The Ramanujan Journal

A functional Identity involving Elliptic Integrals

--Manuscript Draft--

Manuscript Number:	
Full Title:	A functional Identity involving Elliptic Integrals
Article Type:	Research Paper
Keywords:	Elliptic Integral of an elliptic integral; Invariance
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Funding Information:	Consejería de Educación, Junta de Castilla y León (ES) (UIC 0 11) — Professor Lawrence Glasser
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A FUNCTIONAL IDENTITY INVOLVING ELLIPTIC INTEGRALS

M. LAWRENCE GLASSER AND YAJUN ZHOU

Abstract. We show that the following double integral

$$\int_0^\pi \mathrm{d}\,x \int_0^x \mathrm{d}\,y \frac{1}{\sqrt{1-p\cos x}\sqrt{1+q\cos y}}$$

remains invariant as one trades the parameters p and q for $p' = \sqrt{1-p^2}$ and $q' = \sqrt{1-q^2}$ respectively. This invariance property is suggested from symmetry considerations in the operating characterstics of a semiconductor Hall-effect device.

Keywords: Incomplete elliptic integrals, complete elliptic integrals, Landen's transformation.

Subject Classification (AMS 2010): 33E05 (Primary), 78A35 (Secondary)

1. Introduction

When an electron current flows perpendicular to a magnetic field through a conducting medium, the charges are forced to deviate to one side creating an imbalance which results in a measurable electric potential conveying important information about the material. A device based on this, so-called Hall effect, has been studied in detail by Ausserlechner [1] who has found that its operating features are summed up in the Hall-geometry-factor

$$G(\lambda_f,\lambda_p) = \frac{1}{\mathbf{K}'\left(\frac{1-p}{1+p}\right)\mathbf{K}\left(\frac{1-f}{1+f}\right)} \int_0^1 \frac{\int_0^x \frac{\mathrm{d}\,y}{\sqrt{1-\left(\frac{1-p}{1+p}\right)^2(1-y^2)}\sqrt{1-y^2}}}{\sqrt{1-x^2}\sqrt{1-\left[1-\left(\frac{1-f}{1+f}\right)^2\right](1-x^2)}}\,\mathrm{d}\,x.$$

Here p and f are related to the input and output resistances by $\lambda_f = 2\mathbf{K}(f)/\mathbf{K}'(f)$ and $\lambda_p = \mathbf{K}'(p)/[2\mathbf{K}(p)]$, with the complete elliptic integral of the first kind being defined by

$$\mathbf{K}(\sqrt{t}) := \int_0^{\pi/2} \frac{\mathrm{d}\,\theta}{\sqrt{1-t\sin^2\theta}} \equiv \mathbf{K}'(\sqrt{1-t}).$$

Due to the symmetry of the device $G(\lambda_f,\lambda_p)/\sqrt{\lambda_f\lambda_p}$ must be unchanged under the substitution $(\lambda_f,\lambda_p) \to (2/\lambda_f, 2/\lambda_p)$. This can be recast into the remarkable identity that

$$\int_0^\pi \frac{\mathrm{d}\,x}{\sqrt{1-p\cos x}} \int_0^x \frac{\mathrm{d}\,y}{\sqrt{1+q\cos y}}$$

Date: January 24, 2017.

is invariant under $(p,q)\to(\sqrt{1-p^2},\sqrt{1-q^2})$, which is our aim to prove in this note.

2. A Double Integral Identity

Theorem 1. *For parameters $p,q\in(0,1)$, define correspondingly $p'=\sqrt{1-p^2},q'=\sqrt{1-q^2}$, then we have an integral identity $A(p,q)=A(p',q')$, where*

$$\begin{aligned}A(p,q):={}&\int_0^\pi\mathrm{d}\,x\int_0^x\mathrm{d}\,y\frac{1}{\sqrt{1-p\cos x}\sqrt{1+q\cos y}}\\={}&\frac{4}{\sqrt{(1-p)(1+q)}}\int_0^{\pi/2}\frac{\mathrm{d}\,\theta}{\sqrt{1+\frac{2p}{1-p}\sin^2\theta}}\int_0^\theta\frac{\mathrm{d}\,\phi}{\sqrt{1-\frac{2q}{1+q}\sin^2\phi}}.\end{aligned}\tag{1}$$

Before proving the functional equation stated in the theorem above, we need to convert double integrals like $A(p,q)$ into single integrals over the products of elliptic integrals and elementary functions, as described in the lemma below.

Lemma 2. *For $0<\beta<\alpha<1$, the following identity holds:*[1]

$$\begin{aligned}&\int_0^{\pi/2}\frac{\mathrm{d}\,\theta}{\sqrt{1-\alpha\sin^2\theta}}\int_0^\theta\frac{\mathrm{d}\,\phi}{\sqrt{1-\beta\sin^2\phi}}\\&=\frac{1}{\pi}\int_0^\beta\frac{\mathbf{K}(\sqrt{1-\beta})\mathbf{K}(\sqrt{t})}{\sqrt{1-t}+\sqrt{1-\alpha}}\frac{\mathrm{d}\,t}{\sqrt{1-t}}+\frac{1}{\pi}\int_\beta^1\frac{\mathbf{K}(\sqrt{\beta})\mathbf{K}(\sqrt{1-t})}{\sqrt{1-t}+\sqrt{1-\alpha}}\frac{\mathrm{d}\,t}{\sqrt{1-t}},\end{aligned}\tag{2}$$

where the integrations are carried out along straight line-segments joining the end points.

Proof. In what follows, we write $\mathbb{Y}_\lambda(X):=\sqrt{X(1-X)(1-\lambda X)}$ for $X\in(0,1)$ and $\lambda\in(0,1)$, with the square root taking positive values. It is clear that the complete elliptic integral $\mathbf{K}(\sqrt{\lambda}),\lambda\in(0,1)$ satisfies

$$\mathbf{K}(\sqrt{\lambda})=\frac12\int_0^1\frac{\mathrm{d}\,X}{\mathbb{Y}_\lambda(X)}.\tag{3}$$

For $0<\beta<\alpha<1$, we have an addition formula of Legendre type [4, Eq. 2.3.26]

$$\begin{aligned}\frac{\pi}{\mathbb{Y}_\alpha(U)}\int_U^1\frac{\mathrm{d}\,u}{\mathbb{Y}_\beta(u)}={}&\int_0^1\frac{2\alpha\mathbf{K}(\sqrt{1-\beta})}{1-\alpha UV}\frac{V\,\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)}+\int_0^1\frac{2\alpha\mathbf{K}(\sqrt{\beta})}{1-(1-\alpha U)V}\frac{V\,\mathrm{d}\,V}{\mathbb{Y}_{1-\alpha}(V)}\\&-\int_{\frac{1-\alpha}{1-\beta}}^1\frac{\mathrm{d}\,X}{\mathbb{Y}_{1-\beta}(X)}\int_{\frac{1-(1-\beta)X}{\alpha}}^1\frac{\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)}\frac{\alpha V}{1-\alpha UV}.\end{aligned}\tag{4}$$

Integrating over $U\in(0,1)$, we obtain

$$\begin{aligned}&\pi\int_0^1\frac{\mathrm{d}\,U}{\mathbb{Y}_\alpha(U)}\int_U^1\frac{\mathrm{d}\,u}{\mathbb{Y}_\beta(u)}=4\pi\mathbf{K}(\sqrt{\alpha})\mathbf{K}(\sqrt{\beta})-\pi\int_0^1\frac{\mathrm{d}\,U}{\mathbb{Y}_\alpha(U)}\int_0^U\frac{\mathrm{d}\,u}{\mathbb{Y}_\beta(u)}\\&=-2\mathbf{K}(\sqrt{1-\beta})\int_0^1\frac{\log(1-\alpha V)\,\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)}+2\mathbf{K}(\sqrt{\beta})\int_0^1\frac{\log\frac{1-(1-\alpha)V}{1-V}\,\mathrm{d}\,V}{\mathbb{Y}_{1-\alpha}(V)}\\&\quad+\int_{\frac{1-\alpha}{1-\beta}}^1\frac{\mathrm{d}\,X}{\mathbb{Y}_{1-\beta}(X)}\int_{\frac{1-(1-\beta)X}{\alpha}}^1\frac{\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)}\log(1-\alpha V).\end{aligned}\tag{5}$$

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[1]The constraint $0<\beta<\alpha<1$ is needed in the derivation of (2), the validity of which extends to $\alpha=2p/(p-1)<0,\beta=2q/(1+q)\in(0,1)$, by virtue of analytic continuation.

Here, the first two single integrals over V can be evaluated in closed form [4, Eqs. 2.2.3 and 2.2.4]:

$$\int_0^1 \frac{\log(1-\alpha V)\,\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)} = \mathbf{K}(\sqrt{\alpha})\log(1-\alpha), \tag{6}$$

$$\int_0^1 \frac{\log\frac{1-(1-\alpha)V}{1-V}\,\mathrm{d}\,V}{\mathbb{Y}_{1-\alpha}(V)} = \pi\mathbf{K}(\sqrt{\alpha}) + \mathbf{K}(\sqrt{1-\alpha})\log(1-\alpha), \tag{7}$$

while the last double integral satisfies [cf. 4, Eq. 2.3.2]

$$\begin{aligned}&\int_{\frac{1-\alpha}{1-\beta}}^1 \frac{\mathrm{d}\,X}{\mathbb{Y}_{1-\beta}(X)} \int_{\frac{1-(1-\beta)X}{\alpha}}^1 \frac{\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)}\log(1-\alpha V)\\ &= \frac{2\mathbf{K}(\sqrt{1-\beta})}{\pi}\int_0^1 \frac{(1-\beta U)\,\mathrm{d}\,U}{\mathbb{Y}_\beta(U)}\int_0^1 \frac{\mathrm{d}\,W}{\sqrt{W(1-W)}}\frac{\log(1-\alpha W-\beta(1-W))}{1-[\alpha W+\beta(1-W)]U}\\ &\quad -\frac{2\mathbf{K}(\sqrt{\beta})}{\pi}\int_0^1 \frac{[1-(1-\beta)U]\,\mathrm{d}\,U}{\mathbb{Y}_{1-\beta}(U)}\int_0^1 \frac{\mathrm{d}\,W}{\sqrt{W(1-W)}}\frac{\log(1-\alpha W-\beta(1-W))}{1-[1-\alpha W-\beta(1-W)]U}.\end{aligned} \tag{8}$$

Substituting $W=(1-\beta U)V/(1-\beta UV)$ such that

$$\frac{W}{1-W} = \frac{(1-\beta U)V}{1-V}, \tag{9}$$

we obtain

$$\begin{aligned}&\int_0^1 \frac{(1-\beta U)\,\mathrm{d}\,U}{\mathbb{Y}_\beta(U)}\int_0^1 \frac{\mathrm{d}\,W}{\sqrt{W(1-W)}}\frac{\log(1-\alpha W-\beta(1-W))}{1-[\alpha W+\beta(1-W)]U}\\ &= \int_0^1 \frac{\mathrm{d}\,U}{\sqrt{U(1-U)}}\int_0^1 \frac{\mathrm{d}\,V}{\sqrt{V(1-V)}}\frac{\log\left(1-\alpha+\frac{(\alpha-\beta)(1-V)}{1-\beta UV}\right)}{1-\alpha UV},\end{aligned} \tag{10}$$

where

$$\begin{aligned}&\frac{\log\left(1-\alpha+\frac{(\alpha-\beta)(1-V)}{1-\beta UV}\right)-\log(1-\alpha V)}{1-\alpha UV}\\ &= \int_0^\beta \left[\frac{1}{1-tUV} - \frac{1-\alpha}{(1-t)(1-V)+(1-\alpha)(1-tU)V}\right]\frac{\mathrm{d}\,t}{t-\alpha}\end{aligned} \tag{11}$$

allows us to integrate over V and U in a sequel on the right-hand side, leading to

$$\begin{aligned}&\int_0^1 \frac{(1-\beta U)\,\mathrm{d}\,U}{\mathbb{Y}_\beta(U)}\int_0^1 \frac{\mathrm{d}\,W}{\sqrt{W(1-W)}}\frac{\log(1-\alpha W-\beta(1-W))}{1-[\alpha W+\beta(1-W)]U}\\ &= 2\pi\left[\int_0^\beta \frac{\mathbf{K}(\sqrt{t})}{t-\alpha}\left(1-\sqrt{\frac{1-\alpha}{1-t}}\right)\mathrm{d}\,t + \frac{\mathbf{K}(\sqrt{\alpha})}{2}\log(1-\alpha)\right].\end{aligned} \tag{12}$$

Here, in the last step, we have evaluated

$$\begin{aligned}&\int_0^1 \frac{\mathrm{d}\,U}{\sqrt{U(1-U)}}\int_0^1 \frac{\mathrm{d}\,V}{\sqrt{V(1-V)}}\frac{\log(1-\alpha V)}{1-\alpha UV}\\ &= \pi\int_0^1 \frac{\log(1-\alpha V)\,\mathrm{d}\,V}{\mathbb{Y}_\alpha(V)} = \pi\mathbf{K}(\sqrt{\alpha})\log(1-\alpha)\end{aligned} \tag{13}$$

with the aid of (6). Likewise, starting with a variable substitution $W=[1-(1-\beta)U]V/[1-(1-\beta)UV]$ such that

$$\frac{W}{1-W}=\frac{[1-(1-\beta)U]V}{1-V},\tag{14}$$

we may compute

$$\int_0^1\frac{[1-(1-\beta)U]\,\mathrm{d}\,U}{\mathbb{Y}_{1-\beta}(U)}\int_0^1\frac{\mathrm{d}\,W}{\sqrt{W(1-W)}}\frac{\log(1-\alpha W-\beta(1-W))}{1-[1-\alpha W-\beta(1-W)]U}$$
$$=2\pi\left[\int_1^\beta\frac{\mathbf{K}(\sqrt{1-t})}{t-\alpha}\left(1-\sqrt{\frac{1-\alpha}{1-t}}\right)\mathrm{d}\,t-\frac{\pi\mathbf{K}(\sqrt{\alpha})}{2}+\frac{\mathbf{K}(\sqrt{1-\alpha})}{2}\log(1-\alpha)\right].\tag{15}$$

Thus, the claimed identity is verified. □

Exploiting the integral identity in the lemma above, together with some modular transformations of elliptic integrals, we will prove Theorem 1.

Proof of Theorem 1. We recall that the Legendre function of the first kind of degree $-1/4$ is defined by

$$P_{-1/4}(1-2t):={}_2F_1\left(\begin{array}{c}\frac14,\frac34\\1\end{array}\middle|t\right)$$
$$=\frac{1}{\sqrt{2}\pi}\int_0^1\left[\frac{u(1-tu)}{1-u}\right]^{-1/4}\frac{\mathrm{d}\,u}{1-u},\quad t\in\mathbb{C}\smallsetminus[1,+\infty).\tag{16}$$

The following relations between $P_{-1/4}$ and the complete elliptic integral $\mathbf{K}$ are recorded in Ramanujan's notebook [2, Chap. 33, Theorems 9.1 and 9.2]:

$$\mathbf{K}\left(\sqrt{\frac{2q}{1+q}}\right)=\frac{\pi}{2}\sqrt{1+q}P_{-1/4}(1-2q^2),\tag{17}$$

$$\mathbf{K}\left(\sqrt{\frac{1-q}{1+q}}\right)=\frac{\pi}{2}\sqrt{\frac{1+q}{2}}P_{-1/4}(2q^2-1),\tag{18}$$

which are provable by standard transformations of the respective hypergeometric functions, provided that $q\in(0,1)$.

With the information listed in the last paragraph, we see that

$$A(p,q)=\int_0^{2q/(1+q)}\frac{\sqrt{2}P_{-1/4}(2q^2-1)\mathbf{K}(\sqrt{t})}{\sqrt{1-t}\sqrt{1-p}+\sqrt{1+p}}\frac{\mathrm{d}\,t}{\sqrt{1-t}}$$
$$+\int_{2q/(1+q)}^1\frac{2P_{-1/4}(1-2q^2)\mathbf{K}(\sqrt{1-t})}{\sqrt{1-t}\sqrt{1-p}+\sqrt{1+p}}\frac{\mathrm{d}\,t}{\sqrt{1-t}}.\tag{19}$$

On one hand, with $t=4\sqrt{s}/(1+\sqrt{s})^2$ and Landen's transformation [3, item 163.02]

$$\mathbf{K}(\sqrt{s})=\frac{1}{1+\sqrt{s}}\mathbf{K}\left(\frac{2\sqrt[4]{s}}{1+\sqrt{s}}\right),\quad 0<s<1,\tag{20}$$

we have

$$\int_0^{2q/(1+q)}\frac{\mathbf{K}(\sqrt{t})}{\sqrt{1-t}\sqrt{1-p}+\sqrt{1+p}}\frac{\mathrm{d}\,t}{\sqrt{1-t}}$$
$$=2\int_0^{(1-\sqrt{1-q^2})/(1+\sqrt{1-q^2})}\frac{\mathbf{K}(\sqrt{s})}{(1-\sqrt{s})\sqrt{1-p}+(1+\sqrt{s})\sqrt{1+p}}\frac{\mathrm{d}\,s}{\sqrt{s}}. \tag{21}$$

On the other hand, it is clear from a substitution $t=1-s$ that

$$\int_{2q/(1+q)}^{1}\frac{\mathbf{K}(\sqrt{1-t})}{\sqrt{1-t}\sqrt{1-p}+\sqrt{1+p}}\frac{\mathrm{d}\,t}{\sqrt{1-t}}$$
$$=\int_0^{(1-q)/(1+q)}\frac{\mathbf{K}(\sqrt{s})}{\sqrt{s}\sqrt{1-p}+\sqrt{1+p}}\frac{\mathrm{d}\,s}{\sqrt{s}}$$
$$=\int_0^{(1-q)/(1+q)}\frac{\sqrt{2}\mathbf{K}(\sqrt{s})}{(1-\sqrt{s})\sqrt{1-\sqrt{1-p^2}}+(1+\sqrt{s})\sqrt{1+\sqrt{1-p^2}}}\frac{\mathrm{d}\,s}{\sqrt{s}}. \tag{22}$$

Here, the last equality results from a pair of elementary identities for $p\in(0,1)$:

$$\sqrt{\frac{1+\sqrt{1-p^2}}{2}}\pm\sqrt{\frac{1-\sqrt{1-p^2}}{2}}=\sqrt{1\pm p}, \tag{23}$$

which are readily verified by squaring both sides.

Therefore, with $p'=\sqrt{1-p^2}, q'=\sqrt{1-q^2}$, we have

$$A(p,q)=\int_0^{(1-q')/(1+q')}\frac{2\sqrt{2}P_{-1/4}(1-2q'^2)\mathbf{K}(\sqrt{s})}{(1-\sqrt{s})\sqrt{1-p}+(1+\sqrt{s})\sqrt{1+p}}\frac{\mathrm{d}\,s}{\sqrt{s}}$$
$$+\int_0^{(1-q)/(1+q)}\frac{2\sqrt{2}P_{-1/4}(1-2q^2)\mathbf{K}(\sqrt{s})}{(1-\sqrt{s})\sqrt{1-p'}+(1+\sqrt{s})\sqrt{1+p'}}\frac{\mathrm{d}\,s}{\sqrt{s}}, \tag{24}$$

which is evidently equal to $A(p',q')$. □

Acknowledgement

M.L.G. thanks Udo Ausserlechner (Infinion Technologies) and Michael Milgram (Geometrics Unlimited) for insightful correspondence. Financial support of MINECO (Project MTM2014-57129-C2-1-P) and Junta de Castilla y Leon (UIC 0 11) is acknowledged.

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