Annals of Physics 333 (2013) 90–103

Contents lists available at SciVerse ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Algebraic special functions and $SO(3, 2)$

E. Celeghini [a], M.A. del Olmo [b,*]

[a] *Dipartimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy*

[b] *Departamento de Física Teórica and IMUVA, Universidad de Valladolid, E-47011, Valladolid, Spain*

HIGHLIGHTS

- The algebraic ladder structure is constructed for the associated Legendre polynomials (ALP).
- ALP and spherical harmonics support a unitary irreducible $SO(3, 2)$-representation.
- A ladder structure is the condition to get a Lie group representation defining "algebraic special functions".
- The "algebraic special functions" connect Lie algebras and L^2 functions.

ARTICLE INFO

Article history:
Received 7 November 2012
Accepted 24 February 2013
Available online 25 March 2013

Keywords:
Special function
Lie algebra
Square-integrable function

ABSTRACT

A ladder structure of operators is presented for the associated Legendre polynomials and the sphericas harmonics. In both cases these operators belong to the irreducible representation of the Lie algebra $so(3, 2)$ with quadratic Casimir equals to $-5/4$. As both are also bases of square-integrable functions, the universal enveloping algebra of $so(3, 2)$ is thus shown to be homomorphic to the space of linear operators acting on the L^2 functions defined on $(-1, 1) \times \mathbb{Z}$ and on the sphere S^2, respectively.

The presence of a ladder structure is suggested to be the general condition to obtain a Lie algebra representation defining in this way the "algebraic special functions" that are proposed to be the connection between Lie algebras and square-integrable functions so that the space of linear operators on the L^2 functions is homomorphic to the universal enveloping algebra.

The passage to the group, by means of the exponential map, shows that the associated Legendre polynomials and the spherical harmonics support the corresponding unitary irreducible representation of the group $SO(3, 2)$.

* Corresponding author. Tel.: +34 983423146; fax: +34 983423013.
E-mail addresses: celeghini@fi.infn.it (E. Celeghini), olmo@fta.uva.es, marianoadel.olmo@gmail.com (M.A. del Olmo).

0003-4916/$ – see front matter
http://dx.doi.org/10.1016/j.aop.2013.02.010

Annals of Physics 335 (2013) 78–85

Contents lists available at SciVerse ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Coherent orthogonal polynomials

E. Celeghini [a], M.A. del Olmo [b,*]

[a] Dipartimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy

[b] Departamento de Física Teórica and IMUVA, Universidad de Valladolid, E-47005, Valladolid, Spain

HIGHLIGHTS

- Fundamental characteristic of orthogonal polynomials (OP): existence of a Lie algebra.
- Differential recurrence relations of OP determine a unitary representation of a non-compact Lie group.
- 2nd order Casimir originates a 2nd order differential equation that defines the corresponding OP family.
- Generalized coherent polynomials are obtained from OP.

ARTICLE INFO

Article history:
Received 27 June 2012
Accepted 27 April 2013
Available online 6 May 2013

Keywords:
Orthogonal polynomials
Group representation theory
Quantum mechanics
Coherent states

ABSTRACT

We discuss a fundamental characteristic of orthogonal polynomials, like the existence of a Lie algebra behind them, which can be added to their other relevant aspects. At the basis of the complete framework for orthogonal polynomials we include thus – in addition to differential equations, recurrence relations, Hilbert spaces and square integrable functions – Lie algebra theory.

We start here from the square integrable functions on the open connected subset of the real line whose bases are related to orthogonal polynomials. All these one-dimensional continuous spaces allow, besides the standard uncountable basis $\{|x\rangle\}$, for an alternative countable basis $\{|n\rangle\}$. The matrix elements that relate these two bases are essentially the orthogonal polynomials: Hermite polynomials for the line and Laguerre and Legendre polynomials for the half-line and the line interval, respectively.

Differential recurrence relations of orthogonal polynomials allow us to realize that they determine an infinite-dimensional irreducible representation of a non-compact Lie algebra, whose second order Casimir $\mathcal{C}$ gives rise to the second order differential equation that defines the corresponding family of orthogonal polynomials. Thus, the Weyl–Heisenberg algebra $h(1)$ with $\mathcal{C} = 0$ for Hermite

* Corresponding author. Tel.: +34 983423146; fax: +34 983423013.
E-mail addresses: celeghini@fi.infn.it (E. Celeghini), olmo@fta.uva.es, marianoadel.olmo@gmail.com (M.A. del Olmo).

0003-4916/$ – see front matter
http://dx.doi.org/10.1016/j.aop.2013.04.017

Annals of Physics 332 (2013) 27–37

Contents lists available at SciVerse ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

A unified approach to quantum and classical TTW systems based on factorizations

E. Celeghini [a], Ş. Kuru [b], J. Negro [c,*], M.A. del Olmo [c]

[a] Departimento di Fisica, Università di Firenze and INFN–Sezione di Firenze, I50019 Sesto Fiorentino, Firenze, Italy

[b] Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey

[c] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47011 Valladolid, Spain

ARTICLE INFO

Article history:
Received 5 December 2012
Accepted 12 January 2013
Available online 31 January 2013

Keywords:
Superintegrable system
Higher order symmetry operator
Ladder operator
Classical motion
Trajectory
Constant of motion

ABSTRACT

A unifying method based on factorization properties is introduced for finding symmetries of quantum and classical superintegrable systems using the example of the Tremblay–Turbiner–Winternitz (TTW) model. It is shown that the symmetries of the quantum system can be implemented in a natural way to its classical version. Besides, by this procedure we get also other type of constants of motion depending explicitly on time that allow to find directly the motion of the system whose corresponding trajectories coincide with those obtained previously by using its symmetries.

1. Introduction

The goal of this paper is to introduce a unified algebraic technique in order to find symmetries for both quantum and classical systems. This method for higher dimensional quantum systems is based on (one-dimensional) factorizations [1,2]. In the classical case factorization properties already developed for one-dimensional systems in [3] can also be applied in order to obtain the symmetries of the corresponding higher dimensional classical systems. Furthermore, the classical factorization provides constants of motion depending on time explicitly that give solutions to the motion algebraically. This is a natural way to deal with the symmetries for both quantum and classical systems using the same procedure.

* Corresponding author. Tel.: +34 983423040; fax: +34 983423013.
E-mail addresses: celeghini@fi.infn.it (E. Celeghini), kuru@science.ankara.edu.tr (Ş. Kuru), jnegro@fta.uva.es (J. Negro), olmo@fta.uva.es (M.A. del Olmo).

0003-4916/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
doi:10.1016/j.aop.2013.01.008

Annals of Physics 330 (2013) 220–245

Contents lists available at SciVerse ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Pisot q-coherent states quantization of the harmonic oscillator

J.P. Gazeau [a], M.A. del Olmo [b,*]

[a] Laboratoire APC, Univ. Paris Diderot, Sorbonne Paris Cité, 75205 Paris, France
[b] Departamento de Física Teórica and IMEVA, Universidad de Valladolid, E-47005, Valladolid, Spain

ARTICLE INFO

Article history:
Received 12 July 2012
Accepted 22 November 2012
Available online 5 December 2012

Keywords:
Coherent states
q-calculus
Moment problem
Quantization
Symmetric q-deformation
Quadratic unit Pisot number

ABSTRACT

We revisit the quantized version of the harmonic oscillator obtained through a q-dependent family of coherent states. For each q, $0 < q < 1$, these normalized states form an overcomplete set that resolves the unity with respect to an explicit measure. We restrict our study to the case in which q^{-1} is a quadratic unit Pisot number, since then the q-deformed integers form Fibonacci-like sequences of integers. We then examine the main characteristics of the corresponding quantum oscillator: localization in the configuration and in the phase spaces, angle operator, probability distributions and related statistical features, time evolution and semi-classical phase space trajectories.

1. Introduction

In this work we revisit non-linear coherent states obtained from the standard ones through a certain type of q- or $q\,p$-deformations of the integers. Similar deformations have already been considered during the two last decades, since the pioneering work by Arik et al. [1] (see for instance [2–5] and references therein for a list of more recent related works, mostly devoted to algebraic aspects of such deformations). Our aim is to use these deformed coherent states for quantizing à la Berezin–Klauder–Toeplitz elementary classical observables like position, momentum, angle, quadratic Hamiltonian etc., and to examine some interesting features of their quantum counterparts. In particular, we insist on the condition that our (symmetric) q-deformations of integers be still integers, more precisely yield sequences of integers that generalize the famous Fibonacci sequence.

* Corresponding author. Tel.: +34 983423146; fax: +34 983423013.
E-mail addresses: gazeau@apc.univ-paris7.fr (J.P. Gazeau), olmo@fta.uva.es (M.A. del Olmo).

0003-4916/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.aop.2012.11.012

J Math Chem (2013) 51:1515–1520
DOI 10.1007/s10910-013-0151-x

ORIGINAL PAPER

Weak harmonic confinement of the quintet solution of a Moshinsky atom with 4 electrons

M. L. Glasser · N. H. March · L. M. Nieto

Received: 5 December 2012 / Accepted: 15 February 2013 / Published online: 26 March 2013

Abstract Bruch, in early work, treated a spatially free Moshinsky atom with four parallel-spin electrons interacting harmonically. Here we add a harmonic external potential, having an unrelated spring constant k, and use the variational method, with a one-parameter trial wave function, to examine the quintet ground state energy and electron density, particularly in the weak confinement limit $k \to 0$. The results are compared with Bruch's and modest contact is made with the early work of Post.

Keywords Moshinsky atom · Harmonic confinement · Beryllium · Quintet state

1 Introduction

Model two-electron atoms with harmonic confinement, going back at least to the work of Moshinsky [1], have now been solved for arbitrary electron-electron interaction $u(r_{12})$ by Holas et al. [2]. A recent study by Amovilli and March [3] of the Hookean atom with four Coulombically repelling electrons employed diffusion quantum Monte-Carlo (DQMC) simulations. These authors demonstrated that, for an external confining potential $V_{ext}(\vec{r})$ given by

M. L. Glasser (✉)
Department of Physics, Clarkson University, Potsdam, NY 13676, USA
e-mail: laryg@clarkson.edu

N. H. March
Department of Physics, University of Antwerp, Antwerp, Belgium

N. H. March
Oxford University, Oxford, England

L. M. Nieto
Departamento de Física Teórica, Atómica y Óptica, Facultad de Ciencias,
University of Valladolid, 40007 Valladolid, Spain

IOP Publishing Journal of Physics: Condensed Matter
J. Phys.: Condens. Matter **25** (2013) 165301 (9pp) doi:10.1088/0953-8984/25/16/165301

Supersymmetry in spherical molecules and fullerenes under perpendicular magnetic fields

V Jakubský[1], Ş Kuru[2], J Negro[3] and S Tristao[3]

[1] Department of Theoretical Physics, Nuclear Physics Institute, 25068 Rež, Czech Republic
[2] Faculty of Science, Department of Physics, Ankara University, 06100 Ankara, Turkey
[3] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, E-47071 Valladolid, Spain

E-mail: jakub@ujf.cas.cz, kuru@science.ankara.edu.tr, jnegro@fta.uva.es and san_maestriaamericas@yahoo.com.br

Received 14 January 2013, in final form 20 February 2013
Published 3 April 2013
Online at stacks.iop.org/JPhysCM/25/165301

Abstract
Methods of supersymmetric quantum mechanics are used to obtain analytical solutions for massless Dirac electrons in spherical molecules, including fullerenes, in the presence of magnetic fields. The solutions for Dirac massive charges are also obtained via the solutions of the Dirac–Weyl equation.

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the isolation of fullerene in 1990, there have been many studies about its interesting electronic properties. Fullerenes are carbon cages that have different shapes and sizes, the well-known fullerene C_{60} is almost a sphere consisting of 12 pentagons and 20 hexagons. A detailed description of fullerenes can be found for instance in [1, 2]. Fullerenes have attracted renewed interest since the discovery of graphene [3, 4].

The continuum model based on the tight-binding approximation for fullerenes has been developed in [5–8]. The Dirac–Weyl equation for spin-$\frac{1}{2}$ particles on the sphere was employed there to describe the low energy excitations of electrons in the crystal. The defects introduced by the pentagons in the fullerene molecule were taken into account by means of an effective magnetic monopole. This approach was used mainly for the computation of the ground state wavefunctions. Although the excited states were not discussed explicitly, they can be easily derived, as we will see later. The electronic states near the Fermi energy have also been considered for a slightly elliptically deformed sphere (spheroidal) in [9] and for icosahedral fullerenes in [10] together with an effective field due to a magnetic monopole placed at the center of the sphere. In [11], in addition to this effective magnetic field, an external uniform magnetic field was considered to study the electronic structure of spheroidal fullerenes.

The relativistic description of the low energy quasi-particles in graphene and related nanostructures proved to be very valuable for acquiring an insight into the physical properties of these condensed matter systems. All this motivates a rigorous analysis of the settings described by the low-dimensional Dirac–Weyl equation, in particular the study of exactly solvable models. Let us mention in this context e.g. [12] where the solutions of the Dirac–Weyl equation corresponding to a massless spin-$\frac{1}{2}$ charged particle on the two-dimensional sphere S^2 as well as on the hyperbolic plane H^2 were obtained by means of shape invariance. Recently, the potential algebra of two-dimensional Dirac-type operators was analyzed as well as their application to describe condensed matter systems [13]. In both references, the systems were studied in the presence of the magnetic monopole.

In this work, our aim is to set up the Dirac–Weyl equation on the sphere with general perpendicular magnetic fields having a rotational symmetry around a fixed axis. We plan to exploit the tools and concepts used in supersymmetric quantum mechanics, in particular the factorization method. The same procedure based on supersymmetric quantum mechanics techniques was previously applied in the analysis of bound states in graphene induced by magnetic fields with translational invariance [14]. Supersymmetric arguments have also been applied in different carbon structures [15].

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Nuclear Physics B 869 [PM] (2013) 303–314

www.elsevier.com/locate/nuclphysb

Generalizations of Maxwell (super)algebras by the expansion method

J.A. de Azcárraga [a,*], J.M. Izquierdo [b], J. Lukierski [c], M. Woronowicz [c]

[a] *Dept. Theor. Phys. and IFIC (CSIC-UVEG), Valencia Univ., 46100-Burjassot (Valencia), Spain*
[b] *Dept. Theor. Phys., Valladolid Univ., 47011-Valladolid, Spain*
[c] *Institute of Theoretical Physics, Wrocław Univ., 50-204 Wrocław, Poland*

Received 10 October 2012; received in revised form 12 December 2012; accepted 13 December 2012

Available online 20 December 2012

Abstract

The Lie algebras expansion method is used to show that the four-dimensional spacetime Maxwell (super)algebras and some of their generalizations can be derived in a simple way as particular expansions of $o(3,2)$ and $osp(N|4)$.

1. Introduction

There are four methods of obtaining new Lie (super)algebras from a given one: contractions, deformations, extensions and expansions. Contractions and deformations lead to new algebras of the same dimension as the original one. The same can be said of an extension $\widetilde{\mathcal{G}}$ of a Lie algebra $\mathcal{G}$ by another one $\mathcal{A}$ (see *e.g.* [1]) in the sense that $\dim\widetilde{\mathcal{G}} = \dim\mathcal{G} + \dim\mathcal{A}$ since $\widetilde{\mathcal{G}} = \mathcal{G}/\mathcal{A}$. A fourth way of obtaining new Lie (super)algebras from a given $\mathcal{G}$ ($s\mathcal{G}$) is the *expansion of algebras*, first used in [2] and studied in general in [3,4]. In contrast with the first three procedures, expanded algebras have, in general, higher dimension than the original $\mathcal{G}$ because expansions give rise to additional generators (expansions also include contractions as a particular case [3], in which they are dimension-preserving). In this paper we shall consider some basic aspects of the expansion procedure to derive Maxwell (super)algebras and other new generalizations as expanded algebras.

* Corresponding author.
E-mail address: j.a.de.azcarraga@ific.uv.es (J.A. de Azcárraga).

0550-3213/$ – see front matter
http://dx.doi.org/10.1016/j.nuclphysb.2012.12.008

Physics Letters A 377 (2013) 2510–2519

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Unstable quantum oscillator with point interactions: Maverick resonances, antibound states and other surprises

J.J. Alvarez [a], M. Gadella [b,*], L.P. Lara [c], F.H. Maldonado-Villamizar [d]

[a] Escuela Universitaria de Informática, Universidad de Valladolid, 40005 Segovia, Spain
[b] Department of FTAO, Facultad de Ciencias, Universidad de Valladolid, Paseo Belén 7, 47011 Valladolid, Spain
[c] Departamento de Física, FCEIA, Universidad Nacional de Rosario, Avda. Pellegrini 250, Rosario, Argentina
[d] Departamento de Física, Centro de Investigación y Estudios Avanzados del IPN, 07360 México DF, Mexico

ARTICLE INFO

Article history:
Received 19 June 2013
Received in revised form 24 July 2013
Accepted 26 July 2013
Available online 2 August 2013
Communicated by P.R. Holland

Keywords:
Quantum resonances
One-dimensional models
Point potentials
Variable mass

ABSTRACT

In the search for solvable or quasi-solvable models for resonances, we consider a one-dimensional potential, which is a harmonic oscillator for $x < 0$, has a point potential at the origin of the form $a\delta(x) + b\delta'(x)$ and no interaction for $x > 0$. After a study of this model, we add a mass jump at the origin and study the effect of the combination of the mass jump and the point potential. We obtain the behavior of resonances, bound and antibound states in terms of given parameters. In spite of the simplicity of the model, it shows quite interesting and unexpected features.

1. Introduction

The study and analysis of quantum unstable states or resonances is of particular importance in molecular, atomic or particle physics as well as in quantum chemistry [1]. However, the theory of quantum resonances is far from being complete. Along some serious conceptual and computational problems, like for instance the difficulty of defining averages on decaying states [2,3], we note that there are not many explicit solvable or quasi-solvable models. These models are quite important for a theoretical study of resonance behavior.

Roughly speaking, there are essentially two types of theoretical models for resonances. First of all, we should mention the Friedrichs model [4,5] and other Friedrichs type models [6,7]. With the exception of some complicated versions [8], they are exactly solvable and give a reasonably good idea on the behavior of systems with one or a small number of resonances. The analysis of realistic systems with an infinite number of resonances, requires of a second type of models: those which are solvable numerically up to a reasonable accuracy for a limited although sufficient number of resonances. In both cases, these resonances are given by an interaction depending on one or several parameters. We are also interested in their behavior as these parameters change. One-dimensional models are often simple, numerically solvable and give interesting non-trivial features on resonance behavior. This fact make them interesting to study. Realistic one-dimensional models are often restrictions of three-dimensional spherically symmetric ones, when restricted to one value of the orbital angular momentum, in the simplest situation $l = 0$. If this is the case, they exhibit an impenetrable barrier at the origin (the potential is infinite for $x < 0$). Resonance models of this type have been studied [9,10]. However, we shall not limit ourselves to these cases, as in the example developed in the present Letter.

It is also worthy to remark that resonances often appear in scattering processes [11,12]. An interesting review on scattering in one-dimensional systems is given in a paper by L.J. Boya [13].

The analysis of the Friedrichs or other models has been quite useful in order to investigate properties of resonances in practical cases, as in nuclear, particle physics or quantum chemistry. This analysis should open the door for further investigations on realistic systems with resonances.

The present Letter is a contribution to the analysis of resonance phenomena based on the study of one-dimensional explicit models. Indeed, we here discuss a quite interesting example of one-dimensional numerically solvable model showing unexpected results. For instance the existence of one resonance that does not follow the general pattern with changes on parameter values. Or the presence of virtual states where one would had expected bound states. Or the appearance of bound states when we consider the existence of a mass jump at the origin, when some parameter

* Corresponding author. Tel.: +34983423145; fax: +34983423013.
E-mail addresses: manuelgadella1@gmail.com, gadella@fta.uva.es (M. Gadella).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.07.045