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Dynamical algebras of general two-parametric Pöschl–Teller Hamiltonians

J.A. Calzada [a], Ş. Kuru [b], J. Negro [c,*], M.A. del Olmo [c]

[a] Departamento de Matemática Aplicada, Universidad de Valladolid, 47071 Valladolid, Spain
[b] Department of Physics, Faculty of Sciences, Ankara University, 06100 Ankara, Turkey
[c] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain

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ABSTRACT

A class of operators connecting general two-parametric Pöschl–Teller Hamiltonians is found. These operators include the so-called "shift" (changing only the potential parameters) and "ladder" (changing also the energy eigenvalue) operators. The explicit action on eigenfunctions is computed within a simple and symmetric three-subindex notation. It is shown that the whole set of operators close an $su(2,2) \approx so(4,2)$ dynamical Lie algebra. A unitary irreducible representation of this $so(4,2)$ differential realization is characterized.

1. Introduction

The purpose of this paper is to investigate and characterize the algebraic properties of the operators that connect Hamiltonians of the two-parametric Pöschl–Teller (P–T) type. This problem is related to others known under different terms. For instance, we are dealing with the so-called "shape-invariant potentials" [1], since the operators involved will change just the parameters in the same family of potentials. Another name appropriate to the operators in this context is that of "potential algebras" [2], in order to distinguish them from the invariance algebras of a given Hamiltonian. Some relations in our study are referred to by "spectrum-generating algebra" [3] due to the fact that some operators may also change the energy eigenvalues of the Hamiltonians. Perhaps a term that is wide enough to include many of the above considerations is "dynamical algebras" of P–T hierarchy, as we are dealing with general operators connecting Hamiltonians in a given family.

* Corresponding author. Tel.: +34 983423040; fax: +34 983423754.
E-mail address: jnegro@fta.uva.es (J. Negro).

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On the determination of approximate periodic solutions of some non-linear ODE

M. Gadella [a,*], L.P. Lara [b,c]

[a] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain
[b] Departamento de Física, FCEIA, Avda. Pellegrini 250, (2000) Rosario, Argentina
[c] Departamento de Sistemas, FRRO, Zevallos 1345, (2000) Rosario, Argentina

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ABSTRACT

We propose a numerical-symbolic method for the approximation of periodic solutions of a type of non-linear ODE. The efficiency of our method is contrasted with the harmonic balance method and with another one which combines the differential transformation method with Padé approximants on a non trivial example: the relativistic oscillator. It is shown that our method is computationally more reliable.

1. Introduction

As is well known, ordinary differential equations (ODE) are widely used in a range of very different fields including physics, engineering, biology, etc. These equations are often non-linear and the determination of explicit or even approximate solutions is a complex task with a difficulty which depends on the system under study. Fortunately, we have a large number of methods based in simple concepts and showing a great level of ingenuity to obtain approximate analytic solutions [1–5].

The objective of the present work is to obtain approximate periodic solutions of some ordinary non-linear differential equations. For the type of equations under our study the most powerful method to obtain approximate solutions is possibly the harmonic balance method [6–8,10,11]. This technique is useful to determine periods and approximate analytic solutions. However, it requires of cumbersome analytic calculations which could be quite impractical to obtain a high order of precision. Fortunately, the use of a recently developed software and new powerful hardware makes these calculations much faster with a considerable reduction of the CPU times.

In the present paper, we are combining traditional methods of numerical integration [12–14] and symbolic calculus in order to obtain approximate analytic solutions.

2. Periodic solution

Our objective is the determination of a quasi-analytic approximation of the periodic solution of the following differential equation:

$$z''(x) = f(x, z(x), z'(x)), \tag{1}$$

with initial values given by $z(x_0) = z_0,\ z'(x_0) = z'_0$. Here, $x \in (a,b)$. The function $f(x,y,z)$ is analytic on a given domain of $\mathbb{R}^3$, at least in the variables y and z. Under this condition, it is known that the solution $z(x)$ of Eq. (1) is analytic on some interval $I \subset \mathbb{R}$ with x_0 as interior point.[1] Then, we determine the solution of Eq. (1) by determining the coefficients of a Taylor series of the form:

* Corresponding author.
E-mail addresses: manuelgadella1@gmail.com (M. Gadella), lplara@fceia.unr.edu.ar (L.P. Lara).

[1] This is an straightforward application of Theorem 4 on page 267 in [1].

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Curvature-dependent formalism, Schrödinger equation and energy levels for the harmonic oscillator on three-dimensional spherical and hyperbolic spaces

José F Cariñena[1], **Manuel F Rañada**[1] **and Mariano Santander**[2]

[1] Departamento de Física Teórica and IUMA, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain

E-mail: jfc@unizar.es, mfran@unizar.es and msn@fta.uva.es

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Abstract

A nonlinear model representing the quantum harmonic oscillator on the three-dimensional spherical and hyperbolic spaces, S^3_κ ($\kappa > 0$) and H^3_k ($\kappa < 0$), is studied. The curvature κ is considered as a parameter and then the radial Schrödinger equation becomes a κ-dependent Gauss hypergeometric equation that can be considered as a κ-deformation of the confluent hypergeometric equation that appears in the Euclidean case. The energy spectrum and the wavefunctions are exactly obtained in both the three-dimensional sphere S^3_κ ($\kappa > 0$) and the hyperbolic space H^3_k ($\kappa < 0$). A comparative study between the spherical and the hyperbolic quantum results is presented.

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1. Introduction

The study of quantum problems in curved spherical spaces (positive constant curvature) was initiated by Schrödinger [1], Infeld [2] and Stevenson [3], in 1940 and 1941. Infeld and Schild [4] considered in 1945 a similar problem but in a hyperbolic space (negative constant curvature). Later, Barut *et al* studied a path integral treatment for the Hydrogen atom in a curved space of constant curvature, first in the spherical case [5] and then in the hyperbolic case [6]. Since then other authors have studied similar problems on curved spaces with constant curvature making use of different approaches [7–28]. Most of these papers are concerned with fundamental problems (previously studied at the classical level) but some authors have proved that this matter is also important for the study of certain questions related to condensed matter physics as, for example, the existence of Landau levels for the motion of a charged particle in a curved space [29–32] and, more recently, the study of quantum dots [33–37].

Analyticity of the time dependence of resonance poles: Solving the Friedrichs model with a time-dependent interaction

O. Civitarese[1] and M. Gadella[2]
[1]*Department of Physics, University of La Plata, c.c. 67 (1900) La Plata, Argentina*
[2]*Departament of FTAO, Facultad de Ciencias, University of Valladolid, 46071 Valladolid, Spain*
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We extend the standard Friedrichs model with an extra term that includes time-dependent interactions. The time dependence of the poles of the reduced resolvent of the model is explicitly calculated. It is found that these poles behave as analytical functions of the added time-dependent interaction. The present results are compared with the ones reported by Kälbermann, concerning the assisted tunnelling of α particles.

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I. INTRODUCTION

The study of the physical and mathematical aspects of resonance poles has been the subject of continuous effort since the physical consequences of their existence were dramatically stressed by Gamow in the earlier days of quantum mechanics [1]. In the modern literature, the role of resonance poles in scattering of particles by nuclei and in the decay of nuclei has been analyzed intensively [2–4]. The connection between mathematical [5] and physical formulations of the problem was investigated in Ref. [6] and further explored in a series of papers by several authors [7–11]. Relevant applications of the concept of resonance poles in nuclear structure and nuclear reactions can be found in Refs. [3,12,13].

Generally speaking, the concept of resonance poles is tied up to the S-matrix formalism [5,14,15]. The Hamiltonian formulation is, perhaps, better presented in the model of Friedrichs [16]. For a detailed discussion of this model and its solutions the reader is kindly referred to a recent review article [17].

Further generalizations of the Friedrichs model, with applications to nuclear physics, have been introduced in Refs. [18,19].

In this work we shall address, from a mathematical oriented view, the question of the enhancement (or hindrance) of the resonant structure of a state, as suggested by Kälbermann [20,21]. In these papers, Kälbermann has discussed, based on numerical analysis, the assisted tunneling of a wave packet between square barriers, and concluded that the tunneling probability is enhanced by the perturbation [20,21].

In order to verify this finding, we shall frame the questions raised in Refs. [20,21] in the language of the Friedrichs model. We shall add, to the standard Friedrichs model, a time-dependent interaction, and feature the solutions in terms of the parameters of such an interaction. The aim of the paper is, therefore, to probe the conditions under which a resonance can be modified by the interaction with external potentials (or fields), without depending much on the detail of the interactions.

The present paper is organized as follows. In Sec. II we review, for the benefit of the readers, the elements entering the standard Friedrichs model. Section III is devoted to the mathematical formulation of a one-dimensional model, corresponding to a finite square well to which we have added a delta-force-type interaction. Although this is a particular and very specific model, it shows that the enhancement or reduction of the lifetime depends solely on certain parameters of the added interaction. Section IV describes the extension of the Friedrichs model, which we have developed to accommodate a time-dependent interaction. Finally, our conclusions are drawn in Sec. V.

II. FORMALISM

The basic Friedrichs model is the simplest nontrivial exactly solvable test model for resonances [16]. It admits rather simple generalizations that may be used as excellent tools to test a resonance behavior with a wide sort of interactions. For instance, we have used it in the past in order to study couplings of fermions with bosons and boson fields [6,19]. A presentation of some of the most relevant generalizations of this model is given in Ref. [17].

A brief description of the Friedrichs model is presented here for the sake of completeness. This description is given in the energy representation, so that dimensional problems are avoided. As in any process producing resonances one has two Hamiltonians: a *free* or unperturbed Hamiltonian H_0 and a *total* or perturbed Hamiltonian $H = H_0 + \lambda V$, where V is the potential describing the interaction and λ is a coupling constant, which is usually chosen positive. H_0 has a simple continuous spectrum given by $\mathbb{R}^+ = [0, \infty)$ and a bound state with energy $\omega_0 > 0$. The potential V intertwines the discrete and continuous spectrum of H_0. This interaction can be regarded as an interaction between a discrete boson and a boson field. Thus, in the energy representation, we can write

$$\begin{aligned} H_0 &= \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| \, d\omega, \\ V &= \int_0^\infty f(\omega)[|\omega\rangle\langle 1| + |1\rangle\langle\omega|] \, d\omega. \end{aligned} \tag{1}$$

Here, $H_0|1\rangle = \omega_0|1\rangle$, $H_0|\omega\rangle = \omega|\omega\rangle$, with $\omega \in [0, \infty)$, and $f(\omega)$ is a given function, hereafter referred to as the "form factor" of the interaction V.

The quantum free particle on spherical and hyperbolic spaces: A curvature dependent approach. II

José F. Cariñena,[1,a)] Manuel F. Rañada,[1,b)] and Mariano Santander[2,c)]
[1]*Departamento de Física Teórica and IUMA, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain*
[2]*Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain*

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This paper is the second part of a study of the quantum free particle on spherical and hyperbolic spaces by making use of a curvature-dependent formalism. Here we study the analogues, on the three-dimensional spherical and hyperbolic spaces, S^3_κ ($\kappa > 0$) and H^3_k ($\kappa < 0$), to the standard *spherical waves* in E^3. The curvature κ is considered as a parameter and for any κ we show how the radial Schrödinger equation can be transformed into a κ-dependent Gauss hypergeometric equation that can be considered as a κ-deformation of the (spherical) Bessel equation. The specific properties of the spherical waves in the spherical case are studied with great detail. These have a discrete spectrum and their wave functions, which are related with families of orthogonal polynomials (both κ-dependent and κ-independent), and are explicitly obtained.
 [http://dx.doi.org/10.1063/1.4757604]

I. INTRODUCTION

This article can be considered as a sequel or continuation of a previous paper,[1] which was devoted to the study of the quantum free particle on two-dimensional spherical and hyperbolic spaces making use of a formalism that considers the curvature κ as a parameter. Now, we present a similar analysis but introducing two changes related with the dimension of the space and with the states of the quantum free particle we are looking for. Now we work in a three-dimensional space, and we look for the states analogous to the Euclidean spherical waves, which are determined among all free particle states by the condition of being separable in the geodesic polar coordinate system. We follow the approach of Ref. 1, which contains the fundamental ideas and motivations, and we also use the notation, ideas, and results discussed in some related previous studies.[2–5]

There are two articles that are considered of great importance in the study of mechanical systems in a spherical geometry (see Ref. 1 for a more detailed information; we just make here a quick survey in a rather telegraphic way). Schrödinger studied in 1940 the hydrogen atom in a spherical space[6] and then other authors studied this problem (hydrogen atom or Kepler problem)[7–12] or other related questions (as, e.g., the quantum oscillator on curved spaces).[13–16] Higgs studied in 1979 the existence of dynamical symmetries in a spherical geometry[17] and since then a certain number of authors have considered[18–39] the problem of the symmetries or some other properties characterizing the Hamiltonian systems on curved spaces (the studies of Schrödinger and Higgs were concerned with a spherical geometry but other authors applied their ideas to the hyperbolic space). In fact, these two problems, the so-called Bertrand potentials, have been the two problems mainly studied in curved spaces (at the two levels, classical and quantum). Nevertheless, in quantum mechanics there are some previous problems that are of fundamental importance as, for example, the quantum free particle or the particle in a spherical well.

[a)]E-mail: jfc@unizar.es.
[b)]E-mail: mfran@unizar.es.
[c)]E-mail: msn@fta.uva.es.

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NUCLEAR PHYSICS B

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$D = 3$ (p, q)-Poincaré supergravities from Lie algebra expansions

J.A. de Azcárraga [a,*], J.M. Izquierdo [b]

[a] *Dept. Theor. Phys. and IFIC (CSIC-UVEG), Univ. of Valencia, 46100-Burjassot (Valencia), Spain*
[b] *Dept. Theor. Phys., Univ. of Valladolid, 47011-Valladolid, Spain*

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Abstract

We use the expansion of superalgebras procedure (summarized in the text) to derive Chern–Simons (CS) actions for the (p, q)-Poincaré supergravities in three-dimensional spacetimes. After deriving the action for the $(p, 0)$-Poincaré supergravity as a CS theory for the expansion $osp(p|2; \mathbb{R})(2, 1)$ of $osp(p|2; \mathbb{R})$, we find the general (p, q)-Poincaré superalgebras and their associated $D = 3$ supergravity actions as CS gauge theories from an expansion of the simple $osp(p + q|2, \mathbb{R})$ superalgebras, namely $osp(p + q|2, \mathbb{R})(2, 1, 2)$.

1. Introduction and results

Some important limits in physics can be described in terms of Lie algebra contractions [1–4]. For instance, the simple de Sitter $so(4, 1)$ and anti-de Sitter $so(3, 2)$ algebras lead by contraction to the $D = 4$ Poincaré algebra. The contraction parameter may be related to the AdS_4 constant curvature when $SO(3, 2)$ is interpreted as the isometry group of four-dimensional spacetime (it is $1/R$ where R is the radius of the universe), or to the square root of the cosmological constant Λ when the algebra is taken as the starting point for the construction of gravity via gauging (for $so(4, 1)$ the cosmological constant changes sign). The most familiar example — which in fact motivated the idea [1,2] — is the Galilei algebra as a $c \to \infty$ İnönü–Wigner (I-W) contraction of the Poincaré one. Of course, the procedure also applies to superalgebras: for instance, the

* Corresponding author.
E-mail address: j.a.de.azcarraga@ific.uv.es (J.A. de Azcárraga).

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An analytically solvable model of confined electrons in a magnetic field and its relation to Landau diamagnetism

M.L. Glasser [a,b,*], N.H. March [c,d], L.M. Nieto [a]

[a] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47011 Valladolid, Spain
[b] Department of Physics, Clarkson University, Potsdam, NY 13699, USA
[c] Department of Physics, University of Antwerp, Antwerp, Belgium
[d] Oxford University, Oxford, England, United Kingdom

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ABSTRACT

Here we present analytic results for the Slater sum and the magnetic moment for arbitrary magnetic field strengths for an assembly of harmonically confined, but initially free, electrons. The relevance of the results to the generalized Landau diamagnetism of such confined electrons is emphasized.

1. Introduction

In this brief report harmonically confined but initially free electrons will be treated analytically. In particular, results will be presented for (a) the Slater sum for such electrons in two dimensions in a transverse magnetic field of arbitrary strength and (b) the corresponding free energy and derived magnetic moment. The relevance to a generalization of Landau diamagnetism for such an inhomogeneous electron assembly is finally stressed.

As a starting point, we refer to the work of March and Tosi [1] (MT) in which a single Wigner oscillator was studied in a magnetic field of arbitrary strength. The Hamiltonian adopted by MT takes the form

$$\hat{H} = \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + \frac{1}{2}k\left(x^2 + y^2\right), \tag{1}$$

where the vector potential $\vec{A}$ was chosen specifically to be

$$\vec{A} = \left(-\frac{1}{2}Hy, \frac{1}{2}Hx, 0\right). \tag{2}$$

MT then solved the Bloch equation for the canonical density matrix C, namely

$$\hat{H}C(\vec{r}, \vec{r}_0, \beta) = -\frac{\partial C}{\partial \beta} \tag{3}$$

with $\beta = (k_B T)^{-1}$ and subject to the completeness boundary condition $C(\vec{r}, \vec{r}_0, \beta = 0) = \delta(\vec{r} - \vec{r}_0)$, generating thereby the pioneering formula of Sondheimer and Wilson [2], who examined, however, only the case $k = 0$ of (1). We merely record in Appendix A the shape of $C(\vec{r}, \vec{r}_0, \beta)$ as derived by MT and turn next to the partition function

$$pf = \int C(\vec{r}, \vec{r}, \beta)\, d\vec{r} = \int S(\vec{r}, \beta)\, d\vec{r} \tag{4}$$

where $S(\vec{r}, \beta)$ denotes the Slater sum – the first focus of this Letter.

Using the canonical density matrix of MT, it will be convenient to write the Slater sum, first of all, in the form following from Eq. (A.1):

$$S(\vec{r}, \beta) = f(\beta) \exp\left\{-4\left(x^2 + y^2\right)h(\beta)\right\}. \tag{5}$$

Putting the confining potential in (1) as

$$V(\vec{r}) = \frac{1}{2}\left(x^2 + y^2\right) \tag{6}$$

Eq. (5) has the form

$$S(\vec{r}, \beta) = f(\beta) \exp\left[-\frac{8}{k}h(\beta)V(r)\right] \tag{7}$$

where, from the work of MT the function $h(\beta)$ has the explicit form

$$f(\beta) = \frac{B}{\sinh \alpha b} \tag{8}$$

where $B = m\omega b/2\pi\hbar$, $\omega = eH/2mc$, $b = (1 + k/m\omega^2)^{1/2}$, $\alpha = \hbar\omega\beta$ and $\beta = (k_B T)^{-1}$. The other function, $h(\beta)$, entering (7) is given in Eq. (A.3).

This special case of the MT model for the partition function $\int S(\vec{r}, \beta)\, d\vec{r}$ can, in fact, be traced back at least to Darwin [3]. The

* Corresponding author at: Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47011 Valladolid, Spain.
E-mail addresses: laryg@verizon.net (M.L. Glasser), luismi@metodos.fam.cie.uva.es (L.M. Nieto).

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Action–angle variables, ladder operators and coherent states

R. Campoamor-Stursberg [a], M. Gadella [b], Ş. Kuru [c], J. Negro [b,*]

[a] Departamento de Geometría y Topología, Universidad Complutense de Madrid, 28040 Madrid, Spain
[b] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain
[c] Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey

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ABSTRACT

This Letter is devoted to the building of coherent states from arguments based on classical action–angle variables. First, we show how these classical variables are associated to an algebraic structure in terms of Poisson brackets. In the quantum context these considerations are implemented by ladder type operators and a structure known as spectrum generating algebra. All this allows to generate coherent states and thereby the correspondence of classical–quantum properties by means of the aforementioned underlying structure. This approach is illustrated with the example of the one-dimensional Pöschl–Teller potential system.

1. Introduction

Coherent states already appeared in physics in the 1920s, when Schrödinger noticed the existence of superpositions of quantum states that exhibit various features like dynamical behaviours quite similar to those of the classical counterparts. In this sense, coherent states constitute an interesting tool that allows to reproduce, under some constraints, properties of the classical behaviour of the system. Although initially this important observation passed unnoticed, the notion of coherent states was rediscovered in the frame of quantum optics in 1963. From that time onwards, coherent states have been recognized as an essential concept in many different domains of physics, and various types of generalizations have been proposed. Most constructions are either based on group theory [1,2], imposing some algebraic constraints [3] or even combining both approaches [4].

In this Letter, we intend to formulate in general terms the connection of coherent states to action–angle variables and the relevant algebraic structure in this context. Especially, we will focus on classical systems with one degree of freedom possessing a periodic motion. These systems are characterized by means of action–angle (AA in short) variables, describing a harmonic time dependence. The harmonic dependence implies that certain Poisson brackets (PB) rules must be satisfied which are related to algebraic structures known as spectrum generating algebras (SGA). With respect to the quantum systems, the stationary bound states have discrete energy values. The transitions to the next (upper or lower) energy level are characterized by frequencies obtained from the energy difference. The operators realizing this class of transitions are called ladder operators and possess some special commutation relations together with the Hamiltonian of the system. This structure is called the quantum SGA.

In summary, the harmonic motion associated to a bound classical state has a classical SGA together with AA variables, while from the quantum point of view, the frequencies of transitions are related to the quantum SGA. Both types of algebras describe the harmonic behaviour in classical and quantum contexts and therefore should be applied to establish a correspondence. We plan to exploit this algebraic connection in order to define and study the properties of coherent states.

The study of coherent states for periodic systems in terms of action–angle variables is not new (see for instance the approach of [5]). However, our motivation is to introduce a consistent definition of coherent state based on the above mentioned algebraic structure of the corresponding classical and quantum systems, based on AA variables. Therefore we are in the line of Refs. [2,6], but taking into account that the algebraic properties of the classical system constitute a basic ingredient too.

This problem is also related to the evolution of wave packets in quantum mechanics in the region of a bounded spectrum that correspond to periodic classical motion (for a review see [7]). There are some quantum properties such as the spreading and recovering

* Corresponding author.
E-mail addresses: rutwigcs@gmail.com (R. Campoamor-Stursberg), manuelgadella1@gmail.com (M. Gadella), kuru@science.ankara.edu.tr (Ş. Kuru), jnegro@fta.uva.es (J. Negro).

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Classical spectrum generating algebra of the Kepler–Coulomb system and action-angle variables

Ş. Kuru [a], J. Negro [b,*]

[a] Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey
[b] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain

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ABSTRACT

The classical spectrum generating algebra for the one-dimensional Kepler–Coulomb system is computed and a set of two corresponding constants of motion depending explicitly on time is obtained. Such constants supply the solution to the motion in an algebraic way. The connection of the spectrum generating algebra and the action-angle variables of the system is also shown.

1. Introduction

One-dimensional solvable systems in quantum mechanics have received much attention, either as toy models to approach the real world or as patterns providing clues for integrable systems in higher dimensions. There is, however, a limited number of problems that can be solved exactly, for example the harmonic oscillator, Pöshl–Teller, or the Kepler–Coulomb potentials. The most popular method, already applied since the beginning of quantum mechanics [1], to deal with this type of potentials was the factorization method, actualized more recently under the name of supersymmetric quantum mechanics [2,3]. For such systems, this method allows to compute the spectrum and eigenfunctions in an algebraic way through structures called potential algebras and/or spectrum generating algebras [4,5].

However, in classical mechanics the interest in one-dimensional systems is almost inexistent [6] since they are considered as trivial superintegrable systems. Nevertheless, in this case we can also define the concept of spectrum generating algebra (SGA) [7] which will imply that not all these classical systems are at the same level: some will have SGA and others will not. Then, we will see that the one-dimensional classical systems allowing for a SGA will have special constants of motion leading to an algebraic solution of the motion and also they allow to find easily the action-angle variables. These classical systems correspond to the one-dimensional quantum factorizable systems mentioned above and, as their quantum analogues, they should be considered as very special systems. In this work we will consider the example of the Kepler–Coulomb (KC) one-dimensional system where we plan to show the interest and the applications of classical SGA's. This important example was not included in a previous classification given in [8], the reason being that the SGA functions of this system have an special form not included there. The peculiarities of the classical functions are quite similar to the factorization operators of the quantum KC system, this is why we will start this Letter by constructing the SGA of the quantum KC system in Section 2. Next, in Section 3, we take a similar procedure in order to find the classical SGA. The algebraic structures at the quantum and classical levels are very simply related by replacing Poisson brackets by commutators following the rule $\{\cdot,\cdot\} \to -i[\cdot,\cdot]$, plus a certain classical limit as we will detail in the final section. The immediate application of the so obtained SGA is the building of two constants of motion that depend explicitly on time. Such constants of motion determine algebraically the motion for bounded and unbounded trajectories. In the case of bounded motion, we show how the SGA is closely related to the action-angle variables. We end this Letter with some conclusions and remarks on the results here obtained.

2. Ladder operators for the quantum Kepler–Coulomb system

In this section we will derive the spectrum generating algebra of the quantum KC system following the factorization method (for

* Corresponding author.
E-mail addresses: kuru@science.ankara.edu.tr (Ş. Kuru), jnegro@fta.uva.es (J. Negro).

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Solutions of the Dirac Equation in a Magnetic Field and Intertwining Operators*

Alonso CONTRERAS-ASTORGA †, *David J. FERNÁNDEZ C.* † *and Javier NEGRO* ‡

† *Departamento de Física, Cinvestav, AP 14-740, 07000 México DF, Mexico*
E-mail: *acontreras@fis.cinvestav.mx*, *david@fis.cinvestav.mx*

‡ *Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain*
E-mail: *jnegro@fta.uva.es*

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Abstract. The intertwining technique has been widely used to study the Schrödinger equation and to generate new Hamiltonians with known spectra. This technique can be adapted to find the bound states of certain Dirac Hamiltonians. In this paper the system to be solved is a relativistic particle placed in a magnetic field with cylindrical symmetry whose intensity decreases as the distance to the symmetry axis grows and its field lines are parallel to the $x-y$ plane. It will be shown that the Hamiltonian under study turns out to be shape invariant.

Key words: intertwining technique; supersymmetric quantum mechanics; Dirac equation

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1 Introduction

The intertwining technique, also called Supersymmetric Quantum Mechanics (SUSY QM), is a widespread method used to generate exactly solvable Hamiltonians departing from a given initial one and can be employed as well to solve a certain set of Hamiltonians in a closed way, among other applications. In the simplest case (1-SUSY QM) the new potentials have similar spectra as the original one, namely, they might differ at most in the ground state energy. Examples of potentials generated by this technique are those which arise when adding a bound state to the free particle Hamiltonian (hyperbolic Pöschl–Teller) [14] or the Abraham–Moses–Mielnik potentials which are isospectral to the harmonic oscillator [1, 13, 15]. This method has been also applied successfully to the radial part of the hydrogen atom potential [1, 7, 13, 18], the trigonometric Pöschl–Teller potentials [3], among many others.

To apply the technique [8] we start from two one-dimensional Schrödinger Hamiltonians

$$H_i = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_i(x), \qquad i = 0, 1,$$

where H_0 is known. Now let us suppose the existence of a differential operator $A_1^\dagger$ which satisfies

$$H_1 A_1^\dagger = A_1^\dagger H_0, \qquad A_1^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W_1(x)\right). \tag{1}$$

Since the operator $A_1^\dagger$ is of first order, the technique is known as 1-SUSY QM and the function $W_1(x)$ as the superpotential. It is also said that the potentials $V_0(x)$ and $V_1(x)$, whose Hamiltonians are intertwined by the operator $A_1^\dagger$, are supersymmetric partners.

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