being those that allow for a consistent  $\lambda \rightarrow 0$  limit. Once  $f_a = \lambda e_e$  is chosen, consistency of the contraction limit would require a new field,  $e_{a_1 \dots a_5}$  say, with  $f_{a_1 \dots a_5} = \lambda B_{a_1 \dots a_5}$ , so that the  $osp(1|32)$  MC equations

$$df^a \propto \epsilon^{ab_1 \dots b_5 c_1 \dots c_5} f_{b_1 \dots b_5} f_{c_1 \dots c_5} + \dots \tag{1.18}$$

have a well defined  $\lambda \rightarrow 0$  limit. But, if this consistency condition is removed, we may now set  $f_{a_1 \dots a_5} = \omega_{a_1 \dots a_5}$ ,  $[\omega_{a_1 \dots a_5}] = L^0$ , (rather than  $f_{a_1 \dots a_5} = \lambda B_{a_1 \dots a_5}$ , which implies  $[B_{a_1 \dots a_5}] = L^1$ ). Indeed, it will be shown that the  $\omega_{a_1 \dots a_5}$  fields play the role of the  $F_{a_1 \dots a_4}$  (see below eq. (3.61)).

Unfortunately, a calculation shows that the  $\lambda^9$  term in the expansion of this new, generalized CS action is not  $D = 11$  supergravity (in particular, the fermion equation will not correspond to the spinor equation for CJS supergravity). This was to be expected since, again, there is no reason for this term to be invariant under supersymmetry gauge transformations.

Nevertheless, we will show below that our construction for the fields associated with the bosonic part of a  $osp(1|32)$ , supplemented by the three-form  $\mathcal{A}$ , does work for the bosonic sector of  $D = 11$  supergravity. In other words, there are constants  $\alpha, \dots, \sigma$  in (1.17) such that the  $H_9$  term in  $H$  resulting from the re-scalings  $f^a = \lambda e^a$ ,  $f^{ab} = \omega^{ab}$ ,  $f^{a_1 \dots a_5} = \omega^{a_1 \dots a_5}$  and  $\mathcal{A} = \lambda^3 A$  lead to the equations of its bosonic sector. In particular, the  $\omega^{a_1 \dots a_5}$  equation determines  $\omega^{a_1 \dots a_5}$  itself in terms of the coordinates of  $dA = (dA)_{a_1 \dots a_4} e^{a_1} \dots e^{a_4}$ ,

$$\omega_{a_1 \dots a_5} \propto (dA)_{[a_1 \dots a_4} e_{a_5]}, \tag{1.19}$$

so that  $\omega_{a_1 \dots a_5}$  plays the role of the auxiliary zero-forms of  $D = 11$  supergravity. In this way, the fact that the  $D = 11$  supergravity action contains a generalized ‘CS term’ for the field  $A$ , the eleven-form  $A dA dA$ , is incorporated into the full bosonic action through the sum of powers of  $\lambda$  described above. This result also extends others in refs. [12,13] in which standard pure gravity with just  $\omega^{ab}$  and  $e^a$ , without the fields  $\phi$  in  $D = 5$  and  $A$  in  $D = 11$ , is derived from a CS action in these odd dimensions.

The plan of the paper is as follows. The ‘generalized CS action’ is defined in Sec. 2, where its expression in powers of the scale factor  $\lambda$  is given. Then, we study in Sec. 3 the field equations of the model and compare them with those of the bosonic sector of supergravity. We end with some conclusions and further comments. Some calculations are relegated to Appendix A.

## 2. The generalized $sp(32)$ Chern–Simons action

### 2.1. $sp(32)$ Cartan structure equations and gauge transformations

In terms of its MC forms  $f^\alpha_\beta$ ,  $\alpha, \beta = 1, \dots, 32$ , the  $sp(32)$  algebra is defined by

$$df^\alpha_\beta = -f^\alpha_\gamma \wedge f^\gamma_\beta, \quad df = -f^2. \tag{2.20}$$

Using the symplectic metric  $C_{\alpha\gamma} = -C_{\gamma\alpha}$ ,  $f_{\alpha\beta}$  is given by

$$f_{\alpha\beta} = C_{\alpha\gamma} f^\gamma_\beta, \quad f_{\alpha\beta} = f_{\beta\alpha}. \tag{2.21}$$

Since  $f_{\alpha\beta}$  is a  $32 \times 32$  symmetric matrix, it can be expanded in the basis of  $(\alpha\beta)$ -symmetric matrices given by ‘weight one’ antisymmetrized products of  $D = 11$  Dirac matrices as

$$f_{\alpha\beta} = f_a \gamma_{\alpha\beta}^a + \frac{1}{4} f_{ab} \gamma_{\alpha\beta}^{ab} + f_{a_1 \dots a_5} \gamma_{\alpha\beta}^{a_1 \dots a_5}. \tag{2.22}$$

The  $1/4$  factor is introduced to obtain the usual relation between the spin connection and its curvature (eq. (3.71)) as well as the definition of the torsion (eq. (3.45)).

Gauge curvatures are introduced by moving from the MC equations (zero curvature) to the Cartan structure ones, in which the  $sp(32)$  curvatures express the failure of  $f$  to satisfy the  $sp(32)$  algebra MC equations. Let  $\Omega$  be the two-form matrix incorporating the curvatures. Then,

$$\Omega = \mathcal{D}f = df + f^2, \tag{2.23}$$

where  $f$  contains the one-form gauge fields, and

$$d\Omega = \Omega f - f \Omega = [\Omega, f], \tag{2.24}$$

is the Bianchi identity  $\mathcal{D}\Omega = d\Omega + [f, \Omega] \equiv 0$  for the  $sp(32)$  connection  $f$ . As  $f$ , the curvature  $\Omega$  may be similarly expressed as

$$\Omega_{\alpha\beta} = \Omega_a \gamma_{\alpha\beta}^a + \frac{1}{4} \Omega_{ab} \gamma_{\alpha\beta}^{ab} + \Omega_{a_1 \dots a_5} \gamma_{\alpha\beta}^{a_1 \dots a_5} . \tag{2.25}$$

The infinitesimal gauge transformations of  $f, \Omega$  are given by the standard expressions,

$$\delta_b f = db + fb - bf = db + [f, b] , \quad \delta_b \Omega = \Omega b - b\Omega = [\Omega, b] , \tag{2.26}$$

where the zero-form matrix  $b = b^\alpha_\beta$  contains the gauge functions

$$b = b_a \gamma^a + \frac{1}{4} b_{ab} \gamma^{ab} + b_{a_1 \dots a_5} \gamma^{a_1 \dots a_5} . \tag{2.27}$$

### 2.2. Generic expression for a CS-type action

Since the bosonic sector of  $D = 11$  supergravity contains the three-form field  $A$ , we add it explicitly to the one-form  $sp(32)$  fields by introducing the three-form  $\mathcal{A}$  inert under  $sp(32)$   $\delta_b$  gauge transformations and under  $\delta_\Lambda$  ones. Thus, the most general twelve-form  $H(\Omega, \mathcal{A})$ , closed and invariant under both  $\delta_b$  and  $\delta_\Lambda$  gauge transformations, may be written as

$$H = Tr(\Omega^6) + \alpha Tr(\Omega^4) Tr(\Omega^2) + \beta \left( Tr(\Omega^2) \right)^3 + \nu Tr(\Omega^4) d\mathcal{A} + \delta \left( Tr(\Omega^2) \right)^2 d\mathcal{A} + \rho Tr(\Omega^2) (d\mathcal{A})^2 + \sigma (d\mathcal{A})^3 , \tag{2.28}$$

where the bosonic  $\Omega$  has replaced  $\mathbb{F}$  in eq. (1.17), in which fermions were present. Then, the integral

$$I = \int_{\mathcal{M}^{11}} B, \quad dB = H , \tag{2.29}$$

may be used to obtain a CS-type action.

Our task now is to extract from eq. (2.28) the physically relevant terms (it will turn out that only the first term  $Tr(\Omega^6)$  and those in  $\nu$  and  $\sigma$  will contribute) and to fix their corresponding coefficients so that the resulting action determines the equations of motion for the bosonic sector of supergravity. Because of the presence of the three-form  $\mathcal{A}$ , this action will be referred to as the *generalized CS action* for the bosonic sector of  $D = 11$  supergravity.

### 2.3. Generalized CS action for the bosonic sector of $D = 11$ supergravity

Again, the component fields in the one-form  $f$ , the two-form  $\Omega$  and the three-form  $\mathcal{A}$  field are dimensionless. Dimensions are introduced by setting

$$\mathcal{A} = \lambda^3 A , \quad [A] = L^3 , \tag{2.30}$$

$$f = \lambda e_a \gamma^a + \frac{1}{4} \omega_{ab} \gamma^{ab} + \omega_{a_1 \dots a_5} \gamma^{a_1 \dots a_5} , \tag{2.31}$$

where in (2.22) we set



$$\begin{aligned}
 f_a &= \lambda e_a, & [e_a] &= L, \\
 f_{ab} &= \omega_{ab}, & [\omega_{ab}] &= L^0, \\
 f_{a_1 \dots a_5} &= \omega_{a_1 \dots a_5}, & [\omega_{a_1 \dots a_5}] &= L^0.
 \end{aligned}
 \tag{2.32}$$

With our mostly plus metric we use real gamma matrices such that  $\gamma^{a_1 \dots a_{11}} = \epsilon^{a_1 \dots a_{11}}$ . Besides the 1/4 factor in (2.31) that was fixed in (2.22), there is no special reason for the factors accompanying the fields  $e_a, \omega_{a_1 \dots a_5}$  and  $A$ . Different coefficients would lead to different values for the constants  $\alpha, \dots, \sigma$  in (2.28) after requiring that the action corresponds to the bosonic sector of supergravity. Thus, these constants depend on the way the fields are introduced and will not affect the final result. Keeping this in mind, we now look for the relevant terms and their coefficients for the particular choices in (2.31), (2.30).

An action for  $D = 11$  gravity has dimensions  $L^{D-2} = L^9$ . Thus, writing now  $H|_i$  for  $H_i$  and expressing the twelve-form  $H$  in (2.28) and the eleven-form  $B$  in powers of  $\lambda$ , we obtain

$$\begin{aligned}
 H &= H|_0 + \lambda H|_1 + \dots, \\
 B &= B|_0 + \lambda B|_1 + \dots.
 \end{aligned}
 \tag{2.33}$$

Then,  $H|_i = dB|_i$  allows us to write for the different  $I_{GCS}|_i = \int_{\mathcal{M}^{11}} B|_i$ ,

$$I_{GCS} = I_{GCS}|_0 + \lambda I_{GCS}|_1 + \dots.
 \tag{2.34}$$

The physically relevant term is in  $\lambda^9$  since  $[I_{GCS}|_9] = L^9$ . Therefore,  $I_{GCS}|_9 = \int_{\mathcal{M}^{11}} B|_9$ .

We are thus interested in  $H|_9$ . Since  $H$  contains the  $sp(32)$  curvature two-forms  $\Omega_a, \Omega_{ab}, \Omega_{a_1 \dots a_5}$  of (2.25), we need their expressions in terms of  $e_a, \omega_{ab}, \omega_{a_1 \dots a_5}$ . To simplify the calculations, we write

$$\Omega = df + f^2 = \Omega_0 + \lambda \Omega_1 + \lambda^2 \Omega_2,
 \tag{2.35}$$

with  $f$  in (2.31) expressed as

$$f = \lambda e + \omega_L + \omega_5 = \lambda e + \omega,
 \tag{2.36}$$

where  $e = e_a \gamma^a, \omega_L = \frac{1}{4} \omega_{ab} \gamma^{ab}$  is the spin connection,  $\omega_5 = \omega_{a_1 \dots a_5} \gamma^{a_1 \dots a_5}$  and  $\omega = \omega_L + \omega_5$ . In this way, the  $sp(32)$ -valued curvature in (2.35) gives

$$\begin{aligned}
 \Omega &= d(\lambda e + \omega) + (\lambda e + \omega)(\lambda e + \omega) \\
 &= d\omega + \omega^2 + \lambda(de + \omega e + e\omega) + \lambda^2 e^2 \\
 &\equiv R(\omega) + \lambda T + \lambda^2 \Omega_2.
 \end{aligned}
 \tag{2.37}$$

Thus,  $\Omega_0 = R(\omega) = d\omega + \frac{1}{2}[\omega, \omega], \Omega_1 = T(e, \omega) = de + [\omega, e]$  and  $\Omega_2(e) = e^2 = \frac{1}{2}[e, e]$ . Notice that  $T$  contains a piece proportional to  $\gamma^a$  and another proportional to  $\gamma^{a_1 \dots a_5}$ ; similarly, the curvature  $R(\omega)$  contains contributions proportional to  $\gamma^a, \gamma^{ab}$  and  $\gamma^{a_1 \dots a_5}$ , because it depends on both  $\omega_L$  and  $\omega_5$ . The previous equations tell us that to obtain the piece  $H|_9$  that comes e.g. from  $Tr(\Omega^6)$ , one has to consider all the contributions containing a number  $n_0$  of  $R$  factors,  $n_1$  of  $T$  and  $n_2$  of  $\Omega_2$  in such a way that

1.  $n_0 + n_1 + n_2 = 6$  (there are 6 curvatures)
2.  $n_1 + 2n_2 = 9$ ,

where the first condition guarantees that the order of the forms is twelve and the second one that their length dimension is nine. The only two solutions are:

- $n_2 = 4, n_1 = 1, n_0 = 1$ , or
- $n_2 = 3, n_1 = 3, n_0 = 0$

Thus, the  $R, T, \Omega_2$  contributions are of the form

$$Tr(\Omega^6)|_9 = Tr(\mathcal{W}(\Omega_2^4, T, R)) + Tr(\mathcal{W}(\Omega_2^3, T^3, R^0)), \quad (2.38)$$

where e.g.  $\mathcal{W}(\Omega_2^4, T, R)$  is the sum of all nine-form ‘words’ that can be obtained out of four  $\Omega_2$ , one  $T$  and one  $R$ . This would give us the piece  $Tr(\Omega^6)|_9$  of  $H|_9$ . We could now add to (2.38) the contributions to  $H|_9$  coming from the other terms in (2.28), to find an 11-form  $B|_9$  with  $dB|_9 = H|_9$ , and compare with the action of the bosonic sector of  $D = 11$  supergravity. Instead, we will obtain directly the field equations for the action  $\int_{\mathcal{M}} B_9$  from the original, unexpanded  $H$  twelve-form.

### 3. Field equations

The field equations for  $I_{GCS}$  can be obtained directly from  $H$  in a way similar to that used in [16]. To find them, we use the following fact (see [25]): let  $i_{f\alpha\beta}, i_{\Omega\alpha\beta}, i_{\mathcal{A}}$  and  $i_{d\mathcal{A}}$  be the inner derivations associated with the fields and curvatures of the algebra with respect to  $f, \Omega, \mathcal{A}$  and  $d\mathcal{A}$ , defined by

$$i_{f\alpha\beta} f_{\gamma\delta} = \delta_{(\gamma}^{\alpha} \delta_{\delta)}^{\beta}, \quad i_{\Omega\alpha\beta} \Omega_{\gamma\delta} = \delta_{(\gamma}^{\alpha} \delta_{\delta)}^{\beta}, \quad i_{\mathcal{A}} \mathcal{A} = 1, \quad i_{d\mathcal{A}} d\mathcal{A} = 1, \quad (3.39)$$

and zero otherwise. If  $H = dB$  is a form defined on this algebra that defines the action through  $I = \int B$ , then the field equations for  $I$  are given by  $i_{\Omega\alpha\beta} H = 0$  and  $i_{d\mathcal{A}} H = 0$ . Let us denote the equations of motion for  $f$  and  $\mathcal{A}$  by  $E(f) = 0$  and  $E(\mathcal{A}) = 0$  respectively. Then, using (3.39) in (2.28) we obtain

$$E(f) = 6\Omega^5 + 4\alpha Tr(\Omega^2)\Omega^3 + 2\alpha Tr(\Omega^4)\Omega + 6\beta Tr(\Omega^2)^2\Omega + 4\nu d\mathcal{A}\Omega^3 + 4\delta d\mathcal{A}Tr(\Omega^2)\Omega + 2\rho(d\mathcal{A})^2\Omega = 0, \quad (3.40)$$

where  $E(f)$  is a ten-form, and by

$$E(\mathcal{A}) = \nu Tr(\Omega^4) + \delta (Tr(\Omega^2))^2 + 2\rho(d\mathcal{A})Tr(\Omega^2) + 3\sigma(d\mathcal{A})^2 = 0, \quad (3.41)$$

where  $E(\mathcal{A})$  is an eight-form.

We have to extract now from the above the equations for  $e, \omega$  ( $\omega_L$  and  $\omega_5$ ) and  $A$  for the action  $I_{GCS}|_9$ . Proceeding as in [17], where the equations for the dimensionful fields were derived from those for the dimensionless ones by selecting the appropriate powers of  $\lambda$ , they are given by

$$\begin{aligned} E(e) &= (E(f)|_{9-1=8})|_{\gamma^{[1]}}, \\ E(\omega) &= E(f)|_9, \quad E(\omega_L) = (E(f)|_9)_{\gamma^{[2]}}, \quad E(\omega_5) = (E(f)|_9)_{\gamma^{[5]}} \\ E(A) &= E(\mathcal{A})|_{9-3=6} \end{aligned} \quad (3.42)$$

since  $[e] = L^1$ ,  $[\mathcal{A}] = L^3$ ,  $[\omega] = L^0$ , and where the subscripts  $\gamma^{[2,5]}$  refer to the contributions proportional to the antisymmetrization of two and five  $D = 11$  gamma matrices respectively. Eqs. (3.42) constitute the complete set of equations of our bosonic model.

We have to find now  $E(f)|_8, E(\omega_L)|_9, E(\omega_5)|_9$  and  $E(A)|_6$  by taking into account that

$$\Omega R + \lambda T + \lambda^2 \Omega_2, \quad d\mathcal{A} = \lambda^3 dA.$$

### 3.1. Field equation for $\omega$

We need to know the contributions of all terms in equation (3.40), namely all the contributions containing  $n_2$  factors  $\Omega_2$ ,  $n_0$  factors  $R$  and  $n_1$  factors  $T$  in such a way that the order of the form is 10 and its dimension  $L^9$ . Then, we find that the  $\omega$  equation is given by the ten-form expression

$$E(\omega) = E(f)|_9 = 6\mathcal{W}(\Omega_2^4, T) + 4\nu dAe^6 = 0, \tag{3.43}$$

where the first term comes from the first one in eq. (3.40) and the other comes from the  $\nu$  term. Since  $\omega = \omega_L + \omega_5$ , eq. (3.43) contains two different contributions, one proportional to  $\gamma^{a_1 a_2}$  from the first term that gives the equation for  $\omega_L$ , and another proportional to  $\gamma^{a_1 \dots a_5}$  that comes from both terms and gives the equation for  $\omega_5$ . We consider them now.

The first ten-form in (3.43) is

$$\mathcal{W}(\Omega_2^4, T) = e^8 T + e^6 T e^2 + e^4 T e^4 + e^2 T e^6 + T e^8, \tag{3.44}$$

where  $T$  is given (see (2.37), (2.36)) by

$$T = de + [\omega, e] = T_L + [\omega_5, e]; \quad T_L = de + [\omega_L, e], \tag{3.45}$$

and the explicit expression for the torsion  $T_L$  is

$$T_L = T^a \gamma_a = (de^a + \omega^a_b e^b) \gamma_a. \tag{3.46}$$

Then, the first term on the *l.h.s.* of (3.43) can be written as

$$\mathcal{W}(\Omega_2^4, T) = \mathcal{W}((e^2)^2, T_L) + \mathcal{W}(e^9, \omega_5). \tag{3.47}$$

#### 3.1.1. Equation for $\omega_L(\omega_{ab})$

To see how the  $\gamma_{a_1 a_2}$  and  $\gamma_{a_1 \dots a_5}$  contributions come out, note the identity

$$\gamma_a \gamma_{a_1 \dots a_k} = \sum_{i=1}^k (-1)^{i-1} \eta_{aa_i} \gamma_{a_1 \dots \hat{a}_i \dots a_k} + \gamma_{aa_1 \dots a_k}. \tag{3.48}$$

When contracted with the indices of, say  $e^a B^{a_1 \dots a_k}$ , one gets:

$$e^a \gamma_a B^{a_1 \dots a_k} \gamma_{a_1 \dots a_k} = k e^a B_{aa_2 \dots a_k} + \gamma_{aa_1 \dots a_k} e^a B^{a_1 \dots a_k}, \tag{3.49}$$

*i.e.*, all terms in the sum (3.48) add up, and the first term appears  $k$  times. The same pattern exists when two matrices  $\gamma_{a_1 \dots a_k}$ ,  $\gamma_{a_1 \dots a_s}$  are multiplied, but now there are contributions with all possible number of contractions. The  $e^8 T_L$  terms have the structure  $\gamma^{[8]} \cdot \gamma^{[1]}$  (again, the superscripts indicate the number of  $\gamma$ 's in the skewsymmetric products). This gives, schematically,

$$\gamma^{[8]} \cdot \gamma^{[1]} \sim \gamma^{[9]} + \gamma^{[7]} \tag{3.50}$$

where there are no contractions in  $\gamma^{[9]}$  and one in  $\gamma^{[7]}$ . The  $\gamma^{[7]}$  contribution will cancel because only the matrices symmetric in all indices contribute ( $\gamma^{[1,2,5,6,9,10]}$  are symmetric;  $\mathbf{1}$ ,  $\gamma^{[3,4,7,8]}$  skewsymmetric). Thus, only the  $e^8 T_L$  terms appear in the  $\omega_L$  equation since

$$\gamma^{a_1 \dots a_9} \propto \epsilon^{a_1 \dots a_9 ab} \gamma_{ab}. \tag{3.51}$$

In general, since with our metric signature we can choose  $\gamma^{a_1 \dots a_{11}} = \epsilon^{a_1 \dots a_{11}}$ , we have

$$\gamma^{a_{k+1} \dots a_{11}} = \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} \epsilon^{b_1 \dots b_k a_{k+1} \dots a_{11}} \gamma_{b_1 \dots b_k}. \tag{3.52}$$

On the other hand, the terms  $e^9\omega_5$  coming from (3.47) are, again schematically, of the form

$$\gamma^{[9]} \cdot \gamma^{[5]} \sim \gamma^{[14]} + \gamma^{[12]} + \gamma^{[10]} + \gamma^{[8]} + \gamma^{[6]} + \gamma^{[4]} . \tag{3.53}$$

The  $\gamma^{[10]} \sim \gamma^{[1]}$  contribution vanishes because there is no  $\omega_a$ , *i.e.* there is no equation of dimension  $L^9$  with a single Lorentz index. The only symmetric  $\gamma$  is  $\gamma^{[6]}$ . So the  $e^9\omega_5$  terms only appear in the  $\omega_5$  equation. The  $\omega_L$  equations are then

$$(E(f)|_9)_{\gamma^{[2]}} = E(\omega_L) \propto e_{a_1} \dots e_{a_8} (T_L)_{a_9} \gamma^{a_1 \dots a_9} = 0 . \tag{3.54}$$

This equation implies  $T_L = 0$ , which, as usual, can be used to express  $\omega_{ab\mu}$  in terms of  $e^a{}_\mu$  and its derivatives.

3.1.2. Equation for  $\omega_5$  ( $\omega_{a_1 \dots a_5}$ )

This equation has contributions from the two terms in (3.43). One is given by its second term  $4\nu(dA)e^6$  which, due to  $e^6$ , is proportional to  $\gamma^{[6]}$ , and the other is the contribution with four contractions from the terms with nine  $e$  and one  $\omega_5$  from  $\mathcal{W}(e^9, \omega_5)$ , which is also proportional to  $\gamma^{[6]} \sim \gamma^{[5]}$ , contained in the first one,  $6\mathcal{W}(\Omega_2^4, T)$ . A long calculation shows that this second contribution is given by

$$2 \cdot \frac{9!}{4!} e_{a_1} \dots e_{a_5} e^{b_1} \dots e^{b_4} \omega_{b_4 \dots b_1 a_6} \gamma^{a_1 \dots a_6} . \tag{3.55}$$

Taking into account both terms, the  $\omega_5$  equation of motion is found to be

$$(E(f)|_9)_{\gamma^{[5]}} = E(\omega_5) = 12 \cdot \frac{9!}{4!} e_{a_1} \dots e_{a_5} e^{b_1} \dots e^{b_4} \omega_{b_4 \dots b_1 a_6} \gamma^{a_1 \dots a_6} + 4\nu dA e_{a_1} \dots e_{a_6} \gamma^{a_1 \dots a_6} = 0 . \tag{3.56}$$

Let us see what this equation leads to. In terms of the elfbein components of  $dA$ ,

$$dA = (dA)_{b_1 \dots b_4} e^{b_1} \dots e^{b_4} , \tag{3.57}$$

it reads

$$\frac{9!}{2} e_{a_1} \dots e_{a_5} e_{b_1} \dots e_{b_4} e_c \omega^{b_4 \dots b_1}{}_{a_6}{}^c \gamma^{a_1 \dots a_6} + 4\nu (dA)^{b_1 \dots b_4} e_{a_1} \dots e_{a_6} e_{b_1} \dots e_{b_4} \gamma^{a_1 \dots a_6} = 0 , \tag{3.58}$$

where  $\omega^{b_4 \dots b_1}{}_{a_6}{}^c = \omega^{b_4 \dots b_1}{}_{a_6}{}^c e_c$ . We now write the products of ten  $e$ 's above as

$$e_{a_1} \dots e_{a_5} e_{b_1} \dots e_{b_4} e_c = \epsilon_{a_1 \dots a_5 b_1 \dots b_4 c d} E^d ,$$

for some ten-form  $E^d$ . Then, factoring out this form in eq. (3.58) and  $\gamma^{a_1 \dots a_6}$ , we find

$$\frac{9!}{2} \epsilon_{b_1 \dots b_4 c d [a_1 \dots a_5} \omega^{b_4 \dots b_1}{}_{a_6}{}^c + 4\nu \epsilon_{a_1 \dots a_6 b_1 \dots b_4 d} (dA)^{b_1 \dots b_4} = 0 , \tag{3.59}$$

where  $[ ]$  indicates weight one antisymmetrization in  $a_1 \dots a_6$ . It is shown in Appendix A (sec. A.1) that the solution is

$$\omega^{d_1 \dots d_5}{}_d = -\frac{40}{9!} \nu (dA)^{[d_1 \dots d_4} \delta_d^{d_5]} . \tag{3.60}$$

This equation relates the one-form gauge field components  $\omega^{d_1 \dots d_5}$  to those of the four-form  $F = dA$ . It can also be written as

$$\omega^{d_1 \dots d_5} = -\frac{40}{9!} \nu (dA)^{[d_1 \dots d_4} e^{d_5]} . \tag{3.61}$$

Hence,  $\omega^{d_1\dots d_5}$  may be expressed in terms of the coordinates of  $dA$  so that, as anticipated,  $\omega_5$  plays a role analogous to that of the auxiliary zero-forms  $F_{a_1\dots a_4}$  of the first order formulation of  $D = 11$  supergravity, where  $F \propto dA$ .

### 3.2. Field equation for $A$

The sum of the contributions to the field equation (3.41) with the right dimension,  $E(A)|_6 = E(A) = 0$  (see (3.41)), leads to

$$E(A) = 4 \nu 32 e_{a_1\dots a_6} D \omega_{a_7\dots a_{11}} \epsilon^{a_1\dots a_{11}} + 3\sigma (dA)^2 = 0, \tag{3.62}$$

where again  $D$  is the  $\omega_L$  covariant derivative; we see that there is no contribution from the  $\delta$  and  $\rho$  terms. In the  $e^a$  basis, this gives

$$4 \nu 32 e_{a_1\dots a_6} D_{b_1} \omega_{a_7\dots a_{11} b_2} e^{b_1} e^{b_2} \epsilon^{a_1\dots a_{11}} = -3 \sigma (dA)_{b_1\dots b_4} (dA)_{c_1\dots c_4} e^{b_1\dots b_4} e^{c_1\dots c_4}.$$

Now we can introduce the eight-form  $E_{d_1 d_2 d_3} \equiv \epsilon_{d_1\dots d_3 b_1\dots b_8} e^{b_1\dots b_8}$ , and use it to rewrite the factors with eight one-forms  $e^a$ . If the  $E_{d_1 d_2 d_3}$  are then factorized, we obtain

$$4 \nu 32 \cdot 6! \delta_{b_1 b_2 d_1 d_2 d_3}^{a_7\dots a_{11}} D^{b_1} \omega_{a_7\dots a_{11} b_2} = 3 \sigma \epsilon_{b_1\dots b_4 c_1\dots c_4} d_1 d_2 d_3 (dA)^{b_1\dots b_4} (dA)^{c_1\dots c_4}. \tag{3.63}$$

Using the expression (3.60) for  $\omega_{a_7\dots a_{11} b_2}$  in terms of the components of  $dA$ , the *r.h.s* of (3.63) reads

$$\begin{aligned} \delta_{b_1 b_2 d_1\dots d_3}^{a_7\dots a_{11}} D^{b_1} \omega_{a_7\dots a_{11} b_2} &= -\frac{40}{9!} \nu \delta_{b_1 b_2 d_1 d_2 d_3}^{a_7\dots a_{11}} \delta_{a_{11}}^{b_2} D^{b_1} (dA)_{a_7\dots a_{10}} \\ &= -\frac{40}{9!} \nu \delta_{b_1 b_2 d_1 d_2 d_3}^{a_7\dots a_{10} b_2} D^{b_1} (dA)_{a_7\dots a_{10}} \\ &= -(-7) \frac{40}{9!} \nu \delta_{b_1 d_1 d_2 d_3}^{a_7\dots a_{10}} D^{b_1} (dA)_{a_7\dots a_{10}} \\ &= 4! \cdot 7 \frac{40}{9!} \nu D^{b_1} (dA)_{b_1 d_1 d_2 d_3} = \frac{1}{54} \nu D^{b_1} (dA)_{b_1 d_1 d_2 d_3}. \end{aligned}$$

In this way, the final expression for the  $A$  equation of the motion is found to be

$$D^{b_1} (dA)_{b_1 d_1 d_2 d_3} = \left( \frac{9 \sigma}{5120 \nu^2} \right) \epsilon_{b_1\dots b_4 c_1\dots c_4} d_1 d_2 d_3 (dA)^{b_1\dots b_4} (dA)^{c_1\dots c_4}. \tag{3.64}$$

Note that this equation has the form required to reproduce the equations of  $D = 11$  supergravity in the absence of fermions (see [19,21,26]).

### 3.3. Field equation for $e$

We need to know the contributions of all terms in eq. (3.40) again, but now we have to find  $(E(f)|_8)|_{\gamma^{[1]}} = E(e)$  instead of  $E(f)|_9$  in eq. (3.42). Collecting all the possible contributions as explained before, we find that they all come from the first and the  $\nu$  term in eq. (3.40),

$$\begin{aligned} E(e) &= 6\mathcal{W}(\Omega_2^3, T^2)|_{\gamma^{[1]}} + 6\mathcal{W}(\Omega_2^4, R)|_{\gamma^{[1]}} \\ &\quad + 4\nu dA(\Omega_2^2 T + \Omega_2 T \Omega_2 + T \Omega_2^2)|_{\gamma^{[1]}} = 0, \end{aligned} \tag{3.65}$$

where, again,  $|\gamma^{[1]}$  selects the contribution accompanying a single gamma matrix  $\gamma^a$ , or equivalently, a ten indices gamma matrix,  $\gamma^{a_1\dots a_{10}}$ . In particular we need the contributions coming

from the term  $6\mathcal{W}(\Omega_2^3, T^2)|_{\gamma^{[1]}} + 6\mathcal{W}(\Omega_2^4, R)|_{\gamma^{[1]}}$  in (3.65), but this is a very tedious calculation. Instead, it is more convenient to take advantage of the fact that the symmetry of the stress-energy tensor forces its terms to be the result of contracting three or four indices among two  $dA^{\mu\nu\rho\sigma}$  (in the  $dx^\mu$  basis), namely  $(dA)^{\mu\nu\rho}{}_\alpha (dA)_{\mu\nu\rho\beta}$  and  $(dA)^{\mu\nu\rho\sigma} (dA)_{\mu\nu\rho\sigma} g_{\alpha\beta}$ . Hence, Einstein's equations have the form

$$R(\Gamma)_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R(\Gamma) = P (dA)^{\alpha\rho\gamma}{}_\mu (dA)_{\alpha\rho\gamma\nu} + Q (dA)^{\alpha\rho\gamma\delta} (dA)_{\alpha\rho\gamma\delta} g_{\mu\nu} , \tag{3.66}$$

with  $P, Q$  yet to be determined. With the sign for the curvature tensor as in [27],  $R(\Gamma)$  and  $R(\omega_L)$  are related through the elfbein postulate by  $R(\Gamma) = 2R(\omega_L)$ .

The  $P, Q$  constants are now determined using that the covariant derivative of the Einstein tensor is zero,  $\nabla^\mu \left( R(\Gamma)_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R(\Gamma) \right) = 0$ . Then, the *r.h.s* of (3.66) must vanish when the supergravity field equation for the  $A$  field (equivalent to our eq. (3.64)),

$$\nabla^\mu (dA)_{\mu\nu\rho\sigma} \propto \epsilon_{\nu\rho\sigma\lambda_1\dots\lambda_4\tau_1\dots\tau_4} (dA)^{\lambda_1\dots\lambda_4} (dA)^{\tau_1\dots\tau_4} , \tag{3.67}$$

where the proportionality factor is unimportant here, and

$$\partial_{[\mu} (dA)_{\nu\rho\gamma\tau]} = 0 \tag{3.68}$$

( $d(dA) \equiv 0$ ), are used. Indeed, the covariant derivative of the *r.h.s* of eq. (3.66) may be written using (3.68) as a linear combination of  $(dA)^{\rho\sigma\lambda\tau} \nabla_\nu (dA)_{\rho\sigma\lambda\tau}$  and  $(dA)_{\rho\sigma\lambda\nu} \nabla_\mu (dA)^{\rho\sigma\lambda\mu}$ . This last contribution vanishes due to eq. (3.67). Hence, the first contribution also has to vanish and, since it includes a factor  $(P + 8Q)$ , it follows that  $P/Q = -8$  (see, e.g., [28]). Thus, we only need now the overall factor.

To fix it, we take the trace of eq. (3.66) to find the Ricci scalar

$$R(\Gamma)^{\mu\nu}{}_{\mu\nu} = \frac{P}{12} (dA)_{\mu\nu\rho\sigma} (dA)^{\mu\nu\rho\sigma} . \tag{3.69}$$

We still need the value of  $P$  for our action. If we compute the trace of the  $E(e) = 0$  (eq. (3.65)) times  $e^a \gamma_a$ , we obtain

$$0 = 6 Tr(9\omega_5 e \omega_5 e^8 + 9\omega_5^2 e^9 + 9\omega_5 e^2 \omega_5 e^7 + 9\omega_5 e^3 \omega_5 e^6 + 9\omega_5 e^4 \omega_5 e^5) + \frac{30}{4} Tr(R_L e^9) + 4\nu(dA) 6Tr(\omega_5 e^6) , \tag{3.70}$$

where the curvature  $R_L$  is

$$R_L(\omega_L) = d\omega_L + \omega_L \omega_L = \frac{1}{4}(d\omega_{ab} + \omega_a{}^c \omega_{cb})\gamma^{ab} \equiv \frac{1}{4}R(\omega_L)_{ab}\gamma^{ab} . \tag{3.71}$$

This expression leads to an equation for the Ricci scalar  $R(\omega_L)^{ab}{}_{ab}$  that has the advantage that the different contributions are easier to compute. A calculation (Appendix A, eq. (A.95)) shows that  $\nu$  in our action is related to  $P$  by

$$P = 12 \cdot 32 \cdot \frac{4! \cdot 7!}{(9!)^2} \nu^2 . \tag{3.72}$$

Now, to complete the  $E(e) = 0$  equation of supergravity we need to fix the value of  $\nu$  in (2.28), (3.65); to determine the equation  $E(A) = 0$  in (3.64) we further require the value of  $\sigma$ .

### 3.4. The generalized CS action for the bosonic sector of $D = 11$ supergravity

Having found the field equations from our action, we now fix the remaining constants in (2.28) so that the equations of bosonic  $D = 11$  supergravity follow from  $I_{GCS}$  as stated. First, the  $D = 11$  supergravity equation for the  $e$  field is, after taking the trace (see e.g. [27]),

$$R(\Gamma) = \left(\frac{1}{12}\right)^2 (dA)_{a_1\dots a_4}(dA)^{a_1\dots a_4} . \tag{3.73}$$

Comparing with (3.69) we find  $P = \frac{1}{12}$ , which in eq. (3.72) then gives

$$v^2 = \left(\frac{1}{12}\right)^2 \frac{(9!)^2}{32 \cdot 4! \cdot 7!} \tag{3.74}$$

Secondly, the  $D = 11$  supergravity equation for  $A$  is

$$D^{b_1} (dA)_{b_1 d_1 \dots d_3} = \left(\frac{1}{32 \cdot 2^7}\right) \epsilon_{b_1 \dots b_4 c_1 \dots c_4 d_1 \dots d_3} (dA)^{b_1 \dots b_4} (dA)^{c_1 \dots c_4} . \tag{3.75}$$

Comparing with our (3.64) it follows that

$$\sigma = v^2 \left(\frac{40}{81}\right) . \tag{3.76}$$

The value of  $\sigma$  follows using eq. (3.74) in eq. (3.76),

$$\sigma = \frac{5}{4 \cdot (12)^2} \cdot \frac{(8!)^2}{4! \cdot 7!} . \tag{3.77}$$

Thus, the needed values of  $v$  and  $\sigma$  in (2.28) are now fixed; the terms in  $\alpha, \beta, \delta, \rho$  do not appear once the relevant  $H_9$  term is selected. Note that it is possible to obtain  $v$  from (3.74) because its r.h.s. is positive.

Summarizing, the generalized CS action for the bosonic sector of  $D = 11$  supergravity is obtained from

$$H = Tr(\Omega^6) + vTr(\Omega^4)d\mathcal{A} + \sigma(d\mathcal{A})^3 , \tag{3.78}$$

with  $v$  and  $\sigma$  given by eqs. (3.76) and (3.77). After the rescalings (2.30) and (2.31), the action follows from  $B_{19}$  with  $dB_{19} = H_{19}$  and the equations of motion for the  $\omega, A$  and  $e$  fields are given by eqs. (3.43) [eqs. (3.54), (3.56)], (3.62) and (3.65) [(3.66)] respectively, the constants of which have already been fixed. These equations are those of  $D = 11$  supergravity when spinors are ignored, and hence  $B_{19}$  determines the generalized CS action of its bosonic sector.

## 4. Conclusions

We have shown that the bosonic sector of  $D = 11$  supergravity may be obtained from a generalized CS action based on the one-form gauge fields of the  $sp(32)$  subalgebra of  $osp(1|32)$  supplemented with a dimensionless three-form field  $\mathcal{A}$ . The need for  $\mathcal{A}$  could not have been guessed without having in mind  $D = 11$  supergravity: the presence of fermions requires  $\mathcal{A}$  by simply counting the degrees of freedom of the  $D = 11$  supermultiplet. Further, we have also shown (see (3.60)) that the role of the auxiliary zero-form fields  $F_{a_1\dots a_4}$  that appear in the first-order version of  $D = 11$  supergravity [26] is played by specific gauge fields associated with  $sp(32)$ .

The values of the constants that determine our generalized CS bosonic action were obtained by requiring that the equations it leads to are those of the bosonic sector of  $D = 11$  supergravity. It turns out that only three terms in eq. (2.28) are actually needed, the first one and those in  $\nu$  and  $\sigma$ , since the others do not appear in the bosonic equations obtained from the  $\lambda^9$  term in the  $\lambda$  expansion. The other terms and their constants would appear when including fermions, eq. (1.17), but nevertheless (Sec. 1.3) this will not lead to  $D = 11$  supergravity. Hence, there is no generalized CS action based on  $osp(1|32)$  with the addition of the three-form field leading to CJS supergravity. Therefore, although  $D = 3$  supergravity may be described by a CS action, we conclude that this is not so in larger, odd spacetime dimensions.

It was already conjectured in the original paper [19] that  $osp(1|32)$  would provide the lead for a geometric interpretation of  $D = 11$  supergravity. The main obstacle to relate its field contents to the geometric MC fields of a superalgebra in the search for a possible CS action is the appearance of the three-form field  $A$ . As mentioned, it is possible to retain only one-form fields by assuming a composite nature for  $A$  [20] and then using a superalgebra that incorporates the one-form MC components of  $A$ . In fact, there is a whole family of superalgebras related to  $osp(1|32)$  that do just this [23] (another family of algebras structure has recently been shown to exist for  $N = 2$ ,  $D = 7$  supergravity [30]).

Summarizing, we have shown that although there is no CS action for CJS supergravity, its bosonic sector may be described by a generalized CS action in the sense of Sec. 2.2. But, if we insist in including fermions, we conclude that the only geometric way of relating CJS supergravity to the  $osp(1|32)$  superalgebra requires assuming the mentioned composite nature for  $A$  [20,23]. Even so, the connection with  $osp(1|32)$  is rather subtle [23]: the family of algebras that trivialize the three-form  $A$  are deformations of an algebra which is the expansion  $osp(1|32)(2, 3)$  of  $osp(1|32)$  in the sense of [16,17].

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**Appendix A**

This Appendix provides details of some main text calculations.

*A.1. Solving for  $\omega_5$  in the  $\omega_5$  equation*

Let us solve (3.59) for  $\omega_5$ . Contracting the equation with  $\epsilon^{a_1\dots a_6 d_1\dots d_5}$  we find,

$$\frac{9!}{2} \epsilon^{a_1\dots a_5 d_1\dots d_5} \epsilon_{a_1\dots a_5 b_1\dots b_4 c d} \omega^{b_4\dots b_1} a_6^c + 4\nu \epsilon_{a_1\dots a_6 b_1\dots b_4 d} \epsilon^{a_1\dots a_6 d_1\dots d_5} (dA)^{b_1\dots b_4} = 0 . \tag{A.79}$$

Taking into account that

$$\epsilon^{a_1\dots a_k b_1\dots b_{11-k}} \epsilon_{a_1\dots a_k c_1\dots c_{11-k}} = -k! \delta_{c_1\dots c_{11-k}}^{b_1\dots b_{11-k}} \tag{A.80}$$

for our signature choice,  $(- + \dots +)$ , where  $\delta_{a_1\dots a_k}^{b_1\dots b_k} = \sum_{\sigma \in S_k} \delta_{a_{\sigma(1)}}^{b_1} \dots \delta_{a_{\sigma(k)}}^{b_k}$ , we obtain



$$\frac{9!}{2} 5! \delta_{b_1 \dots b_4 c d}^{a_6 d_1 \dots d_5} \omega^{b_4 \dots b_1} a_6^c + 4\nu 6! \delta_{b_1 \dots b_4 d}^{d_1 \dots d_5} (dA)^{b_1 \dots b_4} = 0. \tag{A.81}$$

Now, using

$$\delta_{b_1 \dots b_k}^{a a_1 \dots a_{k-1}} = \sum_{l=1}^k \delta_{b_l}^a \delta_{b_1 \dots b_{l-1} \dots b_k}^{a_1 \dots a_k} \tag{A.82}$$

in the first term with  $a = a_6$ , it follows that

$$\left( \frac{9!}{2} 5! \omega^{b_4 \dots b_1} a_6^c + 4\nu 6! (dA)^{b_1 \dots b_4} \right) \delta_{b_1 \dots b_4 d}^{d_1 \dots d_5} - \frac{9!}{2} 5! \delta_{b_1 \dots b_4 c}^{d_1 \dots d_5} \omega^{b_4 \dots b_1} a_6^c = 0. \tag{A.83}$$

Now, contracting  $d_5$  and  $d$  in (A.83) we get

$$\omega^{b_4 \dots b_1} a_6^c = -\frac{56}{9!} \nu (dA)^{b_1 \dots b_4} \tag{A.84}$$

and, inserting this in (A.83), we find

$$\omega^{[d_1 \dots d_4]_d} = -\nu \frac{4 \cdot 2}{9!} (dA)^{[d_1 \dots d_4]_d} \delta_d^{d_5}. \tag{A.85}$$

We now use this equation to find  $\omega^{d_1 \dots d_4} a_6^{d_5}$  without antisymmetrization. To this end, we use the following trick: first we make eq (A.85) more explicit, with  $d_5$  interchanged with  $d$ , so that the antisymmetrization involves  $d_1, \dots, d_4$  and  $d$ ,

$$\begin{aligned} & \omega^{d_1 d_2 d_3 d_4 d_5} a_6^d - \omega^{d_1 d_2 d_3} a_6^{d_5} a_6^d \\ & - \omega^{d_1 d_2} a_6^{d_5} a_6^{d_4} a_6^d - \omega^{d_1} a_6^{d_3} a_6^{d_4} a_6^{d_5} a_6^d \\ & - \omega^{d_1} a_6^{d_2} a_6^{d_3} a_6^{d_4} a_6^{d_5} = -\frac{4 \cdot 2}{9!} \nu \left( (dA)^{d_1 d_2 d_3 d_4} \delta_d^{d_5} \right. \\ & - (dA)^{d_2 d_3 d_4} \eta^{d_5 d_1} - (dA)^{d_1 d_3 d_4} \eta^{d_5 d_2} \\ & \left. - (dA)^{d_1 d_2} a_6^{d_4} \eta^{d_5 d_3} - (dA)^{d_1 d_2 d_3} a_6^{d_4} \eta^{d_5 d_4} \right). \end{aligned} \tag{A.86}$$

Antisymmetrizing the indices  $d_1 \dots d_5$  with weight one leads to

$$\omega^{d_1 \dots d_5} a_6^d - 4 \cdot \omega^{[d_1 \dots d_4]_d} a_6^{d_5} = -\frac{4 \cdot 2}{9!} \nu (dA)^{[d_1 d_2 d_3 d_4]_d} \delta_d^{d_5}, \tag{A.87}$$

and using (A.85) in (A.87), we finally obtain

$$\omega^{d_1 \dots d_5} a_6^d = -\frac{40}{9!} \nu (dA)^{[d_1 \dots d_4]_d} \delta_d^{d_5}. \tag{A.88}$$

or, equivalently, (3.61).

### A.2. Calculation of the terms in (3.70)

Defining the zero-form matrix  $\widehat{dA} = (dA)_{a_1 \dots a_4} \gamma^{a_1 \dots a_4}$ , eq. (3.61) may be rewritten as

$$\omega_5 = -\frac{20}{9!} (\widehat{dA} e + e \widehat{dA}) \tag{A.89}$$

Inserting this relation into (3.70), we obtain

$$\begin{aligned}
 30Tr(R_L e^9) &= 48 \frac{20}{9!} v^2 dA Tr(\widehat{dA} e^7) \\
 &\quad - 54v^2 \left(\frac{20}{9!}\right)^2 Tr(4\widehat{dA}\widehat{dA}e^{11} + 3\widehat{dA}e\widehat{dA}e^{10} + 4\widehat{dA}e^2\widehat{dA}e^9 \\
 &\quad + 4\widehat{dA}e^3\widehat{dA}e^8 + 4\widehat{dA}e^4\widehat{dA}e^7 + 4\widehat{dA}e^5\widehat{dA}e^6) .
 \end{aligned}
 \tag{A.90}$$

Let us now compute the terms in this equation. First, the trace on the *l.h.s.* is given by

$$\begin{aligned}
 Tr(R_L e^9) &= \frac{1}{4} Tr(R_L^{b_1 b_2}{}_{a_1 a_2} \gamma_{b_1 b_2} e^{a_1} e^{a_2} e^{a_3} \dots e^{a_{11}} \gamma_{a_3 \dots a_{11}}) \\
 &= \frac{1}{4} Tr(\gamma_{b_1 b_2} \gamma_{a_3 \dots a_{11}}) R_L^{b_1 b_2}{}_{a_1 a_2} \epsilon^{a_1 \dots a_{11}} E \\
 &= 8 \epsilon_{b_1 b_2 a_3 \dots a_{11}} \epsilon^{a_1 \dots a_{11}} R_L^{b_1 b_2}{}_{a_1 a_2} E \\
 &= -8 \cdot 9! \delta_{b_1 b_2}^{a_1 a_2} R_L^{b_1 b_2}{}_{a_1 a_2} E \\
 &= -16 \cdot 9! R_L E ,
 \end{aligned}
 \tag{A.91}$$

where  $E$  is an 11-form defined by  $e^{a_1} \dots e^{a_{11}} = \epsilon^{a_1 \dots a_{11}} E$ , and we have written  $R^{b_1 b_2} = R^{b_1 b_2}{}_{a_1 a_2} e^{a_1} e^{a_2}$ . The first term on the r.h.s. of (A.91) contains the form

$$\begin{aligned}
 dA Tr(\widehat{dA} e^7) &= 32(dA)_{b_1 \dots b_4} e^{b_1} \dots e^{b_4} \epsilon_{a_1 \dots a_{11}} (dA)^{a_1 \dots a_4} e^{a_5} \dots e^{a_{11}} E \\
 &= 32(dA)_{b_1 \dots b_4} (dA)^{a_1 \dots a_4} \epsilon_{a_1 \dots a_{11}} \epsilon^{b_1 \dots b_4 a_5 \dots a_{11}} E \\
 &= -7! \cdot 32(dA)_{b_1 \dots b_4} (dA)^{a_1 \dots a_4} \delta_{a_1 \dots a_4}^{b_1 \dots b_4} E \\
 &= -7! \cdot 4! \cdot 32(dA)_{a_1 \dots a_4} (dA)^{a_1 \dots a_4} E ,
 \end{aligned}
 \tag{A.92}$$

where, as before, we have written  $dA = (dA)_{b_1 \dots b_4} e^{b_1} \dots e^{b_4}$ .

The calculation of the remaining terms is slightly more complicated. These terms have the form

$$\begin{aligned}
 Tr(\widehat{dA} e^k \widehat{dA} e^{11-k}) &= Tr(\widehat{dA} \gamma^{a_1 \dots a_k} \widehat{dA} \gamma^{a_{k+1} \dots a_{11}} \epsilon_{a_1 \dots a_{11}} E) \\
 &= \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} Tr(\widehat{dA} \gamma^{a_1 \dots a_k} \widehat{dA} \gamma_{b_1 \dots b_k}) \epsilon^{b_1 \dots b_k a_{k+1} \dots a_{11}} \epsilon_{a_1 \dots a_{11}} E \\
 &= -(-1)^{\frac{k(k-1)}{2}} (11-k)! Tr(\widehat{dA} \gamma^{a_1 \dots a_k} \widehat{dA} \gamma_{a_1 \dots a_k}) E \\
 &= -32(-1)^{\frac{k(k-1)}{2}} 4!(11-k)! N_k (dA)_{a_1 \dots a_4} (dA)^{a_1 \dots a_4} E ,
 \end{aligned}
 \tag{A.93}$$

where we have used the property (3.52) and the numbers  $N_k$  in the equation are defined through

$$\gamma^{a_1 \dots a_k} \widehat{dA} \gamma_{a_1 \dots a_k} = N_k \widehat{dA} .
 \tag{A.94}$$

These numbers may be computed using gamma matrix algebra; alternatively, they can be found in Ref. [29]. Their values are:  $N_0 = 1, N_1 = 3, N_2 = 2, N_3 = 66, N_4 = -144, N_5 = 1680$ . Then, the second trace on the r.h.s. of (A.90) is given by  $-32 \cdot 168 \cdot 9! \cdot 4! (dA)^{a_1 \dots a_4} (dA)_{a_1 \dots a_4} E$ . When this is taken into account, eq. (A.90) reads

$$R_L = 16 \cdot \frac{7! \cdot 4!}{(9!)^2} \gamma^2 (dA)^{a_1 \dots a_4} (dA)_{a_1 \dots a_4} .
 \tag{A.95}$$

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