Applied Mathematics and Computation 217 (2011) 9480–9487

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Iterative solution of some nonlinear differential equations

M. Gadella [a,*], L.P. Lara [b,c], G.P. Pronko [d]

[a] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain
[b] Departamento de Física, FCEIA, Avda. Pellegrini 250, Rosario, Argentina
[c] Departamento Sistemas, FRRO, Zevallos 1345, Rosario, Argentina
[d] Department of Theoretical Physics, IHEP. Protvino, Moscow Region 142280, Russia

ARTICLE INFO

Keywords:
Iterative solutions
Mathieu equation
van der Pole equation
Hill equation

ABSTRACT

We propose an iterative method to solve some non-linear ordinary differential equations. Comparing on the Mathieu, van der Pol and Hill equation of fourth order, we see that this method is much more efficient than the well known methods by Lyapunov or Picard.

1. Introduction

We intend to determine explicit approximate solutions, with given initial values, of equations of the type

$$z''(x) + \omega^2 z(x) = f(x, z, z'), \quad z(x_a) = \alpha, \quad z'(x_a) = \beta. \tag{1}$$

In particular, we shall focus our attention in the search for periodic solutions. There exists several perturbative methods to determine explicit approximate solutions. Most of them require a small perturbative parameter. In this context, we can mention for instance the Lindstedt–Poincaré, Krylov–Bogolubov–Mitropolskii perturbation methods and also the multi-time expansion method [1]. When a perturbative parameter cannot be found, one can use the harmonic balance to obtain periodic approximations [2,3]. The need of efficient ways to find approximate solutions in the case of strongly nonlinear equations has been stressed in [4–6]. Apart from those mentioned general methods, there exist a more specific approach for some equations like the van der Pol equation [7–12]. Different iteration schemes are discussed in [13,14]. The implementation of all these methods of resolution have a certain degree of difficulty that can be somehow overcome with the use of a software like Mathematica. In this article, we present an effective calculation tool, which is conceptually quite simple and is inspired in the method of successive approximations of Picard–Liouville [12–15].

In this kind of articles in which an analytic solution is search using iterations a study of the uniform convergence of the approximate solutions to the exact solution is an absolute requirement. This result is shown in Appendix A.

This paper is organized as follows: In Section 2, we propose our iterative method. Sections 3–5 are devoted to applications to given linear (Mathieu or Hill of fourth order) or non-linear (van der Pol) differential equations. In addition, we compare our results with those obtained with Lyapunov and Picard methods. Finally, in Appendix A, we discuss the convergence of our method.

2. The iterative method

We shall obtain explicit solutions of Eq. (1), where f is a continuous function of three variables continuous on a domain that includes the range $x_a \leqslant x \leqslant x_b$ for the first variable. Inspired in the Picard method [12,15], we try to solve Eq. (1) by iteration. At the kth step (1) looks like

* Corresponding author.
E-mail addresses: manuelgadella1@gmail.com (M. Gadella), lplara@fceia.unr.edu.ar (L.P. Lara), George.Pronko@ihep.ru (G.P. Pronko).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.04.058

Models including electron correlation in relation to Fock's proposed expansion of the ground-state wave function of He-like atomic ions

M. L. Glasser*
Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, ES-47011 Valladolid, Spain and Department of Physics, Clarkson University, Potsdam, New York 13699, USA

N. H. March
Department of Physics, University of Antwerp, BE-2020 Antwerp, Belgium and Department of Theoretical Chemistry, University of Oxford, Oxford OX1 2JD, United Kingdom

L. M. Nieto†
Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, ES-47011 Valladolid, Spain
(Received 13 December 2010; revised manuscript received 9 November 2011; published 22 December 2011)

Here attention is first drawn to the importance of gaining insight into Fock's early proposal for expanding the ground-state wave function for He-like atomic ions in hyperspherical coordinates. We approach the problem via two solvable models, namely, (i) the s-term model put forth by Temkin [Phys. Rev. **126**, 130 (1962)] and (ii) the Hookean atom model proposed by Kestner and Sinanoglu [Phys. Rev. **128**, 2687 (1962)]. In both cases the local kinetic energy can be obtained explicitly in hyperspherical coordinates. Separation of variables occurs in both model wave functions, though in a different context in the two cases. Finally, a **k**-space formulation is proposed that should eventually result in distinctive identifying characteristics of Fock's nonanalyticities for He-like atomic ions when both electrons are close to the nucleus.

DOI: 10.1103/PhysRevA.84.062119 PACS number(s): 03.65.Ge, 01.55.+b, 02.30.Hq

I. INTRODUCTION

Because of the present intractability of the exact analytic solution of the Schrödinger solution for the ground-state of He-like atomic ions, we approach the problem here via two solvable models. The first model we discuss was put forth by Kestner and Sinanoglu [1], who proposed the replacement of the Coulomb confinement form of the external potential $V_{\rm ext}(\vec{r})$ by a harmonic restoring potential given by $\frac{1}{2}kr^2$ but retaining the full Coulombic interaction e^2/r_{12}. It was natural to refer to this model as the two-electron Hookean atom. The model's great merit is that the center-of mass motion can be separated from the relative motion (see, for instance, Ref. [2]). Kais *et al.* quote the exact ground-state wave function $\Psi(\vec{r}_1,\vec{r}_2)$ for the solvable case of the Hamiltonian $H_{\rm Hooke} = -\nabla_1^2 - \nabla_2^2 + e^2/r_{12} + (k/2)(r_1^2 + r_2^2)$ for $k = \frac{1}{4}$ a.u. as

$$\Psi(\vec{r}_1,\vec{r}_2) = N_0\big(1 + \tfrac{1}{2}|\vec{r}_1 - \vec{r}_2|\big)\exp\big[-\tfrac{1}{4}\big(r_1^2 + r_2^2\big)\big], \quad (1)$$

which corresponds to ground-state energy $E_0 = 2$ a.u. ($E_{\rm HF} = 2.039\,325$ a.u., where $E_{\rm HF}$ represents the Hartree-Fock energy). Following Fock's proposal [3], we immediately rewrite Eq. (1) in hyperspherical coordinates R, Θ, and α defined by

$$R = \sqrt{r_1^2 + r_2^2}, \quad r_1 = R\cos\alpha, \quad r_2 = R\sin\alpha, \quad \cos\Theta = \frac{\vec{r}_1\cdot\vec{r}_2}{r_1 r_2} \quad (2)$$

to obtain

$$\Psi(R,\alpha,\Theta) = \Psi_g(R) - \sqrt{1 - \sin 2\alpha\cos\Theta}\,\frac{\partial\Psi_g(R)}{\partial R}, \quad (3)$$

where $N_0\exp[-\frac{1}{4}R^2] = \Psi_g(R)$.

For He-like atomic ions Fock proposed expanding the ground-state wave function for small R in a form involving not only non-negative-integer powers of R, but also positive-integer powers of $\ln R$. We refer to the important subsequent work of Refs. [4–9], which have demonstrated the convergence and usefulness of Fock's expansion. It is also relevant here to note two recent papers on the Hookean model [10,11].

Here we note that while Eq. (3) is not directly separable in hyperspherical coordinates, it is the sum of a part $\Psi_g(R)$, independent of the angles α and Θ, plus a piece with an R dependence determined by $\partial\Psi_g(R)/\partial R$, times the known function $\sqrt{1 - \sin 2\alpha\cos\Theta}$ of angles α and Θ. Clearly, both R-dependent parts can be expanded to all orders in R^2 and there are no logarithmic terms at small R. Since for small R both electrons are near the origin of harmonic confinement, it is clear that e^2/r_{12} alone for small r_{12} does not lead to Fock-like nonanalytic terms for small r. Thus, Coulomb confinement with $V_{\rm ext}(r) = -Z/r$ replacing the harmonic form needs to be invoked in He if Fock's nonanalytic behavior is to be recovered.

This leads us directly to a second model, proposed by Temkin [12], which is now referred to as the Temkin-Poet [13] model for He-like atomic ions. Particularly relevant to this study is Ref. [14], where it is pointed out that the wave function considered by Howard and March [15], namely,

$$\Psi_{\rm HM}(\vec{r}_1,\vec{r}_2) = C\exp\left[-(Z - 1/2)(r_1 + r_2) + \frac{|r_1 - r_2|}{2}\right], \quad (4)$$

*laryg@verizon.net
†luismi@metodos.fam.cie.uva.es

Exotic supersymmetry of the kink-antikink crystal, and the infinite period limit

Mikhail S. Plyushchay,[1,2] Adrián Arancibia,[1] and Luis-Miguel Nieto[2]

[1]*Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*

[2]*Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071, Valladolid Spain*

(Received 30 December 2010; revised manuscript received 28 January 2011; published 23 March 2011)

Some time ago, Thies *et al.* showed that the Gross-Neveu model with a bare mass term possesses a kink-antikink crystalline phase. Corresponding self-consistent solutions, known earlier in polymer physics, are described by a self-isospectral pair of one-gap periodic Lamé potentials with a Darboux displacement depending on the bare mass. We study an unusual supersymmetry of such a second-order Lamé system, and show that the associated first-order Bogoliubov-de Gennes Hamiltonian possesses its own nonlinear supersymmetry. The Witten index is ascertained to be zero for both of the related exotic supersymmetric structures, each of which admits several alternatives for the choice of a grading operator. A restoration of the discrete chiral symmetry at zero value of the bare mass, when the kink-antikink crystalline condensate transforms into the kink crystal, is shown to be accompanied by structural changes in both of the supersymmetries. We find that the infinite period limit may or may not change the index. We also explain the origin of the Darboux-dressing phenomenon recently observed in a nonperiodic self-isospectral one-gap Pöschl-Teller system, which describes the Dashen, Hasslacher, and Neveu kink-antikink baryons.

DOI: 10.1103/PhysRevD.83.065025 PACS numbers: 11.30.Pb, 03.65.−w, 11.10.Kk, 11.10.Lm

I. INTRODUCTION

The Gross-Neveu (GN) model [1–3] is a remarkable $(1+1)$-dimensional theory of self-interacting fermions that has no gauge fields or gauge symmetries, but exhibits some important features of quantum chromodynamics, namely, asymptotic freedom, dynamical mass generation, and chiral symmetry breaking [4]. It has been widely studied over the years and the richness of its properties is still astonishing. Some time ago, Thies *et al.* showed that at finite density, the ground state of the model with a discrete chiral symmetry is a kink crystal [5], while the kink-antikink crystalline phase was found in the GN model with a bare mass term [6]. Then, Dunne and Basar derived a new self-consistent inhomogeneous condensate, the twisted kink crystal in the GN model with continuous chiral symmetry [7,8]. On the other hand, the relation of the GN model with the sinh-Gordon equation and classical string solutions in AdS_3 has been observed recently [9,10].

These two classes of the results seem to be different, but both are rooted in the integrability features of the GN model, and may be related to the Bogoliubov-de Gennes (BdG) equations incorporated implicitly in its structure. It is because of these properties that the model finds many applications in diverse areas of physics. Particularly, the model has provided very fruitful links between particle and condensed matter physics, see [11–13].

The origin of the model itself may also be somewhat related to the BdG equations. We briefly discuss these equations to formulate the aim of the present paper.

The BdG equations [14] in the Andreev approximation [15] is a set of two coupled linear differential equations, which can be presented in the form of a stationary Dirac-type matrix equation,

$$\hat{G}_1\psi = \omega\psi, \qquad \hat{G}_1 = a\sigma_1\frac{1}{i}\frac{d}{dx} - \sigma_2\Delta(x). \tag{1.1}$$

The scalar field $\Delta(x)$ is determined via a self-consistency condition, which is often referred to as a gap equation. Equation (1.1) arose in the theory of superconductivity by linearizing the *nonrelativistic* energy dispersion (in the absence of magnetic field), or, equivalently, by neglecting the second derivatives of the Bogoliubov amplitudes, see [16]. A constant a is proportional there to the Fermi momentum $\hbar k_F$. In what follows, we put $a = 1$ and $\hbar = 1$.

The Lagrangian of the GN model of the N species of self-interacting fermions is

$$\mathcal{L}_{\mathrm{GN}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_0)\psi + \tfrac{1}{2}g^2(\bar{\psi}\psi)^2, \tag{1.2}$$

where g^2 is a coupling constant, the summation in the flavor index is suppressed, and a bare mass term $\sim m_0$, which breaks explicitly the discrete chiral symmetry $\psi \to \gamma_5\psi$ of the massless model, is included.[1] It is the two-dimensional version of the Nambu-Jona-Lasinio model [17] (with continuous chiral symmetry reduced to the discrete one). The latter is based on an analogy with superconductivity, and was introduced as a model of symmetry breaking in particle physics. There are two equivalent methods to seek solutions for the

[1]The investigation of model (1.2) is motivated in [6] by a massive nature of quarks; there, the 't Hooft limit $N \to \infty$, $Ng^2 = \text{const}$, is considered.

Klein tunneling in carbon nanostructures: A free-particle dynamics in disguise

Vít Jakubský,[1] Luis-Miguel Nieto,[2] and Mikhail S. Plyushchay[2,3,*]

[1]*Nuclear Physics Institute, Řež near Prague, 25068 Czech Republic*

[2]*Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071, Valladolid, Spain*

[3]*Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile*

(Received 6 October 2010; published 14 February 2011)

The absence of backscattering in metallic nanotubes as well as perfect Klein tunneling in potential barriers in graphene are the prominent electronic characteristics of carbon nanostructures. We show that the phenomena can be explained by a peculiar supersymmetry generated by a first order Hamiltonian and zero-order supercharge operators. Like the supersymmetry associated with second order reflectionless finite-gap systems, it relates here the low-energy behavior of the charge carriers with the free-particle dynamics.

DOI: 10.1103/PhysRevD.83.047702 PACS numbers: 72.80.Vp, 11.30.Pb, 73.63.Fg

Graphene is a genuine two-dimensional material composed of the carbon atoms that form a honeycomb lattice. Three of the valence electrons of each carbon atom participate in the interatomic interaction, while the fourth one contributes to the conductivity of the crystal. Graphene has been studied theoretically for a long time, see e.g., [1,2]. However, its experimental observation [3] triggered a real boom of both theoretical and experimental analysis [4–6].

The material manifests extraordinary electronic properties, which are the consequence of an unusual dynamics of the low-energy charge carriers. It was pointed out in [2] that the tight-binding description of the system is reduced to the massless Dirac equation in the low-energy approximation. This makes graphene an ideal test field for $(2+1)$-dimensional QED [6]; due to the low Fermi velocity v_F, $c/v_F \sim 300$, it is possible to simulate relativistic effects in condensed matter systems which would be unreachable experimentally otherwise.

It was predicted [7,8] that the scattering of the relativistic electrons on the potential barrier is qualitatively different from the nonrelativistic case. The particles can tunnel the barrier without reflection, provided that its height tends to infinity. This is in contrast to the nonrelativistic regime where the tunneling would be exponentially suppressed [8]. This phenomenon, known as Klein tunneling, is not experimentally realizable with elementary particles nowadays due to the extreme electric field needed to observe the predicted difference between relativistic and nonrelativistic scattering [9].

The scattering of the low-energy quasiparticles in graphene on the barrier with translational symmetry in one dimension was analyzed in [9–11]. The absence of backscattering was noticed for normal incidence. The effect is independent of the height of the barrier and, hence, is testable experimentally [12]. A similar phenomenon was observed earlier [13] and discussed theoretically [14–16] in the context of electron transport in carbon nanotubes. The perfect transmission of the low-energy charge carriers occurs in metallic nanotubes despite the presence of a scattering potential generated by impurities. The absence of backscattering was understood as a consequence of topological singularity identified with a Dirac point, see [17,18], or as a result of the pseudospin conservation [9].

We provide here a *simple, alternative* explanation for the absence of backscattering in the carbon nanostructures within the *framework of supersymmetric quantum mechanics*. We shall discuss a broad class of potentials in graphene as well as in the metallic nanotubes with the range exceeding the interatomic distance.

The honeycomb lattice is a superposition of two triangular sublattices, A and B. The eigenstate Ψ of the Hamiltonian can be then written as $\Psi = c_A\Psi_A + c_B\Psi_B$, where Ψ_A and Ψ_B are atomic wave functions of the sublattices, whereas c_A and c_B are slowly varying amplitudes. Ψ is a Bloch function, which acquires a nontrivial phase factor when shifted by a translation vector $\mathbf{R}$ of the Bravais lattice, $\Psi(\mathbf{k}, \mathbf{x}+\mathbf{R}) = e^{i\mathbf{k}\mathbf{R}}\Psi(\mathbf{k}, \mathbf{x})$. Fermi surface of graphene is formed by discrete points. There are six of them in the first Brillouin zone, situated in its corners, see Fig. 1. In the analysis of the low-energy behavior of the charge carriers, it is sufficient to consider just two of them, denoted as Dirac points $\mathbf{K}$ and $\mathbf{K}' = -\mathbf{K}$. The remaining four Dirac points do not represent distinct electronic states. They can be obtained either from $\mathbf{K}$ or $\mathbf{K}'$ by translations in the reciprocal lattice.

In the vicinity of Dirac points, the behavior of the system is described by the massless Dirac equation. When the effective Hamiltonian is considered in the valley of the point $\mathbf{K}$ and expanded up to the terms linear in the momenta, the energy eigenvalue equation acquires the form [2] (we put $\hbar = 1$)

$$H\psi = -iv_F(\sigma_1\partial_x + \sigma_2\partial_y)\psi = E\psi, \tag{1}$$

where $\sigma_{1,2}$ are Pauli matrices. Spinor ψ reads explicitly $\psi = (c_A, c_B)^t e^{i\delta\mathbf{k}\mathbf{x}}$, where $\delta\mathbf{k} = \mathbf{k} - \mathbf{K}$ and t is a transposition. Degree of freedom associated with the components c_A and c_B is called pseudospin, see [4,5].

*mikhail.plyushchay@usach.cl

Fortschr. Phys., 1–65 (2011) / **DOI** 10.1002/prop.201100038

The Friedrichs model and its use in resonance phenomena

M. Gadella[1,*] and **G. P. Pronko**[2]

[1] Departamento de Física Teórica, Atómica y Optica, Facultad de Ciencias, 47071 Valladolid, Spain
[2] Institute for High Energy Physics, Protvino 142284, Moscow Region, Russia

Received 4 March 2011, revised 20 June 2011, accepted 7 July 2011
Published online 29 July 2011

Key words Resonances, Friedrichs model.

We present here a relation of different types of Friedrichs models and their use in the description and comprehension of resonance phenomena. We first discuss the basic Friedrichs model and obtain its resonance in the case that this is simple or doubly degenerated. Next, we discuss the model with N levels and show how the probability amplitude has an oscillatory behavior. Two generalizations of the Friedrichs model are suitable to introduce resonance behavior in quantum field theory. We also discuss a discrete version of the Friedrichs model and also a resonant interaction between two systems both with continuous spectrum. In an appendix, we review the mathematics of rigged Hilbert spaces.

1 Introduction

The Friedrichs model is a model aimed to describe the basic features of resonance phenomena. The basic idea is considering resonances associated to a Hamiltonian pair $\{H_0, H\}$, where H_0 is the Hamiltonian for the "non perturbed" dynamics. H_0 has a simple non-degenerate absolutely continuous spectrum that coincides with the positive semiaxis. In addition, H_0 has at least an eigenvalue imbedded in the continuous spectrum. The total or "perturbed" Hamiltonian has the form $H = H_0 + \lambda V$, where V is a potential and λ a coupling parameter that it is usually taken to be real (to preserve the self adjointness of H) and positive. The potential depends on a form factor function $f(\omega)$, which determines the existence and properties of the resonance. The action of the potential is to transform the bound state into a resonance, characterized by a point in the complex plane, as shall be described below. This point depends analytically on the coupling parameter λ. This is the basic description of the model as originally introduced by Friedrichs in 1948 [50].

The first step to show that the Fridrichs model is an excellent device in order to understand the machinery of decay in Quantum Mechanics accessible to physicists was given by Horwitz and Marchand [59]. After that, there were given several generalizations of the original model for various purposes including a description of unstable theory of fields.

In the present review, we intend to discuss most of the known versions of The Friedrichs model together their applications to model various situations in which quantum decay appears.

The Fridrichs model was conceived as mathematically rigorous and exactly solvable so that it could well serve as a toy model for a precise description of quantum decay. Also its possible generalizations are enormous in number and vast in applications. The present review is a first step to collect these generalizations. In order not to make this paper excessively long, we have selected some of these generalizations and not included a few ones. Our selection has been biased by our own work in the field. Examples of generalizations of the Fridrichs model that we have not included in our review are:

* Corresponding author E-mail: manuelgadella1@gmail.com

The quantum free particle on spherical and hyperbolic spaces: A curvature dependent approach

José F. Cariñena,[1,a)] Manuel F. Rañada,[1,b)] and Mariano Santander[2,c)]
[1]*Departamento de Física Teórica and IUMA, Facultad de Ciencias Universidad de Zaragoza, 50009 Zaragoza, Spain*
[2]*Departamento de Física Teórica, Facultad de Ciencias Universidad de Valladolid, 47011 Valladolid, Spain*

(Received 2 January 2011; accepted 25 June 2011; published online 26 July 2011)

The quantum free particle on the sphere S^2_κ ($\kappa > 0$) and on the hyperbolic plane H^2_κ ($\kappa < 0$) is studied using a formalism that considers the curvature κ as a parameter. The first part is mainly concerned with the analysis of some geometric formalisms appropriate for the description of the dynamics on the spaces (S^2_κ, $\mathbb{R}^2$, H^2_κ) and with the transition from the classical κ-dependent system to the quantum one using the quantization of the Noether momenta. The Schrödinger separability and the quantum superintegrability are also discussed. The second part is devoted to the resolution of the κ-dependent Schrödinger equation. First the characterization of the κ-dependent "curved" plane waves is analyzed and then the specific properties of the spherical case are studied with great detail. It is proved that if $\kappa > 0$ then a discrete spectrum is obtained. The wavefunctions, that are related with a κ-dependent family of orthogonal polynomials, are explicitly obtained. © *2011 American Institute of Physics*. [doi:10.1063/1.3610674]

I. INTRODUCTION

The correct formulation of quantum mechanics on spaces of constant curvature is a problem that can lead to important difficulties. There are some fundamental quantum questions, well stated in the Euclidean space, that become difficult to formulate on a curved space. The study of these questions is important, not only for extending our knowledge of certain fundamental points of quantum mechanics, but also because it is very convenient for the construction of more general relativistic theories.[1,2] In addition, this matter has also become important for the study of certain questions arising in applied nonrelativistic quantum mechanics. We mention here two examples related with two-dimensional quantum mechanics and with condensed matter physics. In the first case (motion of a particle on a two-dimensional surface) the existence of Landau levels for the motion of a charged particle under perpendicular magnetic fields has been also studied for the case of non-Euclidean geometries.[3–6] Concerning the second point, the study of quantum dots has also lead to the use of models based in quantum mechanics in spaces of constant curvature.[7–11]

The first step was probably given by Schrödinger who made use of a factorization method[12] for the study of the hydrogen atom in a spherical geometry. Then Infeld[13] and Stevenson[14] studied the same system and Infeld and Schild[15] considered this problem in an open universe of constant negative curvature. Other more recent papers on the hydrogen atom in a curved space are

[a)]Electronic mail: jfc@unizar.es.
[b)]Electronic mail: mfran@unizar.es.
[c)]Electronic mail: msn@fta.uva.es.

On a class of *n*-Leibniz deformations of the simple Filippov algebras

José A. de Azcárraga[1,a)] and José M. Izquierdo[2]
[1]*Deptartment of Theoretical Physics and IFIC (CSIC-UVEG), University of Valencia, 46100-Burjassot (Valencia), Spain*
[2]*Deptartment of Theoretical Physics, University of Valladolid, 47011-Valladolid, Spain*

(Received 20 September 2010; accepted 17 January 2011; published online 25 February 2011)

We study the problem of infinitesimal deformations of all real, simple, finite-dimensional Filippov (or *n*-Lie) algebras, considered as a class of *n*-Leibniz algebras characterized by having an *n*-bracket skewsymmetric in its $n-1$ first arguments. We prove that all $n>3$ simple finite-dimensional Filippov algebras (FAs) are rigid as *n*-Leibniz algebras of this class. This rigidity also holds for the Leibniz deformations of the semisimple $n=2$ Filippov (i.e., Lie) algebras. The $n=3$ simple FAs, however, admit a nontrivial one-parameter infinitesimal 3-Leibniz algebra deformation. We also show that the $n\geq 3$ simple Filippov algebras do not admit nontrivial central extensions as *n*-Leibniz algebras of the above class. © *2011 American Institute of Physics*. [doi:10.1063/1.3553797]

I. INTRODUCTION

Lie algebras can be generalized by relaxing the skewsymmetry of the Lie bracket. This leads to the *Leibniz (or Loday's) algebras* $\mathscr{L}$,[1–4] defined as a vector space $\mathscr{L}$ endowed with a bilinear operation $\mathscr{L}\times\mathscr{L}\to\mathscr{L}$ that satisfies the *Leibniz identity*

$$[X,[Y,Z]]=[[X,Y],Z]+[Y,[X,Z]]\qquad \forall X,Y,Z\in\mathscr{L}, \tag{1.1}$$

which states that $ad_X=[X,\ ]$ is a derivation of the Leibniz bracket. Lie algebras $\mathfrak{g}$ are the special class of Leibniz algebras for which $[X,Y]=-[Y,X]\ \forall X,Y$. Since the Leibniz algebra bracket is not skewsymmetric, left and right derivations are not (anti)equivalent and correspondingly there are two possible versions of the Leibniz identity; Eq. (1.1), which we shall adopt, is the left Leibniz identity and correspondingly defines a left Leibniz algebra.

Lie algebra deformations[5,6] can be easily generalized to the Leibniz case. Infinitesimal Leibniz algebra deformations are defined by a deformed bracket $[X_1,X_2]_t$,

$$[X_1,X_2]_t=[X_1,X_2]+t\alpha(X_1,X_2), \tag{1.2}$$

such that $\alpha(X_1,X_2)$ is a bilinear $\mathscr{L}$-valued map $\alpha:\mathscr{L}\otimes\mathscr{L}\to\mathscr{L}$, $\alpha:(X_1,X_2)\mapsto\alpha(X_1,X_2)$ and $[X_1,X_2]_t$ satisfies (1.1) (for deformations of Lie algebras $\mathfrak{g}$, α would be skewsymmetric in its two arguments). The nontrivial inequivalent infinitesimal deformations of Lie and Leibniz algebras are classified by the elements of the second cohomology groups $H^2_{ad}(\mathfrak{g},\mathfrak{g})$ and $H^2_{ad}(\mathscr{L},\mathscr{L})$, respectively. The Leibniz algebra cohomology has been discussed in Refs. 1–3 and 7 (there for right $\mathscr{L}$s) and in Ref. 8. The cohomology complex $(C^\bullet(\mathscr{L},\mathscr{L}),\delta)$ becomes the Lie algebra one $(C^\bullet(\mathfrak{g},\mathfrak{g}),\delta)$ when $\mathscr{L}=\mathfrak{g}$ and, as a result of the antisymmetry, the cochains are also required to be antisymmetric. But, since Lie algebras are also Leibniz, it is also possible to look for Leibniz deformations of Lie algebras when viewed as Leibniz ones. This may result in the appearance of more deformations, a fact recently discussed and observed in Ref. 9 for the nilpotent three-dimensional Heisenberg algebra. In fact, and for a symmetric representation of $\mathscr{L}$ (Refs. 2 and 1), there is a homomorphism[2]

a)Author to whom correspondence should be addressed. Electronic mail: j.a.de.azcarraga@ific.uv.es.

Contractions of Filippov algebras

José A. de Azcárraga,[1,a)] José M. Izquierdo,[2] and Moisés Picón[1,3]
[1]*Department of Theoretical Physics and IFIC (CSIC-UVEG), University of Valencia, 46100-Burjassot, Valencia, Spain*
[2]*Department of Theoretical Physics, University of Valladolid, 47011-Valladolid, Spain*
[3]*C.N.Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794-3840, USA*

(Received 6 September 2010; accepted 14 December 2010; published online 31 January 2011)

We introduce in this paper the contractions $\mathfrak{G}_c$ of n-Lie (or Filippov) algebras $\mathfrak{G}$ and show that they have a semidirect structure as their $n = 2$ Lie algebra counterparts. As an example, we compute the nontrivial contractions of the simple A_{n+1} Filippov algebras. By using the İnönü–Wigner and the generalized Weimar-Woods contractions of ordinary Lie algebras, we compare (in the $\mathfrak{G} = A_{n+1}$ simple case) the Lie algebras Lie $\mathfrak{G}_c$ (the Lie algebra of inner endomorphisms of $\mathfrak{G}_c$) with certain contractions (Lie $\mathfrak{G})_{IW}$ and (Lie $\mathfrak{G})_{W-W}$ of the Lie algebra Lie $\mathfrak{G}$ associated with $\mathfrak{G}$.
 [doi:10.1063/1.3533944]

I. INTRODUCTION

In 1985, Filippov[1,2] initiated the study of certain linear algebras (called n-Lie algebras by him) endowed with a completely antisymmetric bracket with n entries that satisfies a characteristic identity, the Filippov identity. These *n-Lie* or *Filippov algebras* (FAs) $\mathfrak{G}$ reduce for $n = 2$ to ordinary Lie algebras $\mathfrak{g}$.

The properties of Filippov algebras[1] have been studied further in parallel with those of the Lie algebras, specially by Kasymov[3,4] and Ling[5] (see Ref. 6 for a review). It has been shown, for instance, that it is possible to define solvable ideals, simple and semisimple Filippov algebras, etc. Semisimple FAs satisfy a Cartan-like criterion[4] and, as in the Lie algebra case, they are given by the direct sums of simple ones. One result, however, in which FAs differ significantly from their $n = 2$ Lie algebra counterparts is that for each $n > 2$ there is only one complex simple finite Filippov algebra,[1,5] which is $(n + 1)$-dimensional. The real Euclidean simple n-Lie algebras A_{n+1}, which are constructed on Euclidean $(n + 1)$-dimensional vector spaces, are thus the only $(n > 2)$-Lie (Filippov) algebra generalizations of the simple $so(3)$ Lie algebra. Similarly, the simple pseudo-Euclidean ones may be considered as $n > 2$ generalizations of $so(1, 2)$.

Other properties of FAs, such as deformations (or, e.g., central extensions) may be studied. As in the general and Lie algebra cases,[7,8] deformations are associated with FA cohomology. The Filippov algebra cohomology suitable for deformations of Filippov algebras was given in Ref. 9 in the context of Nambu–Poisson algebras (see further Refs. 10, 11, and 12); the FA cohomology generalizes the Lie algebra cohomology complexes (see also Ref. 6). The FA cohomology is not completely straightforward. For instance, for $n > 3$ it turns out that the p-cochains are mappings $\alpha^p : \wedge^{n-1}\mathfrak{G} \otimes \overset{p}{\cdots} \otimes \wedge^{n-1}\mathfrak{G} \wedge \mathfrak{G} \to \mathbb{R}$ (e.g., in the cohomology suitable for central extensions of FAs) rather than $\alpha^p : \wedge^p\mathfrak{g} \to \mathbb{R}$ as they would be for Lie algebras $\mathfrak{g}$. Thus, it is convenient to label the p-cochains by the number p of arguments $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ that they contain rather than by the number of elements of $\mathfrak{G}$ itself (the $\mathcal{X}$'s were called *fundamental objects* in Ref. 12). It has been proved recently[12] that there is a Whitehead lemma for Filippov algebras: semisimple FAs do not have nontrivial central extensions and are moreover rigid, i.e., they do not admit nontrivial deformations. As a result, the Whitehead lemma holds true for all n-Lie semisimple FAs, $n \geq 2$.

a)Electronic mail: j.a.de.azcarraga@ific.uv.es.

JOURNAL OF MATHEMATICAL PHYSICS **52**, 063509 (2011)

Spectrum generating algebras for the free motion in S^3

M. Gadella,[1,a)] J. Negro,[1,b)] L. M. Nieto,[1,c)] G. P. Pronko,[2,d)] and M. Santander[1,e)]

[1]*Department of FTAO, Facultad de Ciencias, Universidad de Valladolid, 47071 Valladolid, Spain*

[2]*Department of Theoretical Physics, IHEP. Protvino, Moscow Region 142280, Russia*

(Received 13 December 2010; accepted 16 May 2011; published online 17 June 2011)

We construct the spectrum generating algebra (SGA) for a free particle in the three-dimensional sphere S^3 for both classical and quantum descriptions. In the classical approach, the SGA supplies time-dependent constants of motion that allow to solve algebraically the motion. In the quantum case, the SGA includes the ladder operators that give the eigenstates of the free Hamiltonian. We study this quantum case from two equivalent points of view. © *2011 American Institute of Physics*. [doi:10.1063/1.3598407]

I. INTRODUCTION

The notion of the spectrum generating algebra (SGA), sometimes called non-invariance algebra, was introduced many years ago.[1–3,19] In the context of quantum mechanics, the idea of the SGA consists in reducing the construction of the whole Hilbert space for a given system to a problem of representation theory. The knowledge of the symmetry (usually called "dynamical symmetry") of a problem allows to solve it only partly: its representations gives the subspace of the whole Hilbert space of eigenstates corresponding to a fixed energy. The further extension to the SGA needs to introduce ladder operators that change the energy, i.e., operators that do not commute with the Hamiltonian (it is the reason to call this construction non-invariance algebra). At the very best, the whole set of operators —those generating the dynamical algebra plus the ladder operators— may form a finite-dimensional non-compact algebra whose representation gives the Hilbert space of the system. In this respect, the symmetry algebra of the Hamiltonian plays the role of the Cartan subalgebra, while the additional operators of the SGA, which do not commute with the Hamiltonian, play the role of the Borel elements.

In the classical frame, the symmetry algebra provides constants of motion which are functions of the dynamical variables characterizing the possible trajectories. However, the motion is obtained from another kind of constants of motion that include explicitly the time. Such constants come from the elements of the SGA "not commuting" (in the sense of Poisson brackets) with the Hamiltonian.[5]

The main purpose of this paper is using the SGA technique to solve the spectral problem related with the quantum Hamiltonian of the free motion in the three-dimensional sphere S^3, embedded in the four-dimensional coordinate space $\mathbb{R}^4$, which has a pure discrete spectrum.[4] This problem is already nontrivial, interesting by itself and will provide important clues for the extension of the SGA in the study of more general quantum systems evolving on configuration spaces with constant curvature. In the case of the free particle in S^3, it is well known that the symmetry algebra is $\mathfrak{so}(4)$ and our task is to construct the ladder operators which do not commute with the Hamiltonian. As we shall see below in order to achieve this goal we will need to involve, apart from symmetry operators, also the elements of the homogeneous space of the group $SO(4)$. The main result obtained in this work is the explicit construction of a SGA isomorphic to $\mathfrak{so}(4, 2)$.

a)Electronic mail: manuelgadella1@gmail.com.
b)Electronic mail: jnegro@fta.uva.es.
c)Electronic mail: luismi@metodos.fam.cie.uva.es.
d)Electronic mail: George.Pronko@ihep.ru.
e)Electronic mail: msn@fta.uva.es.

